2001 VOL. XV(XXIII)

IBRAHIMOV E.J.

ASYMPTOTIC ESTIMATIONS OF APPROXIMATION OF FUNCTIONS BY LINEAR POSITIVE OPERATORS

Abstract

At the given paper the conception of Hegenbauer derivative is introduced and the asymptotic estimations of the approximation of functions which have the Hegenbauer derivative by linear positive operators generated by summation of ultraspheric series by some generalized methods, are established.

Let $L_{p,\mu}[-1,1]$ be a space of functions summable with p-th degree and with the weight $\mu(x)=(1-x^2)^{\lambda-\frac{1}{2}}$.

To the functions $f\in L_{p,\mu}$ associate its Fourier-Hegenbauer series

$$f(x)\sim\sum_{n=0}^{\infty}a_n^{\lambda}(f)P_n^{\lambda}(x), \tag{1}$$

where $P_n^{\lambda}(x)$ are polynomials of Hegenbauer which form the orthogonal system on the segment $[-1,1]$ with the $\mu(x)$ weight.

Let $\Phi=\{\varphi_n(\tau)\}$, $(\varphi_0(\tau)=1,\ n=1,2,...)$ be a sequence of functions defined on the set G and τ_0 is a limit point of this set.

The summation of series (1) by the Φ method leads to the interval of the form [1].

$$L_{\tau}^{\lambda}(f;x)=\frac{\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)}\int_0^{\pi}A_t^{\lambda}f(x)K_{\tau}^{\lambda}(t)\sin^{2\lambda}t\,dt, \tag{2}$$

where

$$K_{\tau}^{\lambda}(t)=\sum_{n=0}^{\infty}\varphi_n(\tau)(n+\lambda)P_n^{\lambda}(\cos t) \tag{3}$$

is the kernel of summation of series (1) by the Φ method.

The expression

$$\Delta_t^n f(x_0)=A_t^{\lambda}f(x_0)-\sum_{\nu=0}^{n-1}\lambda_{\nu}\ (1-\cos t)^{\nu},$$

where $\alpha_0=f(x_0)$ and α_{ν}, $(\nu=1,2,...,n-1)$ are some constants, let's call the n-th difference of Taylor type at the x_0 point.

Here

$$A_t^{\lambda}f(x)=\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(\lambda)}\int_0^{\pi}f\left(x\cos t+\sqrt{1-x^2}\sin t\cos\varphi\right)(\sin\varphi)^{2\lambda-1}d\varphi$$

is a function of generalized shear [2].

Let's give the following

Definition. *If there exists a finite limit of the expression at* $(1-\cos t)^{-n}\Delta_t^n f(x_0)$ *for* $t\to 0$, *then this limit we'll call the* n*-th derivative of Hegenbauer at the point* x_0 *and we'll denote it by* $D_\lambda^{(n)} f(x_0)$ *i.e.*

$$\lim_{t\to 0}\frac{\Delta_t^n f(x_0)}{(1-\cos t)^n}=D_\lambda^{(n)} f(x_0), \tag{4}$$

The next theorem gives us the expansion of the functions of generalized shear by the Hegenbauer derivative and it is an analogue of Taylors theorem

Theorem 1. *If for faction* $f(x)$ *at the point* x_0 *exists the* n*-th Hegenbauer derivative* $D_\lambda^{(n)} f(x_0)$ *then at this point there exists the Hegenbauers derivatives of order* v $(v=1,2,\ldots,n-1)$, *moreover* $\alpha_v=D_\lambda^{(v)} f(x_0)$ *and the following relation*

$$A_t^\lambda f(x_0)=f(x_0)+\sum_{v=1}^{n} D_\lambda^{(v)} f(x_0)(1-\cos t)^v+o\left((1-\cos t)^n\right). \tag{5}$$

Proof. From (4) we have

$$\frac{\Delta_t^n f(x_0)}{(1-\cos t)^{n-1}}=\frac{A_t^\lambda f(x_0)-f(x_0)-\sum_{v=1}^{n-2}\alpha_v(1-\cos t)^v}{(1-\cos t)^{n-1}}-\alpha_{n-1}=o(1-\cos t).$$

Whence it follows that

$$\alpha_{n-1}=\lim_{t\to 0}\frac{\Delta_t^{n-1} f(x_0)}{(1-\cos t)^{n-1}}=D_\lambda^{(n-1)} f(x_0).$$

Continuing so we'll receive

$$\alpha_v=D_\lambda^{(v)} f(x_0),\quad (v=1,2,\ldots,n-1). \tag{6}$$

The statement (5) of theorems follows from (4) and (6).

Let's denote by $W_\lambda^{(r)}[-1,1]$ the set of functions which has the r-th Hegenbauers derivative at the point $x\in[-1,1]$.

Let

$$\mu_\tau^{[k]}=L_\tau^\lambda\left[(1-\cos t)^k;0\right],\quad (k=1,2,\ldots).$$

The next theorem is some analogue R.Mamedov's one theorem ([3], p.79).

Theorem 2. (basic) *Let* $f\in W_\lambda^{(n)}[-1,1]$ *and* $K_\tau^\lambda(t)\geq 0$. *If even at one* $j=1,2,\ldots$

$$\lim_{\tau\to\tau_0}\mu_\tau^{[n+j]}/\mu_\tau^{[n]}=0, \tag{7}$$

then the following relation

$$\lim_{\tau\to\tau_0}\frac{1}{\mu_\tau^{[n]}}\left[L_\tau^\lambda(f;x)-f(x)-\sum_{v=1}^{n} D_\lambda^{(v)} f(x)\mu_\tau^{[v]}\right]=D_\lambda^{(n)} f(x). \tag{8}$$

is true.

Proof. It follows from the theorem 1 that

$$\lim_{t\to 0}\frac{A_t^\lambda f(x)-f(x)-\sum_{v=1}^{n} D_\lambda^{(v)} f(x)(1-\cos t)^v}{(1-\cos t)^n}=D_\lambda^{(n)} f(x).$$

Then according to R.Mamedov's one theorem ([3], p.75) the equality

$$\lim_{\tau\to\tau_0}\frac{L_\tau^\lambda(f;x)-f(x)-\sum_{v=1}^{n-1}D_\lambda^{(v)}f(x)\mu_\tau^{[v]}}{L_\tau^v\left[(1-\cos t)^n;0\right]}=D_\lambda^{(n)}f(x)$$

will hold if we prove that

$$\lim_{\tau\to\tau_0}\frac{\alpha_\tau(\delta)}{L_\tau^v\left[(1-\cos t)^n;0\right]}=0, \tag{9}$$

where

$$\alpha_t(\delta)=L_\tau^\lambda[\beta_\delta(t);0];\ \beta_\delta(t)=v_\delta(t)(1-\cos t)^n,$$

$$v_\delta(t)=\begin{cases}0, & 0\le t<\delta;\\ 1, & \delta\le t\le\pi.\end{cases}$$

Really

$$\alpha_\tau(\delta)=\frac{\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)}\int_\delta^\pi(1-\cos t)^n K_\tau^\lambda(t)\sin^{2\lambda}t\,dt\le$$

$$\le\frac{\Gamma(\lambda)(1-\cos t)^{-j}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)}\int_\delta^\pi(1-\cos t)^{n+j}K_\tau^\lambda(t)\sin^{2\lambda}t\,dt\le$$

$$\le\frac{\Gamma(\lambda)(1-\cos t)^{-j}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)}\int_\delta^\pi(1-\cos t)^{n+j}K_\tau^\lambda(t)\sin^{2\lambda}t\,dt=\frac{\mu_\tau^{[n+j]}}{(1-\cos\delta)^j}.$$

From here it follows that

$$\frac{\alpha_\tau(\delta)}{L_\tau^\lambda\left[(1-\cos t)^n;0\right]}\le\frac{1}{(1-\cos\delta)^j}\frac{\mu_\tau^{[n+j]}}{\mu_\tau^{[n]}}.$$

Passing here to the limit at $\tau\to\tau_0$ and taking into account the condition (7) we'll get (9). Theorem is proved.

Let's consider the integral Vallée-Poussin-Hegenbauer [1]

$$V_n^\lambda(f;x)=\frac{\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)}\int_0^\pi v_n^\lambda(t)A_t^\lambda f(x)\sin^{2\lambda}t\,dt,$$

where

$$V_n^\lambda(t)=\frac{\Gamma(n+2\lambda+1)}{4^\lambda\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(n+\lambda+\frac{1}{2}\right)}\cos^{2n}\frac{t}{2},$$

which is a linear positive operator.

It is easy to count that

$$\mu_n^{[v]}=V_n^\lambda\left[(1-\cos t)^v;0\right]=\frac{2^{v-2}\Gamma(\lambda)\Gamma(n+2\lambda+1)\Gamma\left(v+\lambda+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma^2\left(\lambda+\frac{1}{2}\right)\Gamma(n+v+2\lambda+1)}.$$

From which follows that at $j=1,2,..$ and $n\to\infty$

$$\frac{\mu_n^{[v+j]}}{\mu_n^{[v]}} \sim \frac{2^j \Gamma\left(v+j+\lambda+\frac{1}{2}\right)}{\Gamma\left(v+\lambda+\frac{1}{2}\right)} \frac{1}{n^j} \to 0. \qquad (10)$$

On the other hand

$$\mu_n^{[v]} \sim \frac{2^{v-2}\Gamma(\lambda)\Gamma\left(v+\lambda+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma^2\left(\lambda+\frac{1}{2}\right)} \frac{1}{n^v}. \qquad (11)$$

From (10) it follows that it holds the condition (7) of theorem 2, that is why it is true.

Theorem 3. *If* $f \in W_\lambda^{(n)}[-1,1]$ *then at* $n \to \infty$ *it holds the relation*

$$V_n^\lambda(f;\lambda) - f(x) - \sum_{v=1}^{m-1} D_\lambda^{(v)} f(x) \mu_n^{[v]} = \frac{2^{m-2}\Gamma(\lambda)\Gamma\left(m+\lambda+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma^2\left(\lambda+\frac{1}{2}\right)} \left[D_\lambda^{(m)} f(x) + o(1)\right] \cdot \frac{1}{n^m}.$$

The statement of this theorem follows from theorem 2 and from the relation (11).

Let's consider the Jackson-Hegenbauer integral

$$D_{n,m}^\lambda(f;x) = \frac{1}{J_n^m(\lambda)} \int_0^\pi \left(\frac{\sin\frac{nt}{2}}{n\sin\frac{t}{2}} \right)^{2m} A_t^\lambda f(x) \sin^{2\lambda} t\,dt, \qquad (12)$$

where $\frac{1}{J_n^m(\lambda)}$ is a normalizing set.

The integral (12) is a linear positive operator.

For this operator at $n \to \infty$

$$\mu_n^{[k]} = D_{n,m}^\lambda\left[(1-\cos t)^k; 0\right] \sim \frac{2^k \Gamma\left(k+\lambda+\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)\left(\frac{m}{3}\right)^k} \frac{1}{n^{2k}}. \qquad (13)$$

From this operator at $j = 1,2,\ldots$ and $n \to \infty$

$$\frac{\mu_n^{[k+j]}}{\mu_n^{[k]}} \sim \frac{2^j \Gamma\left(k+j+\lambda+\frac{1}{2}\right)}{\Gamma\left(k+\lambda+\frac{1}{2}\right)} \left(\frac{3}{m}\right)^j \frac{1}{n^{2j}} \to 0,$$

i.e. the condition (7) of theorem 2 is fulfilled, that is why it is true.

Theorem 4. *If* $f \in W_\lambda^{(r)}[-1,1]$ *then at* $n \to \infty$ *it holds the relation*

$$D_{n,m}^\lambda(f;x) - f(x) - \sum_{v=1}^{r-1} D_\lambda^{(v)} f(x) \mu_n^{[v]} = \frac{2^r \Gamma\left(r+\lambda+\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)} \left[D_\lambda^{(r)} f(x) + o(1)\right] \cdot \frac{1}{n^{2r}}.$$

The statement of this theorem follows from theorem 2 and from the relation (13).

Let's calculate $\mu_\tau^{[k]}$ for the operator (2). Using the formula 7.311 (3) at page 840 from [4] we'll get that

$$\mu_\tau^{[k]}=\sum_{\nu=0}^{k}(-1)^\nu\frac{\Gamma(\lambda)2^{k+2\lambda}(\nu+\lambda)\Gamma\left(k+\lambda+\frac{1}{2}\right)\Gamma(\nu+2\lambda)\Gamma(k+1)}{\Gamma\left(\frac{1}{2}\right)\Gamma(2\lambda)\Gamma(\nu+1)\Gamma(k+1-\nu)\Gamma(k+\nu+2\lambda+1)}\varphi_\nu(\tau).$$

From here follows the next equality:

$$\frac{\mu_\tau^{[2]}}{\mu_\tau^{[1]}}=\frac{1}{2\lambda+2}\left[\frac{4(\lambda+1)}{2\lambda+1}-\frac{1-\varphi_2(\tau)}{1-\varphi_1(\tau)}\right], \tag{14}$$

$$\frac{\mu_\tau^{[3]}}{\mu_\tau^{[2]}}=3-\left(\frac{\lambda+1}{\lambda+2}\right)\frac{(2\lambda+1)\varphi_3(\tau)-3(2\lambda+3)\varphi_1(\tau)+4\lambda+8}{(2\lambda+1)\varphi_2(\tau)-4(\lambda+1)\varphi_1(\tau)+2\lambda+3}, \tag{15}$$

$$\begin{gathered}\frac{\mu_\tau^{[4]}}{\mu_\tau^{[3]}}=4-\frac{1}{2(\lambda+3)}\{(\lambda+1)(2\lambda+1)(2\lambda+3)\varphi_4(\tau)-\\ -6(\lambda+2)(2\lambda+1)(2\lambda+5)\varphi_2(\tau)+32(\lambda+1)(\lambda+2)(\lambda+3)\varphi_1(\tau)-\\ -3(\lambda+3)(2\lambda+3)(2\lambda+5)\}\cdot\{(\lambda+1)(2\lambda+1)\varphi_3(\tau)-\\ -3(\lambda+2)(2\lambda+1)\varphi_2(\tau)+3(\lambda+1)(2\lambda+5)\varphi_1(\tau)-(\lambda+2)(2\lambda+5)\}^{-1}.\end{gathered} \tag{16}$$

From (14)-(16) and theorem 2 the following corollaries

Corollary 2.1. *Let* $f\in W_\lambda^{-1}[-1,1]$ *and* $K_\tau^\lambda(t)\geq 0$. *If*

$$\lim_{\tau\to\tau_0}\frac{1-\varphi_2(\tau)}{1-\varphi_1(\tau)}=\frac{4(\lambda+1)}{2\lambda+1},$$

then the equality

$$\lim_{\tau\to\tau_0}\frac{L_\tau^\lambda(f;x)-f(x)}{\mu_\tau^{[1]}}=D_\lambda^{(1)}f(x)$$

is true.

This result has been received by the other way in [5] (see p.70 theorem 2.1.2), i.e. it is special case of theorem 2.

Corollary 2.2. *Let* $f\in W_\lambda^{(2)}[-1,1]$ *and* $K_\tau^\lambda(t)\geq 0$. *If*

$$\lim_{\tau\to\tau_0}\frac{(2\lambda+1)\varphi_3(\tau)-3(2\lambda+3)\varphi_1(\tau)+4\lambda+8}{(2\lambda+1)\varphi_2(\tau)-4(\lambda+1)\varphi_1(\tau)+2\lambda+3}=\frac{3(\lambda+2)}{\lambda+1},$$

then the equality

$$\lim_{\tau\to\tau_0}\frac{1}{\mu_\tau^{[2]}}\left[L_\tau^\lambda(f;x)-f(x)-D_\lambda^{(1)}f(x)\mu_\tau^{[1]}\right]=D_\lambda^{(2)}f(x).$$

is true.

Corollary 2.3. *Let* $f\in W_\lambda^{(3)}[-1,1]$ *and* $K_\tau^\lambda(t)\geq 0$. *If*

$$\begin{gathered}\lim_{\tau\to\tau_0}\{(\lambda+1)(2\lambda+1)(2\lambda+3)\varphi_4(\tau)-\\ -6(\lambda+2)(2\lambda+1)(2\lambda+5)\varphi_2(\tau)+32(\lambda+1)(\lambda+2)(\lambda+3)\varphi_1(\tau)-\\ -3(\lambda+3)(2\lambda+3)(2\lambda+5)\}\cdot\{(\lambda+1)(2\lambda+1)\varphi_3(\tau)-\\ -3(\lambda+2)(2\lambda+1)\varphi_2(\tau)+3(\lambda+1)(2\lambda+5)\varphi_1(\tau)-(\lambda+2)(2\lambda+5)\}^{-1}=8(\lambda+3),\end{gathered}$$

then the equality

$$\lim_{\tau\to\tau_0}\frac{1}{\mu_\tau^{[3]}}\left[L_\tau^\lambda(f;x)-f(x)-D_\lambda^{(1)}f(x)\mu_\tau^{[1]}-D_\lambda^{(2)}f(x)\mu_\tau^{[2]}\right]=D_\lambda^{(3)}f(x).$$

is true.

References

[1]. Jafarova S.A. *The approximations by the linear averages of Fourier-Jacobi series.* Can. Dissertation, Baku, 2000, 105 p.

[2]. Potanov M.K. *On approximation of Jacobi polynomials.* Vesti MSU, 1977, series mathematics mechanics, №5, p.70-82.

[3]. Mamedov R. *The approximation of function by the linear operators.* Azerbneshr, Baku, 1967.

[4]. Gradshkeyn I.S., Rijik I.M. *The tables of integrals, sums, series and derivatives.* Fizmathgiz., M., 1971, 1108 p.

Ibrahimov E.J.
Azerbaijan State Oil Academy.
20, Azadlyg av., 370601, Baku, Azerbaijan.

Received November 9, 2000; Revised April 17, 2001.
Translated Mamedova V.A.