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# ON $F$-SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR ONE CLASS OF OPERATOR-DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER

**Abstract**

*At the given paper the sufficient conditions of $F$-solvability of boundary value problem have been found for one class of operator-differential equations of fourth order whose main part has a multiple characteristics.*

Let $H$ - be a separable Hilbert space, $A$ be a positive defined self-adjoint operator in $H$. Let's denote by $H_\gamma$ the scale of Hilbert spaces generated by the operator $A$, i.e. $H_\gamma = D(A^\gamma)$, $\|x\|_\gamma = \|A^\gamma x\|$, $\gamma \ge 0$, $x \in D(A^\gamma)$. Let $L_2(R_+ : H)$ be Hilbert space of the vector-function $f(t)$ with values from $H$, for which

$$\|f\|_{L_2(R_+:H)} = \left( \int_0^\infty \|f(t)\|^2 dt \right)^{1/2} < \infty$$

Let's suppose that $0 \le \alpha < \frac{\pi}{2}, 0 \le \beta < \frac{\pi}{2}$. Let's denote by $H_2(\alpha, \beta : H)$ the set of vector-function $f(z)$ with values from $H$, which are holomorphic in the sector $S_\alpha = S(\alpha : \beta) = \{z / -\beta < \arg z < \alpha\}$ and at any $\varphi \in [-\beta; \alpha]$ the functions $f(\xi e^{i\varphi}) \in L_2(R_+ : H)$ (see [1]). We can always establish the vector-function $f(z)$ with the help of boundary values $f_{-\beta}(\xi) = f(\xi e^{-i\beta})$ and $f_\alpha(\xi) = f(\xi e^{i\alpha})$ using the integrals formula of Cauchy type:

$$f(z) = \frac{1}{2\pi i} \int_0^\infty \frac{f_{-\beta}(z)}{\xi e^{-i\beta} - z} e^{-i\beta} d\xi - \frac{1}{2\pi i} \int_0^\infty \frac{f_\alpha(z)}{\xi e^{i\alpha} - z} e^{i\alpha} d\xi$$

The set $H_2(\alpha, \beta : H)$ becomes the Hilbert space with respect to the norm

$$\|f\|_{(\alpha:\beta)} = \frac{1}{\sqrt{2}} \left( \|f_{-\beta}(\xi)\|^2_{L_2(R_+:H)} + \|f_\alpha(\xi)\|^2_{L_2(R_+:H)} \right)^{\frac{1}{2}}.$$

So, let's define the space $W_2^4(\alpha, \beta : H)$

$$W_2^4(\alpha, \beta : H) = \left\{ u(z) \middle| A^4 u(z) \in H_2(\alpha, \beta : H), u^{(4)}(z) \in H_2(\alpha, \beta : H) \right\}.$$

With the norm

$$\|u\|_{(\alpha,\beta)} = \left( \|A^4 u\|^2_{(\alpha,\beta)} + \|u^{(4)}\|^2_{(\alpha,\beta)} \right)^{\frac{1}{2}}.$$

Here and in further the derivatives are understood in the meaning of complex analyses in abstract spaces.

Let's consider the next problem

$$P\left(\frac{d}{dz}\right)u(z)=\left(-\frac{d^2}{dz^2}+A^2\right)^2 u(z)+\sum_{j=1}^{3}A_{4-j}u^{(j)}(z)=f(z),\ z\in S_\alpha, \qquad (1)$$

$$u(0)=u'(0)=0 \qquad (2)$$

**Definition 1.** *The vector-function $u(z)\in W_2^4(\alpha,\beta:H)$ will be called a regular solution of boundary problem (1)-(2), if $u(z)$ satisfies the equation (1) identically in $S_\alpha$ and the boundary conditions (2) are fulfilled in the meaning*

$$\lim_{\substack{z\to 0\\ -\beta<\arg z<\alpha}}\left\|u^{(j)}(z)\right\|_{4-j-1/2}=0,\ j=0,1$$

**Definition 2.** *The problem (1)-(2) is called $\Phi$-solvable if at any $f(z)\in H_1\subset H_2(\alpha,\beta:H)$ exists $u(z)\in W_1\subset W_2^2(\alpha,\beta:H)$ such that*

$$\|u\|_{(\alpha,\beta)}\le const\|f\|_{(\alpha,\beta)}. \qquad (3)$$

Moreover the spaces $H_1$ and $W_1$ have the finite dimensional orthogonal complements in the spaces $H_2(\alpha,\beta:H)$ and $W_2^4(\alpha,\beta:H)$, respectively. At the given paper using the method of paper [1] the conditions have been found for the coefficients of the operator-differential equation (1), which provide $\Phi$-solvability of the problem (1)-(2). The analogous problem in a general form when the main part doesn't contain the multiple characteristics are investigated at papers [1], [2]. At paper [3] for $\alpha=\beta=\frac{\pi}{4}$ the conditions of correct and one-valued solvability of this problem have been found.

Let's denote that the results of the article are true for the problems with the boundary conditions $u^{(s_0)}(0)=u^{(s_1)}(0)=0$ where $s_0$ and $s_1$ are integer numbers satisfying the condition $0\le s_0<s_1\le 3$. For the simplicity of the statement we'll consider only the case when $s_0=0,\ s_1=1$.

**Lemma.** *The boundary problem*

$$P_0\left(\frac{d}{dz}\right)u(z)\equiv\left(-\frac{d^2}{dz^2}+A^2\right)^2 u(z)=v(z),\ z\in S_\alpha, \qquad (4)$$

$$u(0)=u'(0)=0 \qquad (5)$$

*has a unique regular solution at any $v(z)\in H_2(\alpha,\beta:H)$, moreover it holds the inequality*

$$\|u\|_{(\alpha;\beta)}\le const\|v\|_{(\alpha;\beta)}.$$

**Proof.** It is easy to see that ([1]) the vector-function

$$u_0(z)=\frac{1}{2\pi i}\int_\Gamma P_0^{-1}(\lambda)\hat{v}e^{\lambda z}d\lambda$$

satisfies equation (4) when $z\in S_\alpha$. Here $\hat{v}(\lambda)$ is a Laplacian transformation vector-function $v(z)$: $\hat{v}(\lambda)=\int_0^\infty v(t)e^{-\lambda t}dt$, which

$$S=\left\{\lambda:\frac{3\pi}{2}-\alpha<\arg\lambda<\frac{\pi}{2}+\beta\right\},$$

and $\Gamma$ is contour of integrating formulated by the rays

$$\Gamma_1 = \left\{ \lambda / \arg \lambda = \frac{\pi}{2} + \beta \right\} \text{ and } \Gamma_2 = \left\{ \lambda / \arg \lambda = \frac{3\pi}{2} - \alpha \right\}$$

Let's denote that $\|v(\lambda)\| \to 0$ when $\lambda \in S$ and $\lambda \to \infty$ [4]

Thus

$$u_0(z) = \frac{1}{2\pi i} \int_{\Gamma_1} P_0^{-1}(\lambda) \hat{v}(\lambda) e^{\lambda z} d\lambda - \frac{1}{2\pi i} \int_{\Gamma_2} P_0^{-1}(\lambda) \hat{v}(\lambda) e^{\lambda z} d\lambda, \; z \in S_\alpha, \qquad (6)$$

On the other hand it is easy to see that on the rays $\Gamma_1$ and $\Gamma_2$ the inequality

$$\left\| \lambda^4 P_0^{-1}(\lambda) \right\| + \left\| A^4 P_0^{-1}(\lambda) \right\| \le const,$$

is fulfilled, consequently $u_0(z) \in W_2^4(\alpha, \beta : H)$.

From formula (6) after the simple transformations we have:

$$u_0(z) = \int_0^\infty \left( \frac{1}{2\pi i} \int_0^{i\infty} P_0^{-1}\left(\lambda e^{i\beta}\right) e^{\lambda\left(ze^{i\beta} - \xi\right)} d\lambda \right) v(\xi)_{-\beta} d\xi -$$
$$- \int_0^\infty \left( \frac{1}{2\pi i} \int_0^{-i\infty} P_0^{-1}\left(\lambda e^{-i\alpha}\right) e^{\lambda\left(ze^{-i\alpha} - \xi\right)} d\lambda \right) v_\alpha(\xi) d\xi, \qquad (7)$$

where $v_\alpha(\xi) = v\left(\xi e^{id}\right)$, $v_{-\beta}(\xi) = v\left(\xi e^{-i\beta}\right)$, and

$$G_1(S) = \frac{1}{2\pi i} \int_0^{i\infty} P_0^{-1}\left(\lambda e^{i\beta}\right) e^{\lambda s} d\lambda,$$
$$G_2(S) = \frac{1}{2\pi i} \int_0^{-i\infty} P_0^{-1}\left(\lambda e^{i\alpha}\right) e^{\lambda s} d\lambda. \qquad (8)$$

Thus, the general regular solution of equation (4) from the spare $W_2^4(\alpha, \beta : H)$ is represented in the next form

$$u(z) = u_0(z) + e^{-zA} c_1 + zAe^{-zA} c_1,$$

where the vectors $c_1, c_2 \in H_{7/2}$, and $e^{-zA}$ is holomorphic in $S_\alpha$ semi-group of the boundary operators generated by the operator $(-A)$. Let's select the vectors $C_1$ and $C_2$ such that $u(z)$ becomes a regular solution of problem (4)-(5). Then we'll get

$$u(z) = u_0(z) - e^{-zA} u_0(0) - zAe^{-zA} u_0(0) - ze^{-zA} u'(0) \qquad (9)$$

it yields from representation (9) and inequality (7) that $u(z)$ satisfies the inequality

$$\|u(z)\|_{W_2^4(\alpha, \beta : H)} \le const \|v(z)\|_{H_2(\alpha, \beta : H)}$$

Lemma is proved.

Now let's prove the basic theorem.

**Theorem.** *Let $A$ be a positive determined self-adjoint operator with quite continuous inverses $A^{-1}$. The resolvent $P^{-1}(\lambda)$ exists on the rays $\Gamma_1 = \left\{ \lambda / \arg \lambda = \frac{\pi}{2} + \beta \right\}$ and $\Gamma_2 = \left\{ \lambda / \arg \lambda = \frac{3\pi}{2} - \alpha \right\}$ and $\left\| \lambda^4 R^{-1}(\lambda) \right\| + \left\| A^4 P^{-1}(\lambda) \right\| \le const$, the operators $B_j = A_j A^{-j}$ $\left(j = \overline{1,3}\right)$ are bounded.*

Then the problem (1)-(2) is $\Phi$-solvable.

**Proof.** Applying to the both parts of equality (9) the operator $P\left(\frac{d}{dz}\right)=P_0\left(\frac{d}{dz}\right)+P_1\left(\frac{d}{dz}\right)$ we'll get

$$v(z)=P_1\left(\frac{d}{dz}\right)u_0(z)-P_1\left(\frac{d}{dz}\right)\left[e^{-zA}u_0(0)+zAe^{-zA}u_0(0)+ze^{-zA}u_0'(0)\right], \quad (10)$$

where

$$P_0\left(\frac{d}{dz}\right)=\left(-\frac{d^2}{dz^2}+A^2\right)^2, \; P_1\left(\frac{d}{dz}\right)=\sum_{i=1}^{3}A_{4-j}\frac{d^j}{dz^j}$$

Allowing for equalities (7) and (8) in equality (10) and passing to the boundary values when $z\to te^{i\alpha}$ and $z\to te^{-i\beta}$ $(t\in R_+=(0,\infty))$ we get the next system of integral equations in the space $L_2(R_+:H)$:

$$v_\alpha(t)+\int_0^\infty\left(k_2(t-\xi)+k_4\left(te^{i\alpha},\xi\right)\right)v_\alpha(\xi)d\xi+$$
$$+\int_0^\infty\left(k_1\left(te^{i(\alpha+\beta)}-\xi\right)+k_3\left(te^{i\alpha},\xi\right)\right)d\xi=f_\alpha(t)$$

and

$$v_{-\beta}(t)+\int_0^\infty\left(k_1(t-\xi)+k_3\left(te^{-i\beta},\xi\right)\right)v_{-\beta}(\xi)d\xi+$$
$$+\int_0^\infty\left(k_2\left(te^{-i(\alpha+\beta)}-\xi\right)+k_4\left(te^{-i\alpha},\xi\right)\right)v_\alpha(\zeta)d\xi \quad (11)$$

Here

$$k_1\left(te^{i\beta}-\xi\right)=P\left(e^{i\beta}\frac{d}{dt}\right)G_1\left(te^{i\beta}-\xi\right),$$

$$k_2\left(te^{-i\alpha}-\xi\right)=P_1\left(e^{-i\beta}\frac{d}{dt}\right)G_2\left(te^{-i\alpha}-\xi\right),$$

$$k_3(t,\xi)=-P_1\left(e^{i\beta}\frac{d}{dt}\right)\left[e^{-t\beta A}G(-\xi)+te^{-i\beta}Ae^{-te^{-i\beta}}A_{G_1}(-\xi)+te^{-i\beta}e^{-te^{-i\beta A}}G_1'(-\xi)\right],$$

$$k_4(t,\xi)=-P_1\left(e^{i\alpha}\frac{d}{dt}\right)\left[e^{-te^{i\alpha}A}G_1(-\xi)+te^{i\alpha}Ae^{-te^{i\alpha}A}G_1(-\xi)+te^{i\alpha}e^{-te^{i\alpha}A}G_1'(-\xi)\right].$$

Since the operator $P_0\left(\frac{d}{dz}\right)$ transform isomorphically the space

$$W_2^4(\alpha,\beta:H;0,1) \text{ on } W_2^4(\alpha,\beta:H), \text{ where}$$
$$W_2^4(\alpha,\beta:H;0,1)=\{u(z)/u(z)\in W_2^4(\alpha,\beta:H),u(0)=u'(0)=0$$

then $\Phi$-solvability of the problem (1)-(2) is equivalent to $\Phi$-solvability of the system of integral equations (11) in the space $L_2(R_+:H)$. Therefore we analyze the solvability of the system of integral equations (11).

Since $P^{-1}(\lambda)$ exists on the rays $\Gamma_1$ and $\Gamma_2$ then each equation

$$\tilde{v}(t)+\int_{-\infty}^{+\infty} k_j(t-\xi)\tilde{v}(\xi)=\tilde{f}(t),\ j=1,2$$

is correct and uniquely solvable in the space $L_2(R_+:H)=L_2(R_+:H)\oplus L_2(R_+:H)$ where

$$\tilde{f}(t)\in L_2(R:H),\ \ \tilde{v}(t)\in L_2(R:H).$$

Therefore for $\Phi$ -solvability of the system of integral equations

$$v_\alpha(t)+\int_0^{+\infty} k_2(t-\xi)v_\alpha d\xi=f_\alpha(\xi)$$

$$v_{-\beta}(t)+\int_0^{+\infty} k_1(t-\xi)v_{-\beta} d\xi=f_{-\beta}(\xi)$$

In the space $L_\alpha(R_+:H)$ it is enough to show the kernels $k_1(t+\xi)$ and $k_2(t+\xi)$ generate the completely continuous operators in $L_2(R:H)$. After this for proving $\Phi$ -solvability of the system of integral equations (11) in the space $L_2(R_+:H)$ it is necessary to prove that the kernels $k_1\left(te^{i(\alpha+\beta)}-\xi\right)$, $k_3\left(te^{i\alpha},\xi\right)$, $k_4\left(te^{i\alpha},\xi\right)$, $k_3\left(te^{-i\beta},\xi\right)$, $k_4\left(te^{-i\beta},\xi\right)$, $k_2\left(te^{-i(\alpha+\beta)}-\xi\right)$, also generate completely continuous operators in $L_2(R_+:H)$. The proof of completely continuity of these operators are analogous, therefore following to paper [1] we'll prove the completely continuity of the operator generated by the kernel $k_1(t+\xi)$.

Since

$$k_1(t+\xi)=\sum_{j=1}^{3}A_{4-j}e^{ij\beta}\frac{d^j}{dt^j}\left(\frac{1}{2\pi i}\int_0^{i\infty}\left(-\lambda e^{2i\beta}E+A^2\right)^{-2}e^{\lambda(t+\xi)}dt\right)$$

since for $\lambda\in[0,i\infty)$ and receiving to it the angle with sufficiently small opening it holds the estimation

$$\left\|\left(-\lambda^2e^{2j\beta}E+A^2\right)^{-2}\right\|\le const\left(1+|\lambda|\right)^{-4}$$

then $k_1(t+\xi)$ we can represent in the following form

$$k_1(t+\xi)=\sum_{j=1}^{3}A_{4-j}e^{ij\beta}\frac{d^j}{dt^j}\left(\frac{1}{2\pi i}\int_0^{i\infty}\left(-\lambda e^{\lambda j\beta}E+A^2\right)^{-2}e^{\lambda(t+\xi)}dt\right)d\lambda=$$

$$=\sum_{j=1}^{3}\frac{B_j}{2\pi i}\int_0^{(i-\varepsilon)\infty}\lambda^{4-j}A^j\left(-\lambda^2e^{2i\beta}E+A^2\right)^{-2}e^{\lambda(t+\xi)}dt\equiv\frac{1}{2\pi i}\sum_{j=1}^{3}\beta_jk_{1,j}(t+\xi),$$

where $\varepsilon>0$ is sufficiently small number and

$$k_{1,j}(t+\xi)=\int_0^{(i-\varepsilon)\infty}\lambda^{4-j}A^j\left(-\lambda^2e^{2j\beta}E+A^2\right)^{-2}e^{\lambda(t+\xi)}d\xi$$

Let's show that $k_{1,j}(t+\xi)$ generates the bounded operator in $L_2(R_+:H)$. Really when $\lambda\in(0,(i-\varepsilon)\infty)$ it is easy to see that $\left\|\lambda^{4-j}A^j\left(-\lambda^2e^{2i\beta}E+A^2\right)^{-2}\right\|\le const$ therefore

$$\left\|k_{1,j}(t+\xi)\right\|_{H\to H}=\left\|\int_0^{(i-\varepsilon)\infty}\lambda^{4-j}A^j\left(-\lambda^2e^{2i\beta}E+A^2\right)^{-2}e^{\lambda(t+\xi)}d\lambda\right\|=$$

$$=\left\|\int_0^{(i-\varepsilon)\infty}(i-\varepsilon)^{s-j}\lambda^{4-j}A^j\left(-\lambda^2(i-\varepsilon)^2e^{2i\beta}E+A^2\right)^{-2}e^{-\varepsilon\lambda(t+\xi)}e^{i\lambda(t+\xi)}d\lambda\right\|\le$$

$$\le|i-\varepsilon|\int_0^{\infty}\left\|((i-\varepsilon)\lambda)^{4-j}A^j\left(-\lambda^2(i-\varepsilon)^2e^{2i\beta}E+A^2\right)^{-2}\right\|\times e^{-\varepsilon\lambda(t+\xi)}d(\lambda\varepsilon)\le$$

$$\le C_\varepsilon\int_0^{\infty}e^{-\varepsilon t(t+\xi)}d(\lambda\xi)\le\frac{c_\varepsilon}{t+\varepsilon}.$$

Then it follows from the Hilbert inequality [5] that $k_{1,j}(t+\xi)$ generates the continuous operator $\widetilde{k}_{1,j}$ in the space $L_2(R_+:H)$. Further let $Ae_n=\mu_ne_n$, $n=\overline{1,m}$; $|\mu_1|\le\ldots\le|\mu_n|\le\ldots$, here $\mu_n$ are the first $m$ eigenvalues of the operator $A$ and let $P_m=\sum_{i=1}^{m}(\cdot,e_i)e_i$ is a orthoprojector on the subspace, strained on $(e_1,e_2,\ldots,e_m)$. Since $B_j$ is completely continuous operator in $H$, then the norm of operators $Q_{m,j}=B_j-B_jP_m$ tens to zero for $m\to\infty$. On the other hand

$$\left\|B_jP_mk_{i,j}(t+\xi)\right\|=\left\|\int_0^{(i-\varepsilon)\infty}\sum_{n=1}^{m}\lambda^{n-j}(i-\varepsilon)^{5-j}\mu_n^j\left(-\lambda^2(i-\varepsilon)^2e^{2i\beta}+\mu_n^2\right)^{-2}\times\right.$$

$$\left.\times(\cdot,e_n)B_je_ne^{i\lambda(t+\xi)}e^{-\lambda\varepsilon(t+\xi)}d\lambda\right\|\le C_\varepsilon\sum_{n=1}^{m}\int_0^{\infty}\frac{|(\varepsilon\lambda)^{4-j}|\mu_n^j}{\left|-\lambda^2(i-\varepsilon)^2e^{2i\beta}+\mu_n^2\right|^2}\times$$

$$\times e^{-\lambda\varepsilon(t+\xi)}d(\varepsilon\lambda)\le C_\varepsilon(m)\int_0^{\infty}\frac{\lambda^{4-j}}{1+\lambda^n}e^{-\lambda(t+\xi)}d\lambda,\quad j=1,2,3.$$

It is easy to see that $\int_0^{\infty}\int_0^{\infty}\left\|B_jK_{i,j}(t+\varepsilon)\right\|^{\prime}dsdt<\infty$.

Hence it follows that the kernel $B_jP_mk_{1,j}(t+\xi)$ generates the operator $T_{1,j,m}$ of Hilbert-Shmidt since for $j=1,2,3$. On the other hand from the equality

$$B_jk_{1,j}(t+\xi)=Q_{m,j}k_{1,j}(t+\xi)+B_jP_mk_{1,j}(t+\xi),$$

from the boundness of the operator $\widetilde{k}_{1,j}$ generated by the kernel $k_{1,j}(t+\xi)$ and from the completely continuity of $T_{1,j,m}$ generated by the kernel $B_jP_mk_{1,j}(t+\xi)$ it follows that

$$\left\|\widetilde{k}_{1,j}-T_{1,j,m}\right\|_{L_2\to L_2}\le\left\|Q_{m,j}\right\|\cdot\left\|\widetilde{k}_{1,j}\right\|_{L_2\to L_2}\to0,\ m\to\infty$$

i.e. $\widetilde{k}_{1,j}$ is a completely continuous operator in $L_2(R_+:H)$.

The theorem is proved.

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