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$B_{k,n}$-BESSEL POTENTIALS AND CERTAIN IMBEDDING THEOREMS IN $B_{k,n}$-SOBOLEV--LIOUVILLE SPACES

Abstract

In this work with help the generalized Fourier-Bessel shift operators ($B_{k,n}$-shift) investigated the Bessel potentials, generated by the Bessel differential operators $B_{k,n}=(B_{k+1},\dots,B_n)$, *where* $B_j=\dfrac{\partial^2}{\partial x_j^2}+\dfrac{\gamma_j}{x_j}\dfrac{\partial}{\partial x_j}$, *($B_{k,n}$-Bessel potentials). The boundedness of $B_{k,n}$-Bessel potentials in spaces* $L_{p,\gamma_{k,n}}(R^n_{k,+})=L_p(R^n_{k,+},x_{k,n}^{\gamma_{k,n}}dx),\ 0\le k\le n-1$ *is proved. And also received certain imbedding theorems in $B_{k,n}$-Sobolev-Liouville spaces.*

For $p\in(1,\infty)$ the spaces of Bessel potentials introduced and studied by Aronszajn, Smith [1] and Calderon [2]. The $B_{n-1,n}$-Bessel potentials was investigated by Aliev, Gadjiev (see [3],[4]).

Let R^n n-dimensional Euclidean space of points $x=(x_1,\cdots,x_n)$, $|x|=\left(\sum_{i=1}^n x_i^2\right)^{1/2}$, $1\le k\le n-1$, $x'=x_{1,k}=(x_1,\cdots,x_k)\in R^k$, $x''=x_{k,n}=(x_{k+1},\cdots,x_n)\in R^{n-k}$, $x=(x',x'')=(x_{1,k},x_{k,n})\in R^n$, $R^n_{k,+}=\{x=(x_{1,k},x_{k,n})\in R^n; x_{k+1}>0,\dots,x_n>0\}$, $E_{k,+}(x,r)=\{y\in R^n_{k,+}; |y-x|<r\}$, $\gamma_{k,n}=(\gamma_{k+1},\dots,\gamma_n)$, $\gamma_{k+1}>0,\dots,\gamma_n>0$, $x_{k,n}^{\gamma_{k,n}}=x_{k+1}^{\gamma_{k+1}}\cdot\ldots\cdot x_n^{\gamma_n}$, $|\gamma_{k,n}|=\gamma_{k+1}+\ldots+\gamma_n$.

In the case $k=0$ $x=x''=x_{0,n}\in R^n$, $R^n_{++}\equiv R^n_{0,+}=\{x\in R^n; x_1>0,\dots,x_n>0\}$ and $\gamma=\gamma_{0,n}=(\gamma_1,\dots,\gamma_n)$.

By $L_{p,\gamma_{k,n}}=L_{p,\gamma_{k,n}}(R^n_{k,+})$ we denote the spaces of measurable functions $f(x)$, $x\in R^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}=\|f\|_{p,\gamma_{k,n}}=\left(\int_{R^n_{k,+}}|f(x)|^p x_{k,n}^{\gamma_{k,n}}dx\right)^{1/p},\quad 1\le p\le\infty .$$

We suppose $L_{\infty,\gamma_{k,n}}(R^n_{k,+})=L_\infty(R^n_{k,+})$ where $L_\infty(R^n_{k,+})$ the space of all essential bounded functions $f(x)$, $x\in R^n_{k,+}$ with finite norm

$$\|f\|_{L_{\infty,\gamma_{k,n}}(R^n_{k,+})}\equiv\|f\|_{L_\infty(R^n_{k,+})}=ess\sup_{x\in R^n_{k,+}}|f(x)|.$$

The operator of generalized shift ($B_{k,n}$-shift operator) is defined by following way (see [8],[9],[10]):

$$T^y f(x)=\frac{\prod_{i=k+1}^{n}\Gamma\left(\frac{\gamma_{i+1}}{2}\right)}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{n-k}\prod_{i=k+1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)}\int_0^\pi\cdots\int_0^\pi f\left(x'-y',\sqrt{x_{k+1}^2-2x_{k+1}y_{k+1}\cos\alpha_{k+1}+y_{k+1}^2},\cdots,\right.$$

$$\left.\cdots,\sqrt{x_n^2-2x_ny_n\cos\alpha_n+y_n^2}\right)\cdot\prod_{i=k+1}^{n}\sin^{\gamma_i-1}\alpha_i\,d\alpha_{k+1}\cdots d\alpha_n .$$

Note, that this shift operators is closely connected with $B_{k,n}$-Bessel's singular differential operators (see [9])

$B_{k,n}=(B_{k+1},\dots,B_n)$, where $B_j=\frac{\partial^2}{\partial x_j^2}+\frac{\gamma_j}{x_j}\frac{\partial}{\partial x_j}$, $\gamma_j>0$ $(j=k+1,\dots,n)$.

By virtue of this shift operators the generalized convolution functions ($B_{k,n}$-convolution) introduced

$$(f*g)_{\gamma_{k,n}}=\int_{R_{k,+}^n}f(x)\left(T^y g(x)\right)x_{k,n}^{\gamma_{k,n}}dx$$

For the $B_{k,n}$-shift operator it is valid

Lemma 1. [10] *Let* $1\le p\le\infty$, $f\in L_{p,\gamma_{k,n}}\left(R_{k,+}^n\right)$. *Then*

$$\left\|T^y f(\cdot)\right\|_{L_{p,\gamma_{k,n}}\left(R_{k,+}^n\right)}\le\|f\|_{L_{p,\gamma_{k,n}}\left(R_{k,+}^n\right)},\qquad\forall y\in R_{k,+}^n. \tag{1}$$

We suppose $E_{k,+}(0,r)=\{y\in R_{k,+}^n;|y|<r\}$, *and* $\left|E_{k,+}(0,r)\right|_{\gamma_{k,n}}=\int_{E_{k,+}(0,r)}x_{k,n}^{\gamma_{k,n}}dx$. *Note, that* $\left|E_{k,+}(0,r)\right|_{\gamma_{k,n}}=Cr^{n+|\gamma_{k,n}|}$, *where* $|A|_{\gamma_{k,n}}=\int_A x_{k,n}^{\gamma_{k,n}}dx$, $A\subset R_{k,+}^n$.

For $B_{k,n}$-maximals functions

$$M_{B_{k,n}}f(x)=\sup_{\varepsilon>0}\left|E_{k,+}(0,\varepsilon)\right|_{\gamma_{k,n}}^{-1}\int_{E_{k,+}(0,\varepsilon)}T^y|f(x)|y_{k,n}^{\gamma_{k,n}}dy$$

and $B_{k,n}$-potentials Riesz

$$I_{B_{k,n}}^{\alpha}f(x)=\int_{R_{k,+}^n}T^y|x|^{\alpha-n-|\gamma_{k,n}|}f(y)y_{k,n}^{\gamma_{k,n}}dy,\quad 0<\alpha<n+|\gamma_{k,n}|$$

the following theorems are valid.

Theorem 1. [10] 1) *If* $f\in L_{1,\gamma_{k,n}}\left(R_{k,+}^n\right)$, *then for all* $\alpha>0$

$$\left|\left\{x\in R_{k,+}^n:M_{B_{k,n}}f(x)>\alpha\right\}\right|_{\gamma_{k,n}}\le\frac{C_1}{\alpha}\int_{R_{k,+}^n}|f(x)|x_{k,n}^{\gamma_{k,n}}dx,$$

where C_1 *does depend on* f.

2) *If* $f\in L_{p,\gamma_{k,n}}\left(R_{k,+}^n\right)$, $1<p\le\infty$, *then* $M_{B_{k,n}}f(x)\in L_{p,\gamma_{k,n}}(R_{k,+}^n)$ *and*

$$\left\|M_{B_{k,n}}f\right\|_{p,\gamma_{k,n}}\le C_{p,\gamma_{k,n}}\|f\|_{p,\gamma_{k,n}},$$

where $C_{p,\gamma_{k,n}}$*-depend only on* $p,\gamma_{k,n}$ *and dimension* n.

Theorem 2. [11] *Let* $0<\alpha<n+|\gamma_{k,n}|$, $1\le p<\frac{n+|\gamma_{k,n}|}{\alpha}$ *and* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n+|\gamma_{k,n}|}$.

a) If $f \in L_{p,\gamma_{k,n}}\left(R_{k,+}^{n}\right)$, *then integral* $I_{B_{k,n}}^{\alpha} f$ *converges absolutely for almost all* $x \in R_{k,+}^{n}$.

b) If $1 < p < \frac{n+|\gamma_{k,n}|}{\alpha}$, $f \in L_{p,\gamma_{k,n}}\left(R_{k,+}^{n}\right)$, *then* $I_{B_{k,n}}^{\alpha} f \in L_{q,\gamma_{k,n}}\left(R_{k,+}^{n}\right)$ *and*

$$\left\| I_{B_{k,n}}^{\alpha} f \right\|_{L_{q,\gamma_{k,n}}\left(R_{k,+}^{n}\right)} \le C_p \|f\|_{L_{p,\gamma_{k,n}}\left(R_{k,+}^{n}\right)}. \tag{2}$$

c) If $f \in L_{1,\gamma_{k,n}}\left(R_{k,+}^{n}\right)$, $\frac{1}{q} = 1 - \frac{\alpha}{n+|\gamma_{k,n}|}$, *then*

$$\left| \left\{ x \in R_{k,+}^{n} : I_{B_{k,n}}^{\alpha} f(x) > \beta \right\} \right|_{\gamma_{k,n}}^{1/q} \le \frac{C_1}{\beta} \|f\|_{1,\gamma_{k,n}}. \tag{3}$$

Remark 1. For the $B_{0,n}$-maximal functions and $B_{0,n}$-Riesz potentials analog theorem 1, 2 is proved in [5], and for B_n-Riesz potentials analog theorem 2 is proved in [3].

We suppose

$$j_\nu(t) = \frac{2^\nu \cdot \Gamma(\nu+1) \cdot J_\nu(t)}{t^\nu} = \sum_{k=0}^{\infty} (-1)^k \frac{2^\nu \cdot \Gamma(\nu+1) \cdot t^{2k}}{2^{2k}\,\Gamma(\nu+k+1) \cdot k!},$$

where $t > 0, \nu > -\frac{1}{2}$ and $J_\nu(t)$-Bessel's function of the first kind of order ν. In mathematical literature, the function $j_\nu(t)$ is call the normed Bessel's functions.

By $S_{k,+} \equiv S_{k,+}\left(R_{k,+}^{n}\right)$ we will denote the space of functions $\varphi(x)$ infinitely differentiable, rapidly with all derivatives and even by variables $x_{k+1}, \cdots, x_n$.

Denote by $S'_{k,+} \equiv S'_{k,+}\left(R_{k,+}^{n}\right)$ the spaces of tempered distributions, which dual to $S_{k,+}$.

It is known, that the spaces $S_{k,+}$ and $S'_{k,+}$ with respect to mixed Fourier-Bessel transformations are invariants (see [6], [7]).

The Fourier-Bessel transformations of function $f \in S_{k,+}$ can be defined by

$$F_{B_{k,n}}[f(x)](z) = \int_{R_{k,+}^{n}} f(x) \cdot e^{-i(x',z')} \prod_{j=k+1}^{n} j_{\frac{\gamma_j - 1}{2}}\left(x_j \cdot z_j\right) x_{k,n}^{\gamma_{k,n}} dx,$$

and its inverse transformation can be given by

$$F_{B_{k,n}}^{-1} f(x) = C_{\gamma_{k,n}} \int_{R_{k,+}^{n}} \left(F_{B_{k,n}} f\right)(x) \prod_{j=k+1}^{n} j_{\frac{\gamma_j - 1}{2}}\left(x_j \cdot z_j\right) x_{k,n}^{\gamma_{k,n}} dx.$$

Using the property of generalized shift, easily show, that (see [8])

$$F_{B,n}(f * g) = F_{B,n}(f) \cdot F_{B,n}(g).$$

The Fourier-Bessel transformations and $B_{k,n}$-differentials operators have following connected (see [12])

$$P\left(\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n}, B_{k+1}, \ldots, B_n\right) F_{B_{k,n}} \varphi = F_{B_{k,n}}\left[P\left(-ix_1, \ldots, -ix_k, -x_{k+1}^2, \ldots, -x_n^2\right) \cdot \varphi\right],$$

$$F_{B_{k,n}}\left[P\left(\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n}, B_{k+1}, \ldots, B_n\right) \varphi\right] = P\left(-ix_1, \ldots, -ix_k, -x_{k+1}^2, \ldots, -x_n^2\right) \cdot F_{B_{k,n}} \varphi.$$

We have (see [12]).

$$F_{B_{k,n}}\left[\left(1+|x|^2\right)^{-\frac{\alpha}{2}}\right](\xi)=\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int_0^\infty F_{B_{k,n}}\left(e^{-\delta|x|^2}\right)(\xi)\cdot e^{-\delta}\cdot\delta^{\frac{\alpha}{2}}\cdot\frac{d\delta}{\delta},$$

$$F_{B_{k,n}}\left[e^{-\delta|x|^2}\right](\xi)=\int_{R_+^{n-k}}\prod_{j=k+1}^{n} j_{\frac{\gamma_j-1}{2}}\left(x_j\cdot\xi_j\right)e^{-\delta x_j^2}x_{k,n}^{\gamma_{k,n}}dx''\int_{R^k}e^{-i(x',\xi')}e^{-\delta|x'|^2}dx'=$$

$$=\frac{1}{2}\cdot\delta^{-\frac{|\gamma_{k,n}|+n-k}{2}}e^{-\frac{|\xi''|^2}{4\delta}}\prod_{j=k+1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)e^{-\frac{|\xi'|^2}{4\delta}}\pi^{\frac{k}{2}}\delta^{-\frac{k}{2}}.$$

Then

$$F_{B_{k,n}}\left[\left(1+|x|^2\right)^{-\frac{\alpha}{2}}\right](\xi)=\frac{1}{2}\cdot\pi^{\frac{k}{2}}\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\prod_{j=k+1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)\int_0^\infty\delta^{-\frac{|\gamma_{k,n}|+n-\alpha}{2}}\cdot e^{-\delta-\frac{|\xi|^2}{4\delta}}\frac{d\delta}{\delta}.$$

We denote

$$G_{B_{k,n}}^{(\alpha)}(x)=C_{\gamma_{k,n}}\cdot\pi^{\frac{k}{2}}\prod_{j=k+1}^{n}\frac{\Gamma\left(\frac{\gamma_j+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\int_0^\infty\delta^{-\frac{|\gamma_{k,n}|+n-\alpha}{2}}\cdot e^{-\delta-\frac{|\xi|^2}{4\delta}}\frac{d\delta}{\delta},$$

where

$$C_{\gamma_{k,n}}=\left[2^{|\gamma_{k,n}|+n-1}\cdot\pi^k\cdot\prod_{j=k+1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)\right]^{-1}.$$

Taking into account, that for $\varphi\in S_{k,+}$ $F_{B_{k,n}}^{-1}\varphi=C_{\gamma_{k,n}}\cdot F_{B_{k,n}}\varphi$ we have

$$F_{B_{k,n}}\left(G_{B_{k,n}}^{(\alpha)}\right)(x)=\left(1+|x|^2\right)^{-\frac{\alpha}{2}}.$$

From here we obtain, $\left\|G_{B_{k,n}}^{(\alpha)}\right\|_{L_{1,\gamma_{k,n}}(R_{k,+}^n)}=1$ Indeed,

$$\left\|G_{B_{k,n}}^{(\alpha)}\right\|_{L_{1,\gamma_{k,n}}(R_{k,+}^n)}=\int_{R_{k,+}^n}G_{B_{k,n}}^{(\alpha)}(x)x_{k,n}^{\gamma_{k,n}}dx=F_{B_{k,n}}\left(G_{B_{k,n}}^{(\alpha)}\right)(x)\Big|_{x=0}=\left(1+|x|^2\right)^{-\frac{\alpha}{2}}\Big|_{x=0}=1.$$

Now we define the function φ_k and ψ. $F_{B_{k,n}}\varphi_k(\xi)=\varphi\left(2^{-k}\xi\right),(k=0,\pm1,\pm2,\cdots)$, $F_{B_{k,n}}\psi(\xi)=1-\sum_{k=1}^{\infty}\varphi\left(2^{-k}\xi\right)$, where $\varphi\in S_{k,+}\left(R_{k,+}^n\right)$ have the following properties:

$$supu\varphi(\xi)=\left\{\xi\,|\,2^{-1}\le|\xi|\le2\right\},\quad \varphi(\xi)>0\ \ for\ \ 2^{-1}<|\xi|<2,\quad \sum_{k=-\infty}^{\infty}\varphi\left(2^{-k}\xi\right)=1\ \ for\ \ \xi\neq0.$$

As since, $\varphi_k(\xi)=F_{B_{k,n}}^{-1}\varphi\left(2^{-k}\xi\right)$ and $\psi(\xi)=F_{B_{k,n}}^{-1}\left\{1-\sum_{k=1}^{\infty}\varphi\left(2^{-k}\xi\right)\right\}$, then $\varphi_k\in S_{k,+}$ and $\psi\in S_{k,+}$.

We define the operator $J^{-\alpha}$ the following way (see [4], [13]),

$$J^\alpha f=F_{B_{k,n}}^{-1}\left\{\left(1+|\cdot|^2\right)^{\alpha/2}F_{B_{k,n}}f\right\}\qquad\left(\alpha\in R,\quad f\in S'_{k,+}\right).$$

It is possible show, that $J^\alpha:S'_{k,+}\to S'_{k,+}$.

The operator $J^{-\alpha}$ we call $B_{k,n}$-Bessel potentials.

Definition 1. *Let* $m(\xi)\in S'_{k,+}$. *The generalized function* $m(\xi)$ *is called* $B_{k,n}$-*Fourier-Bessel-multiplier in* $L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$, *if for all* $f\in S_{k,+}$ $B_{k,n}$-*convolution* $\left(F^{-1}_{B_{k,n}}m(\xi)\right)*f$ *belong to* $L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ *and quantity*

$$\|m(\cdot)\|_{M_{p,\gamma_{k,n}}(R^n_{k,+})}=\sup_{\|f\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}=1}\left\|\left(F^{-1}_{B_{k,n}}m(\xi)\right)*f\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}$$

is finite. This linear spaces we denote by $M_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$.

Theorem 3. [6] **(The** $B_{k,n}$**-analog of Mikhlin's theorem on multipliers).** *Let* $m(\xi)$ *belong to the class of functions* C^N_{even} *(i.e.* $m(\xi)\in C^N\left(R^n\right)$ *and even by variables* $x_{k+1},\cdots,x_n$*), where* N *-least even number,more than* $\frac{1}{2}\left(n+\left|\gamma_{k,n}\right|\right)$, *and exist constant* A *does't depend of* $\alpha=(\alpha',\alpha'')$, *such that for* $\xi\in R^n_{k,+}$,

$$|\xi|^{2|\alpha'|+|\alpha''|}\left|D^{2\alpha'}_{x'}B^{\alpha''}_{k,n}m(\xi)\right|\le A,$$

here $\alpha'=(\alpha_1,\cdots,\alpha_k)$, $\alpha''=(\alpha_{k+1},\ldots,\alpha_n)$ *and* $2|\alpha'|+|\alpha''|\le N$.

Then $m(\xi)\in M_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$.

Here beneath $D^{2\alpha'}_{x'}B^{\alpha''}_{k,n}$ the following differential operator be understood:

$$D^{2\alpha'}_{x'}B^{\alpha''}_{k,n}=\frac{\partial^{2|\alpha'|}}{\partial x_1^{2\alpha_1}\cdots\partial x_k^{2\alpha_k}}B^{\alpha_{k+1}}_{k+1}\cdots B^{\alpha_n}_n.$$

Lemma 2. *If* $1\le p\le q\le 2$, *then* $M_{p,\gamma_{k,n}}\subset M_{q,\gamma_{k,n}}$. *And also , if* $\frac{1}{p}+\frac{1}{p'}=1$, $1\le p\le\infty$ *then* $M_{p,\gamma_{k,n}}\subset M_{p',\gamma_{k,n}}$ *(with equality of norms).*

In the case $k=n-1$ lemma 2 is proved in [13] and in the case $0\le k\le n-1$ the lemma is proved analogously .

Lemma 3. *Let* l *even number,* $l>(n+|\gamma_{k,n}|)/2$ *and let* $m(\xi)\in L_{2,\gamma_{k,n}}\left(R^n_{k,+}\right)$ *and* $D^{2\alpha'}_{x'}B^{\alpha''}_{k,n}m(\xi)\in L_{2,\gamma_{k,n}}\left(R^n_{k,+}\right)$, $2|\alpha'|+|\alpha''|=l$. *Then* $m(\xi)\in M_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$, $1\le p\le\infty$, *and*

$$\|m(\cdot)\|_{M_{p,\gamma_{k,n}}}\le C\|m(\cdot)\|^{1-\theta}_{L_{2,\gamma_{k,n}}(R^n_{k,+})}\left(\sup_{2|\alpha'|+|\alpha''|=l}\left\|D^{2\alpha'}_{x'}B^{\alpha''}_{k,n}m(\cdot)\right\|_{L_{2,\gamma_{k,n}}(R^n_{k,+})}\right)^{\theta},$$

where $\theta=\dfrac{n+|\gamma_{k,n}|}{2l}$.

Proof. Let $t>0$. Applying the Holder's and Parseval's inequalities, we obtain:

$$\int_{|x|>t}\left|F^{-1}_{B_{k,n}}m(\xi)\right|x^{\gamma_{k,n}}_{k,n}dx=\int_{|x|>t}|x|^{-l}|x|^{l}\left|F^{-1}_{B_{k,n}}m(\xi)\right|x^{\gamma_{k,n}}_{k,n}dx\le$$

$$\le\left(\int_{|x|>t}|x|^{-2l}x^{\gamma_{k,n}}_{k,n}dx\right)^{1/2}\left(\int_{|x|>t}|x|^{2l}\left|F^{-1}_{B_{k,n}}m(\xi)\right|^2x^{\gamma_{k,n}}_{k,n}dx\right)^{1/2}=$$

$$=\left[\int_{|x|>t}|x|^{-2l}x^{\gamma_{k,n}}_{k,n}dx=C\int_t^{\infty}r^{n+|\gamma_{k,n}|-2l-1}dr=Ct^{n+|\gamma_{k,n}|-2l}\right]=$$

$$=Ct^{\frac{n+|\gamma_{k,n}|-2l}{2}}\cdot\left(\int_{|x|>t}|x|^{2l}\left|F^{-1}_{B_{k,n}}m(\xi)\right|^2 x_{k,n}^{\gamma_{k,n}}dx\right)^{1/2}.$$

Taking into account, $|x|^l F^{-1}_{B_{k,n}}m(\xi)=F^{-1}_{B_{k,n}}\left(D^{2\alpha'}_{x'}B^{\alpha''}_{k,n}m\right)(x)$ and

$$\left(\int_{|x|>t}|x|^{2l}\left|F^{-1}_{B_{k,n}}m(\xi)\right|^2 x_{k,n}^{\gamma_{k,n}}dx\right)^{1/2}=\left(\int_{|x|>t}\left|F^{-1}_{B_{k,n}}\left(D^{2\alpha'}_{x'}B^{\alpha''}_{k,n}m(\xi)\right)\right|^2 x_{k,n}^{\gamma_{k,n}}dx\right)^{1/2}\le$$

$$\le C\left(\int_{|x|>t}\left(D^{2\alpha'}_{x'}B^{\alpha''}_{k,n}m(\xi)\right)^2 x_{k,n}^{\gamma_{k,n}}dx\right)^{1/2}\le\sup_{2|\alpha'|+|\alpha''|=l}\left\|D^{2\alpha'}_{x'}B^{\alpha''}_{k,n}m(\cdot)\right\|_{L_{2,\gamma_{k,n}}(R^n_{k,+})},$$

we have

$$\int_{|x|>t}\left|F^{-1}_{B_{k,n}}m(\xi)\right|x_{k,n}^{\gamma_{k,n}}dx\le Ct^{\frac{n+|\gamma_{k,n}|-2l}{2}}\cdot\sup_{2|\alpha'|+|2\alpha''|=l}\left\|D^{2\alpha'}_{x'}B^{\alpha''}_{k,n}m(\cdot)\right\|_{L_{2,\gamma_{k,n}}(R^n_{k,+})}.$$

Analogously we proved, that

$$\int_{|x|<t}\left|F^{-1}_{B_{k,n}}m(x)\right|x_{k,n}^{\gamma_{k,n}}dx\le\left(\int_{|x|<t}x_{k,n}^{\gamma_{k,n}}dx\right)^{1/2}\left(\int_{|x|<t}\left|F^{-1}_{B_{k,n}}m(x)\right|^2 x_{k,n}^{\gamma_{k,n}}dx\right)^{1/2}\le$$

$$\le Ct^{\frac{n+|\gamma_{k,n}|}{2}}\left(\int_{|x|<t}\left|F^{-1}_{B_{k,n}}m(x)\right|^2 x_{k,n}^{\gamma_{k,n}}dx\right)^{1/2}\le Ct^{\frac{n+|\gamma_{k,n}|}{2}}\|m(\cdot)\|_{L_{2,\gamma_{k,n}}(R^n_{k,+})}.$$

Taking t so that

$$\|m(\cdot)\|_{L_{2,\gamma_{k,n}}(R^n_{k,+})}=t^{-l}\sup_{2|\alpha'|+|2\alpha''|=l}\left\|D^{2\alpha'}_{x'}B^{\alpha''}_{k,n}m(\cdot)\right\|_{L_{2,\gamma_{k,n}}(R^n_{k,+})},$$

we conclude that for $1\le p\le 2$, by virtue of lemma 2,

$$\|m(\cdot)\|_{M_{p,\gamma_{k,n}}(R^n_{k,+})}\le\|m(\cdot)\|_{M_{1,\gamma_{k,n}}(R^n_{k,+})}=\int_{R^n}\left|F^{-1}_{B_{k,n}}m(\xi)\right|x_{k,n}^{\gamma_{k,n}}dx\le$$

$$\le C\|m(\cdot)\|^{1-\theta}_{L_{2,\gamma_{k,n}}(R^n_{k,+})}\left(\sup_{2|\alpha'|+|2\alpha''|=l}\left\|D^{2\alpha'}_{x'}B^{\alpha''}_{k,n}m(\cdot)\right\|_{L_{2,\gamma_{k,n}}(R^n_{k,+})}\right)^{\theta}.$$

For $2<p\le\infty$, by virtue of lemma 2, we have

$$\|m(\cdot)\|_{M_{p,\gamma_{k,n}}(R^n_{k,+})}=\|m(\cdot)\|_{M_{p',\gamma_{k,n}}(R^n_{k,+})}\le\|m(\cdot)\|_{M_{1,\gamma_{k,n}}(R^n_{k,+})}\le$$

$$\le C\|m(\cdot)\|^{1-\theta}_{L_{2,\gamma_{k,n}}(R^n_{k,+})}\left(\sup_{2|\alpha'|+|2\alpha''|=l}\left\|D^{2\alpha'}_{x'}B^{\alpha''}_{k,n}m(\cdot)\right\|_{L_{2,\gamma_{k,n}}(R^n_{k,+})}\right)^{\theta}.$$

Lemma 4. *Let $f\in S'_{k,+}$ and $\varphi_k * f\in L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$. Then for $1\le p\le\infty,\alpha\in R$, we have*

$$\left\|J^{\alpha}\varphi_k * f\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}\le C2^{k\alpha}\left\|\varphi_k * f\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}\quad for\ k\ge 1. \tag{4}$$

If, furthermore, $\varphi_k * f\in L_{p,\gamma_{k,n}}(R^n_{k,+})$, then

$$\left\|J^{\alpha}\psi * f\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}\le C\left\|\psi * f\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}, \tag{5}$$

where the constant C does't depend of p and k.

Proof. Note, that for all k the equality is satisfied

$$\varphi_k * f = \sum_{l=-1}^{1} \varphi_{k+l} * \varphi_k * f \,.$$

If we prove, that

$$\left\| F_{B_{k,n}}\left(J^{\alpha}\varphi_{k+l}\right)\right\|_{M_{p,\gamma_{k,n}}} \le C2^{k\alpha}, \quad (l=0,\pm 1), \tag{6}$$

then we obtain (4). In order to proved (6), observe, that function

$$F_{B_{k,n}}\left\{J^{\alpha}\varphi_{k+l}\right\}(\xi) = \left(1+|\xi|^2\right)^{\alpha/2} F_{B_{k,n}}\varphi_{k+l}(\xi) = \left(1+|\xi|^2\right)^{\alpha/2} \varphi\left(2^{-(k+l)}\xi\right)$$

has the same norm in $M_{p,\gamma_{k,n}}$, as a function $2^{(k+l)\alpha}\left(2^{-2(k+l)}+|\xi|^2\right)^{\alpha/2}\varphi(\xi)$. Indeed,

$$\left\| F_{B_{k,n}}\left\{J^{\alpha}\varphi_{k+l}\right\}\right\|_{M_{p,\gamma,n}} = \left\| \left(1+|\cdot|^2\right)^{\alpha/2}\varphi\left(2^{-(k+l)}\,\cdot\right)\right\|_{M_{p,\gamma,n}} =$$

$$= \left\| \left(1+2^{(k+l)}|\cdot|^2\right)^{\alpha/2}\varphi(\cdot)\right\|_{M_{p,\gamma,n}} = \left\| 2^{(k+l)\alpha}\left(2^{-2(k+l)}+|\cdot|^2\right)^{\alpha/2}\varphi(\cdot)\right\|_{M_{p,\gamma,n}}$$

and by virtue of lemma 2 it is possible show, that the last function really belong to $M_{p,\gamma_{k,n}}$, and also its norm there not exceed $C2^{k\alpha}$ $(k \ge 1)$; thus the inequality (6) is proved.

In order that to proved (5) observe that

$$\varphi_k * f = \left(\psi + \varphi_1\right) * \psi * f \,.$$

And the fact $F_{B_{k,n}}\left(J^{\alpha}\psi\right) \in M_{p,\gamma_{k,n}}$ obviously, in view of the lemma 3.

Now we define the $B_{k,n}$-Sobolev-Liouville space (see [13], [4], [14]).

Definition 2. [14] *Let* $\alpha \in R, 1 \le p \le \infty$. *The* $B_{k,n}$*-Sobolev-Liouville space* $L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ *be define of relation*

$$L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right) = \left\{ f : f \in S'_{k,+}, \|f\|_{L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)} = \left\|J^{\alpha} f\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)} < \infty \right\}.$$

$B_{k,n}$-Sobolev-Liouville space $L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ is banach space. In the order to proved its complete we suppose, that $\{f_n\}$ is Cauhy sequences in $L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$. Then exist the function $g \in L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ ($L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ is complete space), such, that

$$\left\| f_n - J^{-\alpha} g\right\|_{L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)} = \left\| J^{\alpha} f_n - g\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)} \to 0 \quad (n \to \infty).$$

Clearly, that $J^{-\alpha} g \in S'_{k,+}$ and $L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ is complete space.

In next theorem we given other definition of spaces $L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ for positive integers α in term of derivatives $D^{2\alpha'}_{x'} B^{\alpha''}_{k,n}$ $(|s| \le \alpha)$ of function $f \in L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ (observe, that $L^{0}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right) = L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right), 1 \le p \le \infty$).

Theorem 4. *For* $\alpha_1 < \alpha_2$

$$L^{\alpha_2}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right) \subset_{>} L^{\alpha_1}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right) \quad (1 \le p \le \infty).$$

Further, if $N \ge 1$ *-integer number and* $1 < p < \infty$, *then*

$$L^{\alpha_2}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)=\left\{f\in L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)\Big|\frac{\partial^{2N}f}{\partial x_j^{2N}}\in L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right),\ 1\le j\le k,\right.$$
$$\left.B^N_{k+1}f,\dots,B^N_n f\in L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)\right\}$$

and norm $\|f\|_{L^N_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}$ *is equivalent to*

$$\sum_{i=1}^{k}\left\|\frac{\partial^{2N}f}{\partial x_j^{2N}}\right\|_{p,\gamma_{k,n}}+\sum_{i=k+1}^{n}\left\|B_i^N f\right\|_{p,\gamma_{k,n}}+\|f\|_{p,\gamma_{k,n}} \tag{7}$$

and finally, $S_{k,+}$ *is dense in* $L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ $(1\le p<\infty)$.

Proof. Let $f\in L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$. We show, that $J^{\alpha_1-\alpha_2}$ map $L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ into $L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$.

In order to convince, that $J^{-\varepsilon}:L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)\to L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$, applying the lemma 3 and taking into account, that $f=\psi*f+\sum_{k=1}^{\infty}\varphi_k*f$ we obtain,

$$\left\|J^{-\varepsilon}f\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}\le\left\|J^{-\varepsilon}\psi*f\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}+\sum_{k=1}^{\infty}\left\|J^{-\varepsilon}\varphi_k*f\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}\le$$
$$\le C\left(\|\psi*f\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}+\sum_{k=1}^{\infty}2^{-\varepsilon k}\|\varphi_k*f\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}\right)\le C\left(1+\sum_{k=1}^{\infty}2^{-\varepsilon k}\right)\|f\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)},$$

where $\varepsilon=\alpha_2-\alpha_1>0$.

From here will be follow the first of conclusion of the theorem, as since

$$\|f\|_{L^{\alpha_1}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}=\left\|J^{\alpha_1}f\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}=$$
$$=\left\|J^{\alpha_1-\alpha_2}J^{\alpha_2}f\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}\le C\left\|J^{\alpha_2}f\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}=C\|f\|_{L^{\alpha_2}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}.$$

We prove the second conclusion of the theorem. Using the $B_{k,n}$-analog of Mikhlin's theorem on multipliers [6], we obtain, that $\xi_j^{2N}\left(1+|\xi|^2\right)^{-N}\in M_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ $(1<p<\infty)$. Therefore for $1\le j\le k+1$

$$\left\|\frac{\partial^{2N}f}{\partial x_j^{2N}}\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}=\left\|F^{-1}_{k,n}\left\{\xi_j^{2N}F_{B_{k,n}}f\right\}\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}=$$
$$=\left\|F^{-1}_{k,n}\left\{\xi_j^{2N}\left(1+|\cdot|^2\right)^{-N}F_{B_{k,n}}\left(J^N f\right)\right\}\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}\le C\left\|J^N f\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}=C\|f\|_{L^N_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)},$$

and for $k+1\le j\le n$

$$\left\|B_j^N f\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}=\left\|F^{-1}_{B_{k,n}}\left\{\xi_j^{2N}F_{B_{k,n}}f\right\}\right\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}\le C\|f\|_{L^N_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}.$$

Now we prove the inverse inequality. Once more we apply the $B_{k,n}$-analog of Mikhlin's theorem on multipliers [6]. Further, let χ infinitely differentiable, nonnegative function on R, which $\chi(x)=1$ for $|x|>2$ and $\chi(x)=0$ for $|x|<1$. Then we obtain

$$\left(1+|\xi|^2\right)^N\left(1+\sum_{j=1}^{n}\chi(\xi_j)\left|\xi_j\right|^{2N}\right)^{-1}\in M_{p,\gamma_{k,n}},\quad \chi(\xi_j)\left|\xi_j\right|^{2N}\xi_j^{-2N}\in M_{p,\gamma_{k,n}},$$

where $(1<p<\infty)$.

Thus,

$$\left\|J^N f\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})} \le \left\|F^{-1}_{B_{k,n}}\left\{\left(1+\sum_{j=1}^{k}\chi(\xi_j)|\xi_j|^{2N}+\sum_{j=k+1}^{n}\chi(\xi_j)|\xi_j|^{2N}\right)F_{B_{k,n}}f\right\}\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})} \le$$

$$\le C\left(\|f\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}+\sum_{j=1}^{k}\left\|F^{-1}_{B_{k,n}}\left\{\chi(\xi_j)|\xi_j|^{2N}\xi_j^{-2N}F_{B_{k,n}}\left(\partial^{2N}f/\partial x_j^{2N}\right)\right\}\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}+\right.$$

$$\left.+\sum_{j=k+1}^{n}\left\|F^{-1}_{B_{k,n}}\left\{\chi(\xi_j)|\xi_j|^{2N}\xi_j^{-2N}F_{B_{k,n}}\left(B_j^N f\right)\right\}\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}\right)\le$$

$$\le C\left(\|f\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}+\sum_{j=1}^{k}\left\|\partial^{2N}f/\partial x_j^{2N}\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}+\sum_{j=k+1}^{n}\left\|B_j^N f\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}\right).$$

Now we prove dense the space $S_{k,+}$ in space $L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$. Let $f\in L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ i.e. $J^{\alpha}f\in L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$. As since $S_{k,+}$ densely in $L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ $(1\le p<\infty)$ [3], then find the sequence functions $g_n\in S_{k,+}$, such that satisfied $\left\|J^N f-g_n\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}<\infty$. Then

$$\left\|f-J^{\alpha}g_n\right\|_{L^{\alpha}_{p,\gamma_{k,n}}(R^n_{k,+})}=\left\|J^{\alpha}f-g_n\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}$$

small any given positive number. As since $J^{-\alpha}g\in S_{k,+}$, then we conclude, that $S_{k,+}$ is dense in $L^{\alpha}_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$.

Theorem 4 is proved.

Consider the $B_{k,n}$-Bessel potentials

$$\left(J^{\alpha}_{B_{k,n}}f\right)(x)=\left(G^{(\alpha)}_{B_{k,n}}*f\right)(x)=\int_{R^n_{k,+}}T^y f(x)\cdot\left(G^{(\alpha)}_{B_{k,n}}(y)\right)\cdot y_{k,n}^{\gamma_{k,n}}dy. \tag{8}$$

Note that the following Hardy-Littlewood-Sobolev type theorem is valid

Theorem 5. *Let* $f\in L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$, $1\le p\le\infty$ *and* $J^{\alpha}_{B_{k,n}}f$ *define as in (11). Then:*

a) $\left\|J^{\alpha}_{B_{k,n}}f\right\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}\le\|f\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}$, $1\le p\le\infty$;

b) if $1<p<q<\infty$ *and* $\alpha=\left(n+|\gamma_{k,n}|\right)\left(\frac{1}{p}-\frac{1}{q}\right)$, *then*

$$\left\|J^{\alpha}_{B_{k,n}}f\right\|_{L_{q,\gamma_{k,n}}(R^n_{k,+})}\le A_1\cdot\|f\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}; \tag{9}$$

c) if $\alpha=\left(n+|\gamma_{k,n}|\right)\left(1-\frac{1}{q}\right)$ *then*

$$\left|\left\{x:R^n_{k,+}\left|\left(J^{\alpha}_{B_{k,n}}f\right)(x)>\lambda\right.\right\}\right|_{\gamma_{k,n}}\le\frac{A_2}{\lambda^q}\|f\|^q_{L_{1,\gamma_{k,n}}(R^n_{k,+})},$$

here A_1, A_2 *-the positive constants are independent on function* f.

Remark 2. For B_n-Bessel potentials analog of theorem 5 is proved in [4] other method.

Proof. a) implies from the Young's inequality taking account of, that $\left\|G_{B_{k,n}}^{(\alpha)}\right\|_{L_{1,\gamma_{k,n}}}=1$. In order that to prove б) and в), the kernel $G_{B_{k,n}}^{(\alpha)}(x)$ represent in the form $G_{B_{k,n}}^{(\alpha)}(x)=G_1(x)+G_2(x)$ where

$$G_1(x)=\begin{cases}G_{B_{k,n}}^{(\alpha)}(x), & |x|<t\\ 0, & |x|\ge t\end{cases} \quad\text{and}\quad G_2(x)=\begin{cases}0, & |x|\le t\\ G_{B_{k,n}}^{(\alpha)}(x), & |x|>t\end{cases}$$

Then $J_{B_{k,n}}^{\alpha}f=G_1*f+G_2*f=A(x,t)+C(x,t)$ and hence,

$$\left\|J_{B_{k,n}}^{\alpha}f\right\|_{L_{q,\gamma_{k,n}}(R_{k,+}^n)}\le\|G_1*f\|_{L_{q,\gamma_{k,n}}(R_{k,+}^n)}+\|G_2*f\|_{L_{q,\gamma_{k,n}}(R_{k,+}^n)}.$$

Taking into account the asymptotic equalities, we obtain

$$G_1(x)=C_1\cdot|x|^{-n-|\gamma_{k,n}|+\alpha}+o\left(|x|^{-n-|\gamma_{k,n}|+\alpha}\right),\ \text{for } |x|\to0,$$

$$\text{and } 0<\alpha<n+|\gamma_{k,n}| \text{ and } G_2(x)=O\left(e^{-\frac{1}{2}|x|}\right),\ \text{for } |x|\to\infty.$$

Let k -forall integer number. Summable by all $k<0$, we obtain

$$|A(x,t)|\le\int_{E_{k,+}(0,t)}T^y|f(x)|\left(G_{B_{k,n}}^{(\alpha)}(y)\right)y_{k,n}^{\gamma_{k,n}}dy=\sum_{m=-\infty}^{-1}\int_{2^m t\le|y|<2^{m+1}t}T^y|f(x)|\left(G_{B_{k,n}}^{(\alpha)}(y)\right)y_{k,n}^{\gamma_{k,n}}dy\le$$

$$\le C\sum_{m=-\infty}^{-1}\left(2^m t\right)^{\alpha-n-|\gamma_{k,n}|}\int_{2^m t\le|y|<2^{m+1}t}T^y|f(x)|y_{k,n}^{\gamma_{k,n}}dy\le Ct^{\alpha}\left(M_{B_{k,n}}f\right)(x).$$

Thus, the following estimate is valid

$$|A(x,t)|\le Ct^{\alpha}\left(M_{B_{k,n}}f\right)(x), \qquad (10)$$

where constant C is independent on f,x and t.

Further, applying the Holder inequality and inequality (1), we have

$$|C(x,t)|\le\left(\int_{R_{k,+}^n\setminus E_{k,+}(0,t)}T^y|f(x)|^p y_{k,n}^{\gamma_{k,n}}dy\right)^{1/p}\left(\int_{R_{k,+}^n\setminus E_{k,+}(0,t)}\left(G_{B_{k,n}}^{(\alpha)}(y)\right)^{p'}y_{k,n}^{\gamma_{k,n}}dy\right)^{\frac{1}{p'}}\le$$

$$\le\left\|T^yf\right\|_{L_{p,\gamma_{k,n}}(R_{k,+}^n)}\left(\int_{R_{k,+}^n\setminus E_{k,+}(0,t)}\left(G_{B_{k,n}}^{(\alpha)}(y)\right)^{p'}y_{k,n}^{\gamma_{k,n}}dy\right)^{\frac{1}{p'}}\le C\|f\|_{L_{p,\gamma_{k,n}}(R_{k,+}^n)}\left(\int_{R_{k,+}^n\setminus E_{k,+}(0,t)}e^{-\frac{|y|}{2}p'}y_{k,n}^{\gamma_{k,n}}dy\right)^{\frac{1}{p'}}.$$

Passing to the spherical coordinates, we have

$$\left(\int_{R_{k,+}^n\setminus E_{k,+}(0,t)}e^{-\frac{|y|}{2}p'}y_{k,n}^{\gamma_{k,n}}dy\right)^{\frac{1}{p'}}=C\left(\int_t^{\infty}e^{-\frac{\tau p'}{2}}r^{n+|\gamma_{k,n}|-1}dr\right)^{\frac{1}{p'}},$$

$$\lim_{t\to\infty}e^{-\frac{tp'}{2}}t^{\beta}=0\quad\forall\beta.$$

As since $\lim_{t\to\infty}e^{-\frac{tp'}{2}}t^{n+|\gamma_{k,n}|+\frac{n+|\gamma_{k,n}|}{qp'}}=0$, then exist $C>0$, that $\forall t>1$ it is valid the inequality $e^{-\frac{tp'}{2}}t^{n+|\gamma_{k,n}|+\frac{n+|\gamma_{k,n}|}{qp'}}\le C$. Therefore, taking into account, that

$$\left(\int_t^\infty r^{n+|\gamma_{k,n}|-1-\left(n+|\gamma_{k,n}|+\frac{n+|\gamma_{k,n}|}{qp'}\right)}dr\right)^{\frac{1}{p'}} \le Ct^{-\frac{n+|\gamma_{k,n}|}{qp'}},$$

we obtain

$$\left(\int_{R^n_{k,+}\setminus E_{k,+}(0,t)} e^{-\frac{|y|}{2}p'} y_{k,n}^{\gamma_{k,n}} dy\right)^{\frac{1}{p'}} \le Ct^{-\frac{n+|\gamma_{k,n}|}{q}}.$$

Obviously,

$$|C(x,t)| \le C\|f\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})} t^{-\frac{n+|\gamma_{k,n}|}{q}}. \tag{11}$$

Then from the theorems 1 and 2, we obtain

$$\left|J^\alpha_{B_{k,n}} f(x)\right| \le C\left(t^\alpha\left(M_{B_{k,n}} f\right)(x) + t^{-\frac{n+|\gamma_{k,n}|}{q}} \|f\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}\right).$$

Minimizing by t for $t = C\left[\left(\left(M_{B_{k,n}} f\right)(x)\right)^{-1}\|f\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}\right]^{p/(n+|\gamma_{k,n}|)}$, we have

$$\left|J^\alpha_{B_{k,n}} f(x)\right| \le C\left(\left(M_{B_{k,n}} f\right)(x)\right)^{p/q}\|f\|^{1-p/q}_{p,\gamma_{k,n}}.$$

Taking into account the theorem 1, we have

$$\int_{E_{k,+}(0,t)} \left|J^\alpha_{B_{k,n}} f(y)\right|^q y_{k,n}^{\gamma_{k,n}} dy \le C\|f\|^{q-p}_{L_{p,\gamma_{k,n}}(R^n_{k,+})} \int_{R^n_{k,+}\setminus E_{k,+}(0,t)} \left(\left(M_{B_{k,n}} f\right)(y)\right)^p y_{k,n}^{\gamma_{k,n}} dy \le C\|f\|^q_{p,\gamma_{k,n}}.$$

From here follow the inequality (9).

Let $f \in L_{1,\gamma_{k,n}}(R^n_{k,+})$.

Sufficiently is proved inequality (3) with 2β instead of β in left side this inequality. Further,

$$\left|\left\{x : \left|J^\alpha_{B_{k,n}} f(x)\right| > 2\beta\right\}\right|_{\gamma_{k,n}} \le \left|\left\{x : |A(x,t)| > \beta\right\}\right|_{\gamma_{k,n}} + \left|\left\{x : |C(x,t)| > \beta\right\}\right|_{\gamma_{k,n}}.$$

By virtue of (10) and theorem 1, we have

$$\beta\left|\left\{x \in R^n_{k,+} : |A(x,t)| > \beta\right\}\right|_{\gamma_{k,n}} = \beta \int_{\left\{x\in R^n_{k,+} : |A(x,t)|>\beta\right\}} x_{k,n}^{\gamma_{k,n}} dx \le \beta \int_{\left\{x\in R^n_{k,+} : Ct^\alpha\left(M_{B_{k,n}} f\right)(x)>\beta\right\}} x_{k,n}^{\gamma_{k,n}} dx =$$

$$\beta\left|\left\{x \in R^n_{k,+} : M_{B_{k,n}} f(x) > \frac{\beta}{Ct^\alpha}\right\}\right|_{\gamma_{k,n}} \le C_1 t^\alpha \|f\|_{L_{1,\gamma_{k,n}}(R^n_{k,+})}.$$

And also

$$|C(x,t)| \le \int_{R^n_{k,+}\setminus E_{k,+}(0,t)} T^y|f(x)|\rho(y)^{\alpha-n-|\gamma_{k,n}|} y_{k,n}^{\gamma_{k,n}} dy \le t^{\alpha-n-|\gamma_{k,n}|} \int_{R^n_{k,+}\setminus E_{k,+}(0,t)} T^y|f(x)| y_{k,n}^{\gamma_{k,n}} dy \le$$

$$\le t^{-\frac{n+|\gamma_{k,n}|}{q}} \int_{R^n_{k,+}} T^y|f(x)| y_{k,n}^{\gamma_{k,n}} dy = t^{-\frac{n+|\gamma_{k,n}|}{q}} \int_{R^n_{k,+}} |f(y)| y_{k,n}^{\gamma_{k,n}} dy = t^{-\frac{n+|\gamma_{k,n}|}{q}} \|f\|_{L_{1,\gamma_{k,n}}(R^n_{k,+})}$$

and thus if $t^{-\frac{n+|\gamma_{k,n}|}{q}}\|f\|_{L_{1,\gamma_{k,n}}(R^n_{k,+})} = \beta$, then $|C(x,t)| \le \beta$ and obviously, $\left|\left\{x : |C(x,t)| > \beta\right\}\right|_{\gamma_{k,n}} = 0$. Finally we obtain

$$\left|\left\{x : \left|J^{\alpha}_{B_{k,n}} f(x)\right| > 2\beta\right\}\right|_{\gamma_{k,n}} \le C_1 \frac{t^n \|f\|_{L_{1,\gamma_{k,n}}(R^n_{k,+})}}{\beta} = C_1 t^{\alpha + \frac{n+|\gamma_{k,n}|}{q}} =$$

$$= C_1 \beta^{-q} \|f\|^q_{L_{1,\gamma_{k,n}}(R^n_{k,+})} = C_1 \left(\frac{\|f\|_{L_{1,\gamma_{k,n}}(R^n_{k,+})}}{\beta} \right)^q .$$

This we get the inequality (3). Thus, the map $f \to J^{\alpha}_{B_{k,n}} f$ is $(1,q)$-weak type map.

Note, that

$$\|G_1 * f\|_{L_{q,\gamma_{k,n}}(R^n_{k,+})} \le C_3 \cdot \|G_3 * f\|_{L_{r,\gamma_{k,n}}(R^n_{k,+})},$$

where

$$G_3(x) = |x|^{-n-|\gamma_{k,n}|+\alpha}, \quad x \in R^n_{k,+}.$$

The same discusses which were applied in the proving of the theorem 3 show, that $\|G_3 * f\|_{L_{q,\gamma_{k,n}}(R^n_{k,+})} \le C_4 \cdot \|f\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})}$ for $1 < p < q < \infty$ and $\alpha = \left(n + |\gamma_{k,n}|\right)\left(\frac{1}{p} - \frac{1}{q}\right)$ and also

$$\left|\{x : |G_3 * f| > \lambda\}\right|_{\gamma_{k,n}} \le \left(\frac{C \|f\|_{L_{1,\gamma_{k,n}}(R^n_{k,+})}}{\lambda} \right)^q ,$$

for $\alpha = \left(n + |\gamma_{k,n}|\right)\left(1 - \frac{1}{q}\right)$.

Combining estimates for $B^{*}_{k,n}$-convolutions with kernels G_1 and G_2 we conclude the proving of points b) and c).

Theorem 5 is proved.Authors express thanks to acad. A.D.Gadjiev for discussing of results.

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