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THE SCATTERING PROBLEM FOR A SYSTEM OF THE FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS ON A SEMI-AXIS

Abstract

In the paper the direct scattering problem is studied for a system of $n \geq 3$ ordinary differential equations on a semi-axis when one incident wave exists. The integral representations of solutions are found too.

On a semi-axis consider the following system of differential equations

$$-i\frac{dy_k(x)}{dx} + \sum_{j=1}^{n} C_{kj}(x) y_j(x) = \lambda \xi_k y_k(x), \quad x \geq 0 , \qquad (1)$$
$$(\xi_1 > 0 > \xi_2 > \cdots > \xi_n),$$

where $C_{kj}(x)\,(k, j = 1,2,...,n)$ are measurable complex-valued functions, $C_{kk}(x) \equiv 0$ and satisfy the conditions

$$\int_0^{+\infty} |C_{kj}(x)| dx < +\infty . \qquad (2)$$

The scattering problem for system of the equations (1) when $n = 2$ on a semi-axis has been studied in paper [1, 2], on all axis-in [3, 4]. The inverse scattering problem for the general system of the equations (1) $(\xi_1 > \xi_2 > \cdots > \xi_n)$ on all axis has been considered in paper [5-8], and when $\operatorname{Im} \xi_i \neq 0$, $\xi_i \neq \xi_k$ in [9]. The scattering problem on a semi-axis for $n \geq 3$ $(\xi_1 > \cdots \xi_{n-1} > 0 > \xi_n)$, $n = 3$ $(\xi_1 > 0 > \xi_2 > \xi_3)$ has been investigated in papers [10, 11].

The scattering problem on a semi-axis is in determination of a solution of the system (1) by given asymptotics and boundary conditions for $x = 0$.

Consider the problem for the system (1) on the semi-axis $n-1$: the k-th problem is in the determination of a solution of the system (1) under the following boundary conditions

$$y_1^k(0, \lambda) = y_{k+1}^k(0, \lambda) ,$$
$$y_i^k(0, \lambda) = 0 , \ i \neq 1; \ \ i \neq k+1 , \qquad (3)$$

and the given asymptotics

$$y_1^k(x, \lambda) = A_k \exp(i\lambda \xi_1 x) + o(1), \quad x \to \infty , \qquad (4)$$
$$i, k = 1,2,...,n-1 .$$

The joint consideration of these problems is called the scattering problem for the system (1) on a semi-axis.

Theorem 1. *Let the coefficients of the system (1) satisfy the condition (2) and λ be a real number. Then there exists a unique boundary solution of the scattering problem for the system (1) on a semi-axis.*

Proof. The scattering problem for the k-th $(k = 1,2,...,n-1)$ problems on a semi-axis is equivalent to the following system of integral equations

$$y_1^k(x,\lambda)=A_k\exp(i\lambda\xi_1 x)+i\int_x^{\infty}\sum_{j=1}^{n}c_{1j}(x')y_j^k(x',\lambda)\exp(i\lambda\xi_1(x-x'))dx',$$

$$y_p^k(x,\lambda)=B_{p-1}^k\exp(i\lambda\xi_p x)+i\int_x^{\infty}\sum_{j=1}^{n}c_{pj}(x')y_j^k(x',\lambda)\exp(i\lambda\xi_p(x-x'))dx', \tag{5}$$

$$(p=2,3,...,n;\ k=1,2,...,n-1)$$

where

$$B_k^k=A_k-i\int_0^{+\infty}\sum_{j=1}^{n}\left[c_{k+1,j}(x')\exp(-i\lambda\xi_{k+1}x')-c_{1j}(x')\exp(-i\lambda\xi_1 x')\right]y_j^k(x',\lambda)dx',$$

$$B_{p-1}^k=-i\int_0^{+\infty}\sum_{j=1}^{n}c_{pj}(x')y_j^k(x',\lambda)\exp(-i\lambda\xi_p x')dx', \tag{6}$$

$(k=1,2,...,n-1;\ p=2,...,n;\ p\neq k+1)$.

Using the method of successive approximations to the system of integral equations (6) we obtain the existence and uniqueness in a class of bounded functions of solutions of these systems. The theorem is proved.

On the other hand from the relation (5) with regard to (5) the validity of the following equalities follows

$$y_1^k(x)=A_k\exp(i\lambda\xi_1 x)+o(1),$$
$$y_p^k(x)=B_{p-1}^k\exp(i\lambda\xi_p x)+o(1),\ k=1,2,...,n-1; \tag{7}$$
$$p=2,3,...,n;\quad x\to+\infty.$$

On the basis of Theorem 1 according to (7) the elements

$$B_i^k=S_{i,k}(\lambda)A_k\ \ (i,k=1,2,..,n-1), \tag{8}$$

generating the matrix $S(\lambda)=\|S_{ik}(\lambda)\|_{i,k=1}^{n-1}$ are determined.

The matrix $S(\lambda)$ is called a scattering matrix for the system of equations (1).

Note that from the definition of $S(\lambda)$ it follows

$$\begin{pmatrix} B_1^1+B_1^2+\cdots+B_1^{n-1}\\ B_2^1+B_2^2+\cdots+B_2^{n-1}\\ \cdots\\ B_{n-1}^1+B_{n-1}^2+\cdots+B_{n-1}^{n-1}\end{pmatrix}=S(\lambda)\begin{pmatrix}A_1\\ A_2\\ \cdots\\ A_{n-1}\end{pmatrix}. \tag{9}$$

If the coefficients of the system (1) $c_{ij}(x)\equiv 0\ (i,j=1,2,...,n)$ from (6) and (9) we obtain that $S(\lambda)$ is a unit matrix

$$S(\lambda)=I.$$

More exact description of structure of scattering matrix is obtained with attraction of integral representations of solutions. We can express a bounded solution of the system (1) on a semi-axis by their asymptotics (i.e. the functions $(A\exp(i\lambda\xi_1 x),\ B_1\exp(i\lambda\xi_2 x),\ \dots,\ B_{n-1}\exp(i\lambda\xi_n x))$), by values of solutions for $x=0$ (i.e. by $y_1(0),\ y_2(0),\ y_3(0)$) or by some combinations of these quantities. With this aim consider $2n$- vector-functions

$$h^1(x,\lambda)=\{y_1(0)\exp(i\xi_1\lambda x),y_2(0)\exp(i\xi_2\lambda x),...,y_n(0)\exp(i\xi_n\lambda x)\},$$

$$h^2(x,\lambda)=\{A\exp(i\xi_1\lambda x),y_2(0)\exp(i\xi_2\lambda x),\dots,y_n(0)\exp(i\xi_n\lambda x)\},$$
$$h^k(x,\lambda)=\{A\exp(i\xi_1\lambda x),B_1\exp(i\xi_2\lambda x),\dots,B_{k-2}\exp(i\xi_{k-1}\lambda x),$$
$$y_k(0)\exp(i\lambda\xi_k x),\dots,y_n(0)\exp(i\lambda\xi_n x)\},\quad (k=3,\dots,n),$$
$$h^{n+1}(x,\lambda)=\{A\exp(i\xi_1\lambda x),B_1\exp(i\lambda\xi_2 x),\dots,B_{n-1}\exp(i\lambda\xi_n x)\},$$
$$h^{n+k}(x,\lambda)=\{y_1(0)\exp(i\lambda\xi_1 x),\dots,y_{k-1}(0)\exp(i\lambda\xi_{k-1}x),B_{k-1}\exp(i\lambda\xi_k x),$$
$$\dots,B_{n-1}\exp(i\lambda\xi_n x)\},\ (k=2,\dots,n). \tag{10}$$

Lemma. *Let the coefficients of the system (1) satisfy the condition (2). Then every bounded solution has the integral representation*

$$y_k(x)=h_k^1(x,\lambda)+\sum_{j=1}^{n}\int_{\xi_n x}^{\xi_1 x}A_{kj}^1(x,\tau)\exp(i\lambda\tau)d\tau\cdot y_j(0,\lambda), \tag{11}$$

$$y_k(x)=h_k^2(x,\lambda)+A\int_{-\infty}^{\xi_1 x}A_{k1}^2(x,\tau)\exp(i\lambda\tau)d\tau+\sum_{j=2}^{n}\int_{-\infty}^{\xi_2 x}A_{kj}^2(x,\tau)\exp(i\lambda\tau)d\tau\cdot y_k(0,\lambda), \tag{12}$$

$$y_k(x)=h_k^p(x,\lambda)+\sum_{j=1}^{p-2}\int_{-\infty}^{+\infty}A_{kj}^p(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_j^p(0,\lambda)+\int_{-\infty}^{\xi_{p-1}x}A_{k,p-1}(x,\tau)\exp(i\lambda\tau)d\tau\times$$
$$\times h_{p-1}^p(0,\lambda)+\sum_{j=p}^{n}\int_{-\infty}^{\xi_p x}A_{kj}^p(x,\tau)\exp(i\lambda\tau)d\tau h_j^p(0,\lambda),\ (3\le p\le n), \tag{13}$$

$$y_k(x)=h_k^{n+1}(x,\lambda)+\int_{\xi_1 x}^{+\infty}A_{k1}^{n+1}(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_1^{n+1}(0,\lambda)+$$
$$+\sum_{j=2}^{n-1}\int_{-\infty}^{\infty}A_{kj}^{n+1}(x,\tau)\exp(i\lambda\tau)d\tau h_j^{n+1}(0,\lambda)+\int_{-\infty}^{\xi_n x}A_{kj}^{n+1}(x,\tau)\exp(i\lambda\tau)h_j^{n+1}(0,\lambda)+$$
$$+\int_{-\infty}^{\xi_n x}A_{kn}^{n+1}(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_n^{n+1}(0,\lambda), \tag{14}$$

$$y_k(x)=h_k^{n+p}(x,\lambda)+\sum_{j=1}^{p-1}\int_{\xi_{p-1}x}^{+\infty}A_{kj}^{n+p}(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_j^{n+p}(0,\lambda)+$$
$$+\int_{\xi_p x}^{\infty}A_{kp}^{n+p}(x,\tau)\exp(i\lambda\tau)d\tau h_p^{n+p}(0,\lambda)+\sum_{j=p+1}^{n}\int_{-\infty}^{+\infty}A_{kj}^{n+p}(x,\tau)\exp(i\lambda\tau)d\tau\, h_j^{n+p}(0,\lambda),$$
$$(2\le p\le n-1), \tag{15}$$

$$y_k(x)=h_k^{2n}(x,\lambda)+\sum_{j=1}^{n-1}\int_{\xi_{n-1}x}^{+\infty}A_{kj}^{2n}(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_j^{2n}(0,\lambda)+$$
$$+\int_{\xi_n x}^{+\infty}A_{kn}^{2n}(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_n^{2n}(0,\lambda). \tag{16}$$

The kernels of these representations are uniquely determined by the coefficients $C_{kj}(x)$ $(k,j=1,\dots,n)$.

Proof. We prove the lemma, for example, for the representations (11). The solution $y(x,\lambda)=\{y_1(x,\lambda),...,y_n(x,\lambda)\}$ with the initial condition $h^1(0,\lambda)=\{y_1(0,\lambda),...,y_n(0,\lambda)\}$ satisfies the system of the integral equations

$$y_k(x)=y_k(0)\exp(i\lambda\xi_k x)-i\int_0^x\sum_{j=1}^n C_{kj}(x')y_j(x',\lambda)\exp(i\xi_k\lambda(x-x'))dx'\ ,k=1,2,...,n\ . \quad (17)$$

We'll search a bounded solution of the system (17) in the form of (11). Substituting (11) in (17) and subject to the arbitrary of $y_1(0),...,y_n(0)$ we obtain the system of the integral equations

$$A_{kj}^1(x,\tau)+\frac{i}{\xi_j-\xi_k}c_{kj}\left(\frac{\tau-\xi_k x}{\xi_j-\xi_k}\right)+i\int_{\max\left(\frac{\tau-\xi_j x}{\xi_1-\xi_j},\frac{\tau-\xi_j x}{\xi_n-\xi_j}\right)}^{x}\sum_{p=1}^n c_{kp}(x')\times$$

$$\times A_{pj}^1(x',\tau-\xi_k(x-x'))dx'=0\ ,\ (k\neq j;k,j=2,...,n-1), \quad (18)$$

$$A_{11}^1(x,\tau)+i\int_{\frac{\tau-\xi_1 x}{\xi_1-\xi_n}}^{x}\sum_{p=1}^n c_{1p}(x')A_{p1}^1(x',\tau-\xi_1(x-x'))dx'=0\ , \quad (19)$$

$$A_{nn}^1(x,\tau)+i\int_{\frac{\tau-\xi_n x}{\xi_1-\xi_n}}^{x}\sum_{p=1}^n c_{np}(x')A_{pn}^1(x',\tau-\xi_n(x-x'))dx'=0\ ,\ \ (\xi_n x\leq\tau\leq\xi_1 x). \quad (20)$$

If the kernels $A_{kj}^1(x,s)$, $k,j=1,2,...,n$ satisfy the system of the equations (18)-(20), then representations (11) give solutions of the equations (17). Thus for proof of the lemma it's sufficient to establish the solvability of the systems (18)-(20). The solvability of these systems follows from Volterra property.

The lemma is proved.

Note that assuming $\tau=\xi_1 x,\ \tau=\xi_n x$ in the system of integral equations (18) we obtain

$$A_{k1}^1(x,\xi_1 x)=\frac{i}{\xi_k-\xi_1}c_{k1}(x),$$

$$A_{kn}^1(x,\xi_n x)=\frac{i}{\xi_k-\xi_n}c_{kn}(x),\ k=2,...,n-1\ .$$

By solving the inverse problems for Sturm-Liouville equations or system of Dirack equations the facts of analiticity of solutions of Yost play the basic role. In our case the solutions (11)-(16) in general don't allow analytical extension with the real axis λ. However the equation (1) has analytical in the upper half plane λ solution and by this

$$y_k^1(x,\lambda)=\delta_{k1}C_1\exp(i\xi_1\lambda x)+C_1\int_{-\infty}^{0}\theta(\tau-(\xi_n-\xi_1)x)A_{k1}^1(x,\tau+\xi_1 x)\exp(i\lambda\tau)d\tau\ ,$$

$$y_k^2(x,\lambda)=\delta_{k2}C_2\exp(i\xi_2\lambda x)+C_2\int_{-\infty}^{0}A_{k2}^2(x,\tau-\xi_2 x)\exp(i\lambda\tau)d\tau\ ,$$

....

$$y_k^n(x,\lambda)=\delta_{kn}C_n\exp(i\xi_n\lambda x)+C_n\int_{-\infty}^{0}A_{kn}^n(x,\tau+\xi_n x)\exp(i\lambda\tau)d\tau,\ (k=1,2,...,n),\qquad(21)$$

where δ_{kj} are Cronocker's symbol, and

$$\theta(x)=\begin{cases}1 & x>0,\\ 0 & x<0.\end{cases}$$

Analogously, we can construct the solution

$$y_k^1(x,\lambda)=\delta_{k1}D_1\exp(i\xi_1\lambda x)+D_1\int_0^{+\infty}A_{k1}^{n+1}(x,\tau+\xi_1 x)\exp(i\lambda\tau),$$

$$y_k^2(x,\lambda)=\delta_{k2}D_2\exp(i\xi_2\lambda x)+D_2\int_0^{+\infty}A_{k2}^{n+1}(x,\tau+\xi_2 x)\exp(i\lambda\tau),\qquad(22)$$

....

$$y_k^n(x,\lambda)=\delta_{kn}D_n\exp(i\xi_n\lambda x)+D_n\int_0^{+\infty}A_{kn}^{n+1}(x,\tau+\xi_n x)\exp(i\lambda\tau),\ (k=1,2,...,n)$$

allowing analytical extension in lower halfplane.

We can write the formulas (21) and (22) in the form of

$$y(x,\lambda)=(I+W_-(x,\lambda))\exp(i\lambda Jx)\begin{pmatrix}C_1\\ C_2\\ \vdots\\ C_n\end{pmatrix},\qquad(23)$$

$$y(x,\lambda)=(I+W_+(x,\lambda))\exp(i\lambda Jx)\begin{pmatrix}D_1\\ D_2\\ \vdots\\ D_n\end{pmatrix},\qquad(24)$$

where

$$y(x)=(y_1(x),...,y_n(x))^t,\ J=diag(\xi_1,...,\xi_n),$$

$$W_-(x,\lambda)=\int_{-\infty}^{0}\widetilde{W}_-(x,\tau)e^{i\lambda\tau}d\tau,$$

$$W_+(x,\lambda)=\int_0^{+\infty}\widetilde{W}_+(x,\tau)e^{i\lambda\tau}d\tau,$$

$$\widetilde{W}_-(x,\tau)=\left\|\widetilde{A}_{jk}^k(x,\tau+\xi_k x)\right\|_{j,k=1}^n,$$

$$\widetilde{A}_{j1}^1(x,\tau+\xi_1 x)=\theta(\tau-(\xi_n-\xi_1)x)A_{j1}^1(x,\tau+\xi_1 x),$$

$$\widetilde{A}_{jk}^k(x,\tau+\xi_k x)=A_{jk}^k(x,\tau+\xi_k x),\ k=2,...,n,$$

$$\widetilde{W}_+(x,\tau)=\left\|A_{jk}^{n+k}(x,\tau+\xi_k x)\right\|_{j,k=1}^n.$$

If we equate the formula (23) to the formula (24) we'll have

$$\exp(i\lambda Jx)S^{\Phi}(\lambda)\exp(i\lambda Jx)=(I+W_+(x,\lambda))^{-1}(I+W_-(x,\lambda)),\qquad(25)$$

here the conversion matrix $S^{\Phi}(\lambda)$ transfer $C=(C_1,...,C_n)^t$ in $D=(D_1,...,D_n)^t$,

$$S^{\Phi 2}C = D.$$

The coefficients of the system (1) are determined by the kernel $\widetilde{W}_{\pm}(x,\tau)$ of the relations

$$\mp i\left[J,\widetilde{W}_{-}(x,0)\right] = \mp C(x), \qquad (26)$$

where $C(x)=\left\|c_{kj}(x)\right\|_{k,j=1}^{n}$ and square brackets $[\cdot,\cdot]$ denote commutator.

Note that the conversion matrix $S^{\Phi}(\lambda)$ is tightly bound with scattering matrix on the semi-axis $S(\lambda)$.

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