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ON SOLVABILITY OF A FIRST BOUNDARY VALUE PROBLEM FOR NON-UNIFORMLY DEGENERATED SECOND ORDER ELLIPTIC EQUATIONS OF NON-DIVERGENT STRUCTURE

Abstract

At the paper a first boundary value problem is considered for a class of second order elliptic equations of non-divergent structure with non-uniform degeneration. The unique simple strong (almost everywhere) solvability of this problem in corresponding Sobolev weights spaces is proved.

Introduction. Let's D be a bounded domain of n-dimensional euclidean space $\mathbf{R}_n$, $(n\ge 3)$, ∂D be its boundary where $\partial D\in \mathbf{C}^2$. Let's denote by $W^p_{2,\alpha}(D)$ a Banach space of functions $u(x)$ given on D with the norm

$$\|u\|_{W^p_{2,\alpha}(D)}=\left[\int_D\left(|u|^p+\sum_{i=1}^n(\lambda_i(x))^{p/2}|u_i|^p+\sum_{i,j=1}^n(\lambda_i(x)\lambda_j(x))^{p/2}|u_{ij}|^p\right)dx\right]^{1/p},$$

and by $\overset{\bullet}{W}{}^p_{2,\alpha}(D)$ - the subspace of $W^p_{2,\alpha}(D)$, in which a set of all infinitely differentiable on $\overline{D}$ functions, vanishing to zero on ∂D is a dense set. Here $p\in(1,\infty)$, $\lambda_i(x)=\left(|x|_\alpha\right)^{\alpha_i}$, $|x|_\alpha=\sum_{i=1}^n|x_i|^{2/(2+\alpha_i)}$. In further we'll use everywhere the following notations:

$$u_i=\frac{\partial u(x)}{\partial x_i},\quad u_{ij}=\frac{\partial^2u(x)}{\partial x_i\partial x_j}.$$

Let $\alpha=(\alpha_1,...,\alpha_n)$ be a vector, $|\alpha|=\alpha_1+...+\alpha_n$ and

$$\alpha^-=\min_i\{\alpha_i\},\quad \alpha^+=\min_i\{\alpha_i\}.$$

For $R\le 1$, $k>0$, $x^0\in\mathbf{R}_n$ we'll denote by $E_R^{x^0}(k)$ the ellipsoid

$$\left\{x:\sum_{i=1}^n\frac{\left(x_i-x_i^0\right)^2}{R^{\alpha_i}}<(kR)^2\right\}.$$

Let's consider in D the Dirichlet problem

$$Lu=\sum_{i,j=1}^n a_{ij}(x)u_{ij}+\sum_{i=1}^n b_i(x)u_i+c(x)u(x)=f(x),\tag{1}$$

$$u|_{\partial D}=0,\tag{2}$$

where $\|a_{ij}(x)\|$ is a real symmetric matrix, $c(x)\le 0$.

Let's suppose that $0\in D$ and for all $x\in D$ and $\xi\in\mathbf{R}_n$ the condition

$$\mu\sum_{i=1}^{n}\lambda_i(x)\xi_i^2 \le \sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j \le \mu^{-1}\sum_{i=1}^{n}\lambda_i(x)\xi_i^2, \tag{3}$$

is fulfilled where $\mu \in (0,1]$ is constant.

We'll suppose also that the following conditions:

$$\max\left\{-\frac{2}{n-2};-2+\frac{4p}{n}\right\}<\alpha_i<2,\ (i=1,...,n), \tag{4}$$

$$f(x)\in L_q(D), \tag{5}$$

$$c(x)\in L_{\frac{pq_1}{(q_1-p)}}(D), \tag{6}$$

$$\frac{b_i(x)}{\sqrt{\lambda_i(x)}}\in L_{q_2^+}(D),\ (i=1,...,n), \tag{7}$$

$$\widetilde{a}_{ij}(x)=\frac{a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}}\in C(\overline{D}),\ (i,j=1,...,n), \tag{8}$$

are fulfilled, where

$$\frac{n(2+\alpha^+)}{2-\alpha^+}<q<\infty,\ 1<p<q_1<q^*=\frac{nm_1}{n-m_1},$$

$$q_2^+=\max\left\{\frac{pq_2}{q_2-p};\frac{n(6+\alpha^+)}{2-\alpha^+}\right\},\ 1<p<q_2<q^{**}=\frac{nm_2}{n-2m_2},$$

$$m_1=\begin{cases}p-\varepsilon^*, & for\ \alpha^+\le 0;\ \varepsilon^*>0\\ \dfrac{p(2n+|\alpha|)}{2n+|\alpha|+p\alpha^+}, & for\ \alpha^+>0,\end{cases} \tag{9}$$

$$m_2=\begin{cases}p-\varepsilon^{**}, & for\ \alpha^+\le 0;\ \varepsilon^{**}>0\\ \dfrac{p(2n+|\alpha|)}{2n+|\alpha|+p(2\alpha^+-\alpha^-)}, & for\ \alpha^+>0,\end{cases} \tag{10}$$

$$1<p<n/2. \tag{11}$$

Let $x'\in\partial E_R^0(1+r/2)$. Let's introduce the following notations: $E=E_R^{x'}(r/2)$, $E_1=E_R^{x'}(r/4)$, $A_R=E_R^0(1+r)\setminus\overline{E_R^0(1)}$. We'll assume that $r\in(0,\frac{1}{n-1}]$. By $mesD$ we'll denote Lebesgue measure of set D.

From the condition (8) it follows that there exists a non-negative, continuous and non-decreasing on $[0,diam\,D]$ function $h(t)$ such that $h(0)=0$ and

$$|\widetilde{a}_{ij}(x)-\widetilde{a}_{ij}(y)|\le h(|x-y|),\ x,y\in\overline{D};\ i,j=1,...,n,$$

and from the condition (3) it follows that the matrix $\|\widetilde{a}_{ij}(x)\|$ is uniformly positive defined and there exists the positive constant a_0 such that

$$|\widetilde{a}_{ij}(x)|\le a_0;\ i,j=1,...,n. \tag{8'}$$

Let D_ρ be a set of points $x\in D$, for which the distance from boundary of domain D is greater than $\rho>0$:

$$D_\rho = \{x : x \in D,\ dist(x, \partial D) > \rho\}.$$

As was shown in [8] if $x \in A_R$ then

$$C_1(n,\alpha)R^{\alpha_i} \le \lambda_i(x) \le C_2(n,\alpha)R^{\alpha_i}\ ;\ i = 1,...,n. \tag{12}$$

Here and in further by $C(....)$ we'll denote the positive constants only on the content of parentheses.

The aim of present paper is the proof of the unique simple strong (almost everywhere) solvability of the first boundary value problem (1)-(2) in the space $\overset{\bullet}{W}{}^{p}_{2,\alpha}(D)$ for any $f(x) \in L_q(D)$. Let's denote that in case of uniform elliptic equations the analogous result was received in papers [1]-[2], and for equations whose coefficients satisfy Cordes conditions - in [3]-[4]. Relating to non-uniform degenerated elliptic equations not containing minor coefficients and non-negative α_i, we note paper [5].

Let $L_0 = \sum\limits_{i,j=1}^{n} a_{ij}(x)\dfrac{\partial^2}{\partial x_i dx_j}$. Analogously to [5] lemmas 1,2 are proved.

Lemma 1. *Let the conditions (3), (8) and (11) be fulfilled. Then for any function $u(x) \in C_0^\infty(E)$ the estimation*

$$\int\limits_E \sum_{i,j=1}^{n} (\lambda_i(x)\lambda_j(x))^{p/2} |u_{ij}|^p dx \le C_3(n,p,\alpha,\mu,h,a_0) \int\limits_E |L_0 u|^p dx,$$

is true where $0 < r < r_0(n,p,\alpha,\mu,h,a_0)$ and $r_0 \le \dfrac{1}{n-1}$.

Not stipulating specially we'll assume $r = r_0$, and $\overline{E_R^0(1+r)} \subset D$.

Lemma 2. *Let the conditions (3), (8) and (11) be fulfilled. Then for any function $u(x) \in C_0^\infty(E)$ the estimation*

$$\|u\|_{W^p_{2,\alpha}(E)} \le C_4(n,p,\alpha,\mu,h,a_0) \|L_0 u\|_{L_p(E)}$$

is true.

1⁰. Imbedding theorems.

Theorem 1. *Let the conditions (4) be fulfilled. If $1 < p < q_1 < q^*$ then $W^p_{1,\alpha}(E)$ boundly embedded in $L_{q_1}(E)$ where m_1 is defined by formula (9).*

Proof. According to [7] if $m_1 \le n$ and $1 \le q_1 < q^*$, then $W_1^{m_1}(E)$ is completely continuously embedded in $L_{q_1}(E)$:

$$\|u\|_{L_{q_1}(E)} \le C_5 \|u\|_{W_1^{m_1}(E)},$$

where $C_5 = const$.

For simplicity let's restrict by the case $u(x) \in \overset{\bullet}{W}{}^{p}_{1,\alpha}(E)$. According to the Friedrichs inequality [7]:

$$\int\limits_E |u|^{m_1} dx \le C_6 \int\limits_E \sum_{i=1}^{n} |u_i|^{m_1} dx, \text{ where } C_6 = const.$$

Then

$$\|u\|^{m_1}_{W_1^{m_1}(E)}=\int_E\left(|u|^{m_1}+\sum_{i=1}^n|u_i|^{m_1}\right)dx\le C_7\int_E\sum_{i=1}^n|u_i|^{m_1}dx.$$

Consequently

$$\|u\|_{L_{q_1}(E)}\le C_8\left(\int_E\sum_{i=1}^n|u_i|^{m_1}dx\right)^{1/m_1}. \tag{13}$$

The right hand side of this inequality we'll estimate above:

$$\left(\sum_{i=1}^n\int_E|u_i|^{m_1}dx\right)^{1/m_1}=\left(\sum_{i=1}^n\int_E\lambda_i^{-p/2s}(x)\lambda_i^{p/2s}(x)|u_i|^{m_1}dx\right)^{1/m_1}\le$$

$$\le\left[\sum_{i=1}^n\left(\int_E\lambda_i^{-ps'/2s}(x)dx\right)^{1/s'}\cdot\left(\int_E\lambda_i^{p/2}(x)|u_i|^p dx\right)^{1/s}\right]^{1/m_1}\le$$

$$\le\left(\sum_{i=1}^n\int_E\lambda_i^{-ps'/2s}(x)dx\right)^{1/m_1s'}\cdot\left(\sum_{i=1}^n\int_E\lambda_i^{p/2}(x)|u_i|^p dx\right)^{1/p},$$

where $s=p/m_1$, $s'=s/(s-1)$.

Since the condition (4) is satisfied and m_1, is determined by formula (9), then the first integral on the right hand side will be bounded. Indeed according to (12) and

$$mesE\le C_9\prod_{i=1}^n R^{1+\alpha_i/2}=C_9R^{n+|\alpha|/2}, \tag{14}$$

we have

$$\int_E\lambda_i^{-pm_1/2(p-m_1)}(x)dx\le C_{10}R^{-\frac{\alpha_i pm_1}{2(p-m_1)}}\int_E dx\le C_{11}R^{-\frac{\alpha_i pm_1}{2(p-m_1)}+\frac{|\alpha|}{2}+n} \tag{15}$$

From this it's clear that for this integral to be bounded the following must be fulfilled:

$$-\frac{\alpha_i pm_1}{2(p-m_1)}+\frac{|\alpha|}{2}+n\ge 0$$

or

$$m_1(2n+|\alpha|+\alpha_i p)\le p(2n+|\alpha|).$$

According to (4) and (9) the last inequality is always fulfilled.

Taking into account the estimation (15) in (13) we'll get

$$\|u\|_{L_{q_1}(E)}\le C_{12}\|u\|_{W^p_{1,\alpha}(E)}. \tag{16}$$

The theorem is proved.

Theorem 2. *Let the condition (4) be fulfilled. If* $1<p<q_2<q^{**}$ *then*

$$\sum_{i=1}^n\left\|\sqrt{\lambda_i}u_i\right\|_{L_{q_2}(E)}\le C_{13}\|u\|_{W^p_{2,\alpha}(E)},$$

where $C_{13}=const$ *and* m_2 *is defined by the formula (10).*

Proof. According to [7], if $m_2\le n$ and $1\le q_2<q^{**}$ then $W_2^{m_2}(E)$ is completely continuously embedded in $W_1^{q_2}(E)$

$$\|u\|_{W_1^{q_2}(E)} \le C_{14} \|u\|_{W_2^{m_2}(E)}.$$

Again for simplicity let's restrict by the case $u(x) \in \overset{\bullet}{W}{}^{p}_{2,\alpha}(E)$. On the right hand side of this inequality we'll twice successively apply Friedrichs inequality [7]. We have:

$$\left(\int_E \sum_{i=1}^n |u_i|^{q_2} dx\right)^{1/q_2} \le \left(\int_E \left(|u|^{q_2} + \sum_{i=1}^n |u_i|^{q_2}\right) dx\right)^{1/q_2} \le C_{15} \left(\int_E \sum_{i,j=1}^n |u_{ij}|^{m_2} dx\right)^{1/m_2}. \quad (17)$$

The left side of this inequality we'll estimate from below and the right hand side – from above:

$$\left(\int_E \sum_{i=1}^n |u_i|^{q_2} dx\right)^{1/q_2} = \left(\int_E \sum_{i=1}^n \lambda_i^{-q_2/2}(x) \lambda_i^{q_2/2}(x) |u_i|^{q_2} dx\right)^{1/q_2} \ge$$

$$\ge C_{16} R^{-\alpha^-/2} \left(\int_E \sum_{i=1}^n \lambda_i^{q_2/2}(x) |u_i|^{q_2} dx\right)^{1/q_2}, \quad (18)$$

$$\left(\sum_{i,j=1}^n \int_E |u_{ij}|^{m_2} dx\right)^{1/m_2} = \left(\sum_{i,j=1}^n \int_E (\lambda_i(x)\lambda_j(x))^{-p/2s} (\lambda_i(x)\lambda_j(x))^{p/2s} |u_{ij}|^{m_2} dx\right)^{1/m_2} \le$$

$$\le \left(\sum_{i,j=1}^n \left(\int_E (\lambda_i(x)\lambda_j(x))^{-ps'/2s} dx\right)^{1/s'} \cdot \left(\int_E (\lambda_i(x)\lambda_j(x))^{p/2} |u_{ij}|^p dx\right)^{1/s}\right)^{1/m_2} \le$$

$$\le \left(\sum_{i,j=1}^n \int_E (\lambda_i(x)\lambda_j(x))^{-ps'/2s} dx\right)^{1/m_2 s'} \cdot \left(\sum_{i,j=1}^n \int_E (\lambda_i(x)\lambda_j(x))^{p/2} |u_{ij}|^p dx\right)^{1/p}, \quad (19)$$

where $s = p/m_2$.

Taking into account (18) and (19) in (17) we have

$$\left(\int_E \sum_{i=1}^n \lambda_i^{q_2/2}(x) |u_i|^{q_2} dx\right)^{1/q_2} \le C_{17} R^{\alpha^-/2} \left(\sum_{i,j=1}^n \int_E (\lambda_i(x)\lambda_j(x))^{-ps'/2s} dx\right)^{1/m_2 s'} \times$$

$$\times \left(\sum_{i,j=1}^n \int_E (\lambda_i(x)\lambda_j(x))^{p/2} |u_{ij}|^p dx\right)^{1/p} \le$$

$$\le C_{18} R^{\alpha^-/2} \left(\sum_{i,j=1}^n \int_E (\lambda_i(x)\lambda_j(x))^{-ps'/2s} dx\right)^{1/m_2 s'} \cdot \|u\|_{W_{2,\alpha}^p(E)}.$$

In order to complete the proof it's necessary to show that the first integral on the right hand side is bounded. According to (12) and (14)

$$R^{m_2 s' \alpha^-/2} \int_E (\lambda_i(x)\lambda_j(x))^{-ps'/2s} dx \le C_{19} R^{\frac{pm_2(\alpha^- - \alpha_i - \alpha_j)}{2(p-m_2)} + \frac{|\alpha|}{2} + n}.$$

This integral will be bounded if

$$\frac{pm_2(\alpha^- - \alpha_i - \alpha_j)}{2(p - m_2)} + \frac{|\alpha|}{2} + n \geq 0$$

or

$$m_2\left[2n + |\alpha| + p\left(\alpha_i + \alpha_j - \alpha^-\right)\right] \leq p(2n + |\alpha|).$$

The last inequality is always fulfilled according to (4) and (10). Thus we receive that

$$\left[\int_E \sum_{i=1}^{n} \lambda_i^{q_2/2}(x)|u_i|^{q_2}\, dx\right]^{1/q_2} \leq C_{20}\|u\|_{W_{2,\alpha}^p(E)}$$

or

$$\sum_{i=1}^{n}\left\|\sqrt{\lambda_i}\, u_i\right\|_{L_{q_2}(E)}^{q_2} \leq C_{20}^{q_2}\|u\|_{W_{2,\alpha}^p(E)}^{q_2}. \tag{20}$$

It is easy to calculate that

$$\sum_{i=1}^{n}\left\|\sqrt{\lambda_i}\, u_i\right\|_{L_{q_2}(E)}^{q_2} \geq \left(\sum_{i=1}^{n}\left\|\sqrt{\lambda_i}\, u_i\right\|_{L_{q_2}(E)}\right)^{q_2}.$$

Take it into account in (20):

$$\sum_{i=1}^{n}\left\|\sqrt{\lambda_i}\, u_i\right\|_{L_{q_2}(E)} \leq C_{20} n^{(q_2-1)/q_2}\|u\|_{W_{2,\alpha}^p(E)},$$

we'll denote $C_{13} = C_{20} n^{(q_2-1)/q_2}$ and the theorem is proved.

2^0. Integral estimations.

Lemma 3. *Let the conditions (3), (4), (6)-(11) be fulfilled. Then for any function $u(x) \in C_0^\infty(E)$ at every $r \leq r_1(n, p, \alpha, \mu, h, a_0)$ the estimation*

$$\|u\|_{W_{2,\alpha}^p(E)} \leq C_{21}\|Lu\|_{L_p(E)}$$

is true.

Proof. According to lemma 2:

$$\|u\|_{W_{2,\alpha}^p(E)} \leq C_4\|L_0 u\|_{L_p(E)} = C_4\|L_0 u - Lu + Lu\|_{L_p(E)} \leq$$
$$\leq C_4\left(\|L_0 u - Lu\|_{L_p(E)} + \|Lu\|_{L_p(E)}\right). \tag{21}$$

We have

$$\|L_0 u - Lu\|_{L_p(E)} \leq \left\|\sum_{i=1}^{n} b_i u_i\right\|_{L_p(E)} + \|cu\|_{L_p(E)} \leq \sum_{i=1}^{n}\|b_i u_i\|_{L_p(E)} + \|cu\|_{L_p(E)}. \tag{22}$$

Further we have

$$\|cu\|_{L_p(E)} = \left(\int_E |c|^p |u|^p\, dx\right)^{1/p} \leq \left(\int_E |u|^{ps}\, dx\right)^{1/ps} \cdot \left(\int_E |c|^{ps'}\, dx\right)^{1/ps'} =$$
$$= \|u\|_{L_{q_1}(E)}\|c\|_{L_{\frac{pq_1}{q_1-p}}(E)},$$

where $q_1 = ps$.

According to (16) we have:

$$\|cu\|_{L_p(E)} \le C_{12}\|u\|_{W^p_{1,\alpha}(E)}\|c\|_{L_{\frac{pq_1}{q_1-p}}(E)}.$$

Analogously we have:

$$\sum_{i=1}^{n}\|b_i u_i\|_{L_p(E)} = \sum_{i=1}^{n}\left(\int_E |b_i u_i|^p\, dx\right)^{1/p} = \sum_{i=1}^{n}\left(\int_E \left|\frac{b_i}{\sqrt{\lambda_i}}\right|^p \left|\sqrt{\lambda_i}u_i\right|^p dx\right)^{1/p} \le$$

$$\le \sum_{i=1}^{n}\left(\int_E \left|\frac{b_i}{\sqrt{\lambda_i}}\right|^{ps'} dx\right)^{1/ps'} \cdot \left(\int_E \left|\sqrt{\lambda_i}u_i\right|^{ps} dx\right)^{1/ps} =$$

$$= \sum_{i=1}^{n}\left(\left\|\frac{b_i}{\sqrt{\lambda_i}}\right\|_{L_{\frac{pq_2}{q_2-p}}(E)} \left\|\sqrt{\lambda_i}u_i\right\|_{L_{q_2}(E)}\right) \le$$

$$\le \left(\sum_{i=1}^{n}\left\|\frac{b_i}{\sqrt{\lambda_i}}\right\|^{q_2'}_{L_{\frac{pq_2}{q_2-p}}(E)}\right)^{1/q_2'} \left(\sum_{i=1}^{n}\left\|\sqrt{\lambda_i}u_i\right\|^{q_2}_{L_{q_2}(E)}\right)^{1/q_2},$$

where $q_2 = ps$.

Here we use (20):

$$\sum_{i=1}^{n}\|b_i u_i\|_{L_p(E)} \le C_{20}\|u\|_{W^p_{2,\alpha}(E)}\left(\sum_{i=1}^{n}\left\|\frac{b_i}{\sqrt{\lambda_i}}\right\|^{q_2/(q_2-1)}_{L_{\frac{pq_2}{q_2-p}}(E)}\right)^{(q_2-1)/q_2}.$$

Let's choose $r_1 \le r_0$ so small, that the following

$$\|c\|_{L_{\frac{pq_1}{q_1-p}}(E)} \le \delta_1, \quad \left\|\frac{b_i}{\sqrt{\lambda_i}}\right\|_{L_{\frac{pq_2}{q_2-p}}(E)} \le \delta_2,$$

are fulfilled where $\delta_1 > 0$ and $\delta_2 > 0$ will be chosen later. Then we have

$$\|cu\|_{L_p(E)} \le \delta_1 C_{12}\|u\|_{W^p_{1,\alpha}(E)} \le \delta_1 C_{12}\|u\|_{W^p_{2,\alpha}(E)},$$

$$\sum_{i=1}^{n}\|b_i u_i\|_{L_p(E)} \le \delta_2 C_{20} n^{(q_2-1)/q_2}\|u\|_{W^p_{2,\alpha}(E)}.$$

Taking into account these estimations in (22):

$$\|L_0 u - Lu\|_{L_p(E)} \le C_{22}\|u\|_{W^p_{2,\alpha}(E)}.$$

where $C_{22} = \delta_1 C_{12} + \delta_2 C_{20} n^{(q_2-1)/q_2}$.

Finally from (21) we get:

$$\|u\|_{W^p_{2,\alpha}(E)} \le C_4\|Lu\|_{L_p(E)} + C_4\|L_0u - Lu\|_{L_p(E)} \le$$
$$\le C_4\|Lu\|_{L_p(E)} + C_4C_{22}\|u\|_{W^p_{2,\alpha}(E)}.$$

Let's choose δ_1 and δ_2 so sufficiently small that $C_{22} \le \frac{1}{2C_4}$. Let's denote $C_{21} = C_4/(1 - C_4C_{22})$ and the lemma is proved.

Lemma 4. *Let the conditions (3), (4), (6)-(11) be fulfilled. Then for any function* $u(x) \in C^\infty(\overline{E})$ *the estimation*

$$\|u\|^p_{W^p_{2,\alpha}(E_1)} \le C_{23}\int_E |Lu|^p dx + \varepsilon\|u\|^p_{W^p_{2,\alpha}(E)} + \frac{C_{24}}{\varepsilon R^{2p}}\int_E |u|^p dx,$$

is true where ε *is any positive number.*

Proof. We'll introduce the function $\theta(x) \in C_0^\infty(E)$ such that

$$\theta(x) = \begin{cases} 1, & for \quad x \in E_1 \\ 0, & for \quad x \notin E_R^{x'}(3r/8), \end{cases} \qquad 0 \le \theta(x) \le 1$$

and

$$|\theta_i| \le C_{25}R^{-1-\alpha_i/2}, \quad |\theta_{ij}| \le C_{26}R^{-2-(\alpha_i+\alpha_j)/2}. \tag{23}$$

Let $\omega(x) = u(x)\theta(x)$. Then $\omega(x) \in C_0^\infty(E)$ and according to lemma 3:

$$\|u\|_{W^p_{2,\alpha}(E_1)} \le C_{21}\|L\omega\|_{L_p(E)}. \tag{24}$$

It is easy to calculate that

$$L\omega = \theta Lu + u\left(\sum_{i,j=1}^n a_{ij}\theta_{ij} + \sum_{i=1}^n b_i\theta_i\right) + 2\sum_{i,j=1}^n a_{ij}u_i\theta_j.$$

According to (8), (8'), (12) and (23) we have

$$\left|\sum_{i,j=1}^n a_{ij}\theta_{ij}\right| = \left|\sum_{i,j=1}^n \tilde{a}_{ij}(x)\sqrt{\lambda_i(x)\lambda_j(x)}\theta_{ij}\right| \le \frac{C_{27}}{R^2},$$

$$2\left|\sum_{i,j=1}^n a_{ij}u_i\theta_j\right| \le 2\sqrt{\sum_{i,j=1}^n a_{ij}u_iu_j \sum_{i,j=1}^n a_{ij}\theta_i\theta_j} \le$$

$$\le 2\mu^{-1}\sqrt{\sum_{i=1}^n \lambda_i(x)u_i^2}\sqrt{\sum_{i=1}^n \lambda_i(x)\theta_i^2} \le \frac{C_{28}}{R}\sqrt{\sum_{i=1}^n \lambda_i(x)u_i^2},$$

$$\left|\sum_{i=1}^n b_i\theta_i\right| = \left|\sum_{i=1}^n \frac{b_i}{\sqrt{\lambda_i}}\sqrt{\lambda_i}\theta_i\right| \le \sum_{i=1}^n\left|\frac{b_i}{\sqrt{\lambda_i}}\right|\left|\sqrt{\lambda_i}\theta_i\right| \le \frac{C_{29}}{R}\sum_{i=1}^n\left|\frac{b_i}{\sqrt{\lambda_i}}\right|.$$

Consequently

$$|L\omega|^p \le C_{30}|Lu|^p + |u|^p\left(\frac{C_{31}}{R^{2p}} + \frac{C_{32}}{R^p}\sum_{i=1}^n\left|\frac{b_i}{\sqrt{\lambda_i}}\right|^p\right) + $$

$$+\frac{C_{33}}{R^p}\sum_{i=1}^n\left(\sqrt{\lambda_i}|u_i|\right)^p$$

or

$$\int_E|L\omega|^p dx \le C_{30}\int_E|Lu|^p dx + \frac{1}{R^{2p}}\left(C_{31}\int_E|u|^p dx + C_{32}R^p\sum_{i=1}^n\int_E|u|^p\left|\frac{b_i}{\sqrt{\lambda_i}}\right|^p dx\right) +$$

$$+\frac{C_{33}}{R^p}\sum_{i=1}^n\int_E\left(\sqrt{\lambda_i}|u_i|\right)^p dx .$$

Using interpolating inequality [9] we'll get

$$\int_{\tilde{E}}\sum_{i=1}^n\left|\frac{\partial\tilde{u}(y)}{\partial y_i}\right|^p dy \le \varepsilon_1\int_{\tilde{E}}\sum_{i,j=1}^n\left|\frac{\partial^2\tilde{u}(y)}{\partial y_i\partial y_j}\right|^p dy + \frac{C_{34}}{\varepsilon_1}\int_{\tilde{E}}|\tilde{u}|^p dy ,$$

where $y_i = R^{-\alpha_i/2}x_i$ $(i=1,\ldots,n)$, $\tilde{E}$ and $\tilde{u}(y)$ are images of the ellipsoid E and the function $u(x)$ respectively.

Returning to the variables x we'll get:

$$\int_E\sum_{i=1}^n(\lambda_i(x))^{p/2}|u_i|^p dx \le \varepsilon_1 C_{35}\int_E\sum_{i,j=1}^n\left(\lambda_i(x)\lambda_j(x)\right)^{p/2}|u_{ij}|^p dx + \frac{C_{36}}{\varepsilon_1}\int_E|u|^p dx .$$

From here we have

$$\int_E|L\omega|^p dx \le C_{30}\int_E|Lu|^p dx + \left(\frac{C_{31}}{R^{2p}} + \frac{C_{33}C_{36}}{\varepsilon_1 R^p}\right)\int_E|u|^p dx +$$

$$+\frac{\varepsilon_1 C_{33}C_{35}}{R^p}\|u\|^p_{W^p_{2,\alpha}(E)} + \frac{C_{32}}{R^p}\sum_{i=1}^n\int_E|u|^p\left|\frac{b_i}{\sqrt{\lambda_i}}\right|^p dx \le C_{30}\int_E|Lu|^p dx +$$

$$+\left(\frac{C_{31}}{R^{2p}} + \frac{C_{33}C_{36}}{\varepsilon_1 R^p}\right)\int_E|u|^p dx + \frac{\varepsilon_1 C_{33}C_{35}}{R^p}\|u\|^p_{W^p_{2,\alpha}(E)} +$$

$$+\frac{C_{32}}{R^p}\sum_{i=1}^n\left(\int_E\left|\frac{b_i}{\sqrt{\lambda_i}}\right|^{ps'} dx\right)^{1/s'}\cdot\left(\int_E|u|^{ps} dx\right)^{1/s} .$$

Let $s = n/(n-p)$, then according to (16) and (7):

$$\|u\|_{L_{\frac{np}{n-p}}(E)} \le C_{12}\|u\|_{W^p_{1,\alpha}(E)},\quad \left\|\frac{b_i}{\sqrt{\lambda_i}}\right\|_{L_n(E)} \le C_{37} .$$

Consequently,

$$\int_E |L\omega|^p dx \le C_{30} \int_E |Lu|^p dx + \left(\frac{C_{31}}{R^{2p}} + \frac{C_{33}C_{36}}{\varepsilon_1 R^p}\right)\int_E |u|^p dx +$$

$$+\frac{\varepsilon_1 C_{33}C_{35}}{R^p}\|u\|^p_{W^p_{2,\alpha}(E)} + \frac{nC_{32}(C_{12}C_{37})^p}{R^p}\|u\|^p_{W^p_{1,\alpha}(E)}.$$

Let's apply again the interpolating inequality [9]:

$$\int_E |L\omega|^p dx \le C_{30} \int_E |Lu|^p dx + \left(\frac{C_{31}}{R^{2p}} + \frac{C_{33}C_{36}}{\varepsilon_1 R^p}\right)\int_E |u|^p dx +$$

$$+\frac{\varepsilon_1 C_{33}C_{35}}{R^p}\|u\|^p_{W^p_{2,\alpha}(E)} + \frac{\varepsilon_1 C_{38}}{R^p}\|u\|^p_{W^p_{2,\alpha}(E)} + \frac{C_{39}}{\varepsilon_1 R^p}\int_E |u|^p dx =$$

$$= C_{30} \int_E |Lu|^p dx + \left(\frac{C_{31}}{R^{2p}} + \frac{C_{33}C_{36} + C_{39}}{\varepsilon_1 R^p}\right)\int_E |u|^p dx +$$

$$+\frac{\varepsilon_1 (C_{33}C_{35} + C_{38})}{R^p}\|u\|^p_{W^p_{2,\alpha}(E)}.$$

Take this into account in (24):

$$\|u\|^p_{W^p_{2,\alpha}(E_1)} \le C^p_{21} C_{30} \int_E |Lu|^p dx +$$

$$+\left(\frac{C^p_{21}C_{31}}{R^{2p}} + \frac{(C_{33}C_{36} + C_{39})C^p_{21}}{\varepsilon_1 R^p}\right)\int_E |u|^p dx + \frac{\varepsilon_1 C^p_{21}(C_{33}C_{35} + C_{38})}{R^p}\|u\|^p_{W^p_{2,\alpha}(E)}. \qquad (25)$$

Let $\varepsilon > 0$ be an arbitrary number. Without loss of generality we assume that $\varepsilon \le 1$. Let ε_1 be so that $\varepsilon = \varepsilon_1 C^p_{21}(C_{33}C_{35} + C_{38})/R^p$ then from (25) follows

$$\|u\|^p_{W^p_{2,\alpha}(E_1)} \le C^p_{21} C_{30} \int_E |Lu|^p dx + \varepsilon\|u\|^p_{W^p_{2,\alpha}(E)} +$$

$$+\frac{C^p_{21}C_{31} + \dfrac{C^{2p}_{21}(C_{33}C_{36} + C_{39})(C_{33}C_{35} + C_{38})}{\varepsilon}}{R^{2p}}\int_E |u|^p dx.$$

Lemma has been proved.

Analogously to [5] lemmas 5-8 are proved.

Lemma 5. *Let the conditions (3), (4), (6)-(11) be satisfied. Then for any functions* $u(x) \in W^p_{2,\alpha}\left(E^0_R(1+r)\right)$ *the estimation*

$$\|u\|^p_{W^p_{2,\alpha}\left(E^0_R\left(1+\frac{r}{2}+\frac{r^2}{64}\right)\right)} \le C_{40} \int_{E^0_R(1+r)} |Lu|^p dx + \varepsilon\|u\|^p_{W^p_{2,\alpha}\left(E^0_R(1+r)\right)} + \frac{C_{41}}{\varepsilon}\left(\operatorname*{ess\,sup}_{E^0_R(1+r)}|u|\right)^p,$$

is true, where ε *is any positive number.*

Remark. Since the operator L is degenerated only at the point 0, then the present lemma is true and in the ellipsoids $E^{x''}_R\left(1+\frac{r}{2}+\frac{r^2}{64}\right)$ and $E^{x''}_R(1+r)$ if $\overline{E^{x''}_R(1+r)} \subset D$ and $E^{x''}_R(1+r) \cap E^0_R(r) = \varnothing$.

3^0. Main coercive inequality.

Lemma 6. *Let the conditions (3), (4), (6)-(8) and (11) be fulfilled. Then for any function* $u(x)\in \overset{\bullet}{W}{}^p_{2,\alpha}(D)$ *at every* $\varepsilon>0$ *and* $\rho>0$ *the estimation*

$$\|u\|^p_{W^p_{2,\alpha}(D_\rho)}\le C_{42}\left(\int_D |Lu|^p dx+\varepsilon\|u\|^p_{W^p_{2,\alpha}(D)}+\frac{1}{\varepsilon}\left(ess\sup_D|u|\right)^p\right)$$

is true.

Lemma 7. *Let the conditions (3), (4), (6)-(11) be fulfilled. Then for any function* $u(x)\in \overset{\bullet}{W}{}^p_{2,\alpha}(D)$ *at every* $\rho>0$ *the estimation*

$$\|u\|^p_{W^p_{2,\alpha}(D\setminus D_\rho)}\le C_{43}\left(\int_D |Lu|^p dx+\left(ess\sup_D|u|\right)^p\right)$$

is true.

Lemma 8. *Let the conditions (3), (4), (6)-(11) be fulfilled. Then for any function* $u(x)\in \overset{\bullet}{W}{}^p_{2,\alpha}(D)$ *the estimation*

$$\|u\|_{W^p_{2,\alpha}(D)}\le C_{44}\left(\|Lu\|_{L_p(D)}+\left(ess\sup_D|u|\right)^p\right)$$

is true.

Theorem 3. *Let the conditions (3)-(11) be fulfilled. Then for any function* $u(x)\in \overset{\bullet}{W}{}^p_{2,\alpha}(D)$ *the estimation*

$$\|u\|_{W^p_{2,\alpha}(D)}\le C_{45}\|Lu\|_{L_q(D)} \tag{26}$$

is true.

Proof. According to our assumption $c(x)\le 0$ is true, then according to the estimation of A.D.Aleksandrov [6]:

$$ess\sup_D|u|\le C_{46}\left\|\frac{f}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}\cdot F_n\left(\left\|\frac{b}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}\right). \tag{27}$$

It's evident that

$$\det(a_{ij})\ge C_{47}\prod_{i=1}^n\lambda_i(x)=C_{47}\prod_{i=1}^n\left(|x|_\alpha\right)^{\alpha_i}\ge C_{47}\prod_{i=1}^n|x_i|^{2\alpha_i/(2+\alpha_i)}.$$

From here we have:

$$\left\|\frac{f}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}=\left(\int_D\frac{|f|^n}{\det(a_{ij})}dx\right)^{1/n}\le$$

$$\le\left(\int_D|f|^{ns}dx\right)^{1/ns}\cdot\left(\int_D\frac{|f|^n}{\left(\det(a_{ij})\right)^{s'}}dx\right)^{1/ns'}.$$

Let $q=ns$, then

$$\left\| \frac{f}{\sqrt[n]{\det(a_{ij})}} \right\|_{L_n(D)} \leq \|f\|_{L_q(D)} \cdot \left(\int_D \prod_{i=1}^{n} |x_i|^{-\frac{2\alpha_i}{(2+\alpha_i)} \frac{q}{(q-n)}} \right)^{\frac{(q-n)}{nq}}.$$

Since $q > n(2+\alpha^+)/(2-\alpha^+)$ then integral in right side of the last inequality is finite.

Now let's prove, that the second multiplier in the right hand side of (27) is finite. It's evident that

$$\left\| \frac{b}{\sqrt[n]{\det(a_{ij})}} \right\|_{L_n(D)} = \left\| \sqrt{\sum_{i=1}^{n} \left(\frac{b_i}{\sqrt[n]{\det(a_{ij})}} \right)^2} \right\|_{L_n(D)} = \left[\int_D \left(\sqrt{\sum_{i=1}^{n} \left(\frac{b_i}{\sqrt[n]{\det(a_{ij})}} \right)^2} \right)^n dx \right]^{1/n} \leq$$

$$\leq C_{48} \left(\sum_{i=1}^{n} \int_D \frac{|b_i|^n}{\det(a_{ij})} dx \right)^{1/n}.$$

From here we have

$$\int_D \frac{|b_i|^n}{\det(a_{ij})} dx = \int_D \left| \frac{b_i}{\sqrt{\lambda_i}} \right|^n \cdot \frac{\left(\sqrt{\lambda_i}\right)^n}{\det(a_{ij})} dx \leq \left(\int_D \left| \frac{b_i}{\sqrt{\lambda_i}} \right|^{ns_0} dx \right)^{1/s_0} \cdot \left(\int_D \frac{\left(\sqrt{\lambda_i}\right)^{ns_0'}}{\left(\det(a_{ij})\right)^{s_0'}} dx \right)^{1/s_0'} \leq$$

$$\leq C_{49} \left\| \frac{b_i}{\sqrt{\lambda_i}} \right\|_{L_{ns_0}(D)}^n \cdot \left(\int_D \frac{\left(|x|_\alpha\right)^{\alpha_i n s_0'/2}}{\prod_{j=1}^{n} \left(|x|_\alpha\right)^{\alpha_j s_0'}} dx \right)^{1/s_0'}.$$

Let $s_0 = (6+\alpha^+)/(2-\alpha^+)$. Then according to (7) the first multiplier and according to (4) the second multiplier in the right hand side of the last inequality will be finite.

Thus,

$$\operatorname*{ess\,sup}_D |u| \leq C_{50} \|Lu\|_{L_q(D)}. \tag{28}$$

On the other hand it's known that

$$\|Lu\|_{L_p(D)} \leq (mesD)^{(q-p)/qp} \|Lu\|_{L_q(D)}.$$

We'll take into account (28) and the last inequality in the lemma 8:

$$\|u\|_{W_{2,\alpha}^p(D)} \leq C_{44} \left[(mesD)^{(q-p)/qp} \|Lu\|_{L_q(D)} + C_{50} \|Lu\|_{L_q(D)} \right] = C_{45} \|Lu\|_{L_q(D)},$$

where $C_{45} = C_{44} \left[(mesD)^{(q-p)/qp} + C_{50} \right]$.

The theorem is proved.

4^0. The Strong Solvability of the first boundary value problem.

Theorem. *Let the conditions (3)-(11) be fulfilled. Then the problem (1)-(2) is uniquely strong solvable in* $\overset{\bullet}{W}{}_{2,\alpha}^{p}(D)$. *And for the solution* $u(x)$ *of the problem (1)-(2) the estimation*

$$\|u\|_{W^{p}_{2,\alpha}(D)} \le C_{45}\|f\|_{L_q(D)}$$

is true.

The proof of the theorem is conducted by the scheme, used at the paper [5], subject to the coercive estimation (26).

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