2001 VOL. XV(XXIII)

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ON MULTIPLE COMPLETENESS OF A SYSTEM OF EIGEN AND ADJOINT ELEMENTS OF OPERATOR SHEAF IN HILBERT SPACE

Abstract

In the paper the theorem on multiple completeness of a system of eigen and adjoined elements of operator sheaf are proved.

Basis of spectral theory of operators sheaf was found in fundamental articles of M.V. Keldysh [1,2], in which the notion of multiple completeness of a system of eigen and adjoint vectors of polynomial operator sheaf was given and the fundamental theorem on multiple completeness of these systems for some classes of operator sheafs was proved.

Results of articles of M.V. Keldysh found further development in papers of J.E. Allahverdiyev [3], M.G. Gasymov [4], M.G. Krein and G.K. Langer [5], G.V. Radziyevsky [6,7], R.M. Jabbarzadeh [8], I.V. Goriuk [9], V.I. Matsaev and E.Z. Mogulsky [10].

In this paper we discuss $2n$-fold completeness of eigen and adjoined vectors of an operator sheaf of type

$$A(\lambda)=\sum_{j=0}^{n}\lambda^{j}A_{j},$$

where λ is a complex parameter, A_j $(j=0,1,...,n)$ are linear operators acting in Hilbert space H.

Later we'll need some definitions and facts. Let's mention them.

Suppose $L(\lambda)=E-A(\lambda)$.

Definition 1. *The number λ_0 is called an eigenelement of the operator sheaf $L(\lambda)$, if there exists non-zero vector $\varphi_0\in H$ such that the following inequality is satisfied*

$$L(\lambda_0)\varphi_0=0.$$

Here φ_0 is called an eigenvectr of the sheaf $L(\lambda)$, responding to the eigenelement λ_0.

If $\varphi_1,\varphi_2,...,\varphi_k$ satisfy equations

$$\sum_{p=0}^{j}\frac{1}{p!}L^{(p)}(\lambda_0)\varphi_{j-p}=0 \quad (j=1,2,...,k),$$

then they are called vector train, adjoined to the eigenvector φ_0 of the sheaf $L(\lambda)$.

Let φ_0 be an eigenvector of the sheaf $L(\lambda)$, responding to the eigenvalue λ_0, and $\varphi_1,..,\varphi_k$ be corresponding adjoined elements.

We define elements $\tilde{\varphi}_s\in H^n$, where H^n is a direct sum of n-copies of spaces H by the following way

$$\tilde{\varphi}_s=\left(\varphi_s^{(0)},\varphi_s^{(1)},...,\varphi_s^{(n-1)}\right) \text{ in addition } \varphi_0^{(0)}=\varphi_0, \quad \varphi_s^{(0)}=\varphi_s \ (s=1,...,k),$$

and

$$\varphi_1^{(0)}=\lambda_0^{\nu}\varphi_0^{(0)}=\lambda_0^{\nu}\varphi_0, \quad \varphi_s^{\nu}=\lambda_s\varphi_s^{(\nu-1)}+\varphi_{s-1}^{(\nu-1)}, \quad \nu=1,...,k.$$

Definition 2. *A system of eigen and adjoint vectors is called n-fold complete in H, if the system $\widetilde{\varphi}_s$, constructed for all eigen values, is complete in H^n.*

In theorem 1 from the article [8] of R.M. Dzabarzadeh twofold completeness of a system of eigen adjoined elements of operators, quadratically dependent on spectral parameter was proved. Let's mention it.

R.M. Dzabarzadeh's [8] theorem: *Let C be a complete self-adjoint operator of the finite order ρ, acting in a Hilbert space H; A, B are completely continuous operators in H, where B has the form:*

$$B = B_1 C^{1/2} + C_1,$$

where B_1 is a completely continuous operator, and C_1 is a self-adjoint operator of the finite order ρ_1.

Then system of eigen adjoined elements of the operator

$$A + \lambda B + \lambda^2 C$$

is twofold complete in Hilbert space H.

(We'll remind, that a class of completely continuous operators is denoted by G_∞. Let $A \in G_\infty$. Then the operator

$$B = \left(A^* A\right)^{1/2} \in G_\infty .$$

Eigenvaleus of the operator B are called s-numbers of the operator A.

Completely continuous operator B has a finite order, if its s-numbers are such that

$$\sum_{i=0}^{\infty} |s_i|^\rho < \infty \text{ for some } \rho > 0 .$$

Lower boundary of numbers ρ $\rho_0 = \inf \rho$ is the order of the operator B.

The class of completely continuous operators with finite order ρ is denoted by G_ρ).

The foregoing R.M. Dzabarzadeh's theorem has generalization which can be formulated in the following way:

Theorem 1. *Let conditions*

a) C and C_1 are normal completely continuous operators; $C \in_{\rho_c}$, $C_1 \in G_{\rho_{c_1}}$; eigenvalues each of them lie on a finite number of rays, where $\ker C = D$;

b) A and B_1 are completely continuous operators;

c) $CC_1 = C_1 C$ and $C^* C = CC_1^*$ be fulfilled.

Then the two fold completeness of eigen adjoined elements of the operator

$$A + \lambda B_1 C^{1/2} + \lambda C_1 + \lambda^2 C \tag{1}$$

holds.

Proof of theorem 1. As in the proof of theorem 1 from [8] consider in space $H^2 = H \oplus H$ the sheaf

$$\left(\widetilde{A} + \lambda \widetilde{B}\right)\widetilde{x} = \widetilde{x} , \tag{2}$$

where

$$\widetilde{A} = \begin{pmatrix} A & B_1 \\ 0 & 0 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} C_1 & C \\ C & 0 \end{pmatrix}.$$

Operator $\widetilde{B}$ is normal, since

$$\begin{pmatrix} C_1 & C \\ C & 0 \end{pmatrix}\begin{pmatrix} C_1^* & C^* \\ C^* & 0 \end{pmatrix} = \begin{pmatrix} C_1C_1^* + CC^* & C_1C^* \\ CC_1^* & CC^* \end{pmatrix} =$$

$$= \begin{pmatrix} C_1^*C_1 + C^*C & C_1^*C \\ C^*C_1 & C^*C \end{pmatrix} = \begin{pmatrix} C_1^* & C^* \\ C^* & 0 \end{pmatrix}\begin{pmatrix} C_1 & C \\ C & 0 \end{pmatrix},$$

i.e. $\widetilde{B}\widetilde{B}^* = \widetilde{B}^*\widetilde{B}$.

Eigenvalues of the operator $\begin{pmatrix} C_1 & C \\ C & 0 \end{pmatrix}$ are arranged near a finite number of rays.

Really, eigenvalues of the operator $\begin{pmatrix} C_1 & C \\ C & 0 \end{pmatrix}$ coincide with the spectrum of the sheaf $L_1(\lambda) = -\lambda^2 E + C^2 + \lambda C_1$. Let's show it. Let λ be an eigenvalue of the operator $\begin{pmatrix} C_1 & C \\ C & 0 \end{pmatrix}$. Then there exists an eigenelement (x, y) such that

$$\begin{pmatrix} C_1 & C \\ C & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}.$$

It means that

$$C_1x + Cy = \lambda x, \tag{3}$$

and

$$Cx = \lambda y. \tag{4}$$

Multiplying (3) by λ and acting on (4) by the operator C from (3) and (4) we'll obtain:

$$\left(\lambda C_1 + C^2 - \lambda^2 E\right)x = 0,$$

i.e.

$$L_1(\lambda)x = 0.$$

From Allahverdiyev's theorem (see [3], theorem 2) it follows that spectrum of the sheaf $L_1(\lambda)$ is a arranged near a finite number of rays.

From spectral theory of sheafs it follows, that eigenvalues of the last operator and consequently of the operator $\widetilde{B}$ arbitrarily close approach to a finite number of rays, outgoing from origin. It means that excluding the finite number, eigenvalues of operator $\widetilde{B}$ lie inside small angles, bisectrix of which are these rays.

Taking into account that the operator $\widetilde{A}$ is completely continuous by virtue of the condition b) of the theorem we can apply J.E. Allahverdiev's theorem 1 from [3] for equation (2). Consequently, the completeness of a system of eigen adjoined elements of equation (2) in space H^2 holds.

The proof of double completeness of system of eigen adjoined elements of the equation (1) in the space H is led analogously to the proof of R.M. Dzabarzadeh's theorem 1 from [8].

Theorem 1 is proved.

Theorem 2. *Let*

a) operators $A_i \in G_\infty$ $\left(i = \overline{0, n-1}\right)$;

b) BeG_{ρ_1}, $C \in G_{\rho_2}$, $0 < \rho_1, \rho_2 < \infty$.

B and C be normal operators whose eigenvalues lie on a finite number of rays, moreover $B^*C = BC^*$, $BC = CB$.

Then a system of eigen adjoined elements of the sheaf

$$L(\lambda)=A_0+\lambda A_1(C+\lambda B)+\lambda^2A_2(C+\lambda B)+\cdots\lambda^{n-1}A_{n-1}(C+\lambda B)^{n-1}+\lambda^n(C+\lambda B)^n, \quad (5)$$

generates $2n$ *fold complete system in the Hilbert space* H.

Proof of theorem 2. Let H^n be a direct sum of n-Hilbert spaces, i.e. space, whose elements are ordered systems of n elements of the space H. Scalar product in H^n is defined by the following way for

$$\overline{x}=(x_1,x_2,...,x_n)\in H \text{ and } \overline{y}=(y_1,y_2,...,y_n)\in H$$

$$[\overline{x},\overline{y}]_{H^n}=\sum_{i=1}^{n}(x_i,y_i)_H .$$

In space H^n consider the quadratic equation

$$\left(\widetilde{A}+\lambda^2\widetilde{B}\right)\widetilde{x}+\lambda\widetilde{C}\widetilde{x}=\widetilde{x}, \quad (6)$$

where operators $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ are given by means of matrices in a direct sum of spaces H^n,

$$\widetilde{A}=\begin{pmatrix} A_0 & A_1 & \dots & A_{n-1} \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad \widetilde{B}=\begin{pmatrix} 0 & 0 & \dots & 0 & B \\ B & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & B & 0 \end{pmatrix},$$

$$\widetilde{C}=\begin{pmatrix} 0 & 0 & \dots & 0 & C \\ C & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & C & 0 \end{pmatrix}.$$

The operator $\widetilde{A}$ is completely continuous, since operators A_i are completely continuous; operators $\widetilde{B}$ and $\widetilde{C}$ are operators of finite orders ρ_1 and ρ_2, respectively, since operators B and C are the same. Besides, $\widetilde{B}$ and $\widetilde{C}$ are normal operators. Since

$$\widetilde{B}\widetilde{B}^*=\widetilde{B}^*\widetilde{B},$$

$$\widetilde{C}\widetilde{C}^*=\widetilde{C}^*\widetilde{C}.$$

So, all conditions of theorem 1, proved above are fulfilled and double completeness of system of eigen adjoined elements of sheaf (6) in the space H^n holds.

Let's show now connection between the system of eigen adjoined vectors of the sheaf (5) and the system of eigen adjoined vectors of the sheaf (6).

Let $\widetilde{x}=(x_0,x_1,...,x_n)$ be an eigenelement of the sheaf (6), and λ be an eigenvalue of this sheaf. Then we can write

$$\begin{cases} A_0x_0 + A_1x_1 + \cdots + A_{n-1}x_{n-1} + \lambda^2 Bx_{n-1} + \lambda Cx_{n-1} = x_0, \\ x_1 - \lambda^2 Bx_0 - \lambda Cx_0 = 0, \\ x_2 - \lambda^2 Bx_1 - \lambda Cx_1 = 0, \\ \dots\dots\dots \\ \dots\dots\dots \\ x_{n-1} - \lambda^2 Bx_{n-2} - \lambda Cx_{n-2} = 0. \end{cases}$$

After successive substitutions we'll obtain

$$x_{n-1} = \lambda(\lambda B + C)x_{n-2} = \lambda^2(\lambda B + C)^2 x_{n-3} = \dots = \lambda^{n-1}(\lambda B + C)^{n-1}x_0.$$

Consequently,

$$\begin{cases} A_0x_0 + \lambda A_1(\lambda B + C)x_0 + \lambda^2 A_2(\lambda B + C)^2 x_0 + \cdots + \lambda^{n-1}A_{n-1}(\lambda B + C)^{n-1}x_0 + \\ + \lambda^n(\lambda B + C)^n x_0 = x_0, \\ x_1 = (\lambda^2 B + \lambda C)x_0, \\ x_2 = (\lambda^2 B + \lambda C)^2 x_0, \\ \dots\dots\dots \\ \dots\dots\dots \\ x_{n-1} = (\lambda^2 B + \lambda C)^{n-1} x_0. \end{cases} \tag{7}$$

Let now $\widetilde{y} = (y_0, \dots, y_{n-1})$ be the first adjoined to the eigenelement $\widetilde{x}$, responding to the eigenvales λ of the operator sheaf (6).

Then

$$\widetilde{y} = (\widetilde{A} + \lambda\widetilde{C} + \lambda^2\widetilde{B})\widetilde{y} + \widetilde{C}\widetilde{x} + 2\lambda\widetilde{B}\widetilde{x}.$$

Consequently, allowing for (7), we can write:

$$\begin{cases} y_0 = A_0y_0 + A_1y_1 + \dots + A_{n-1}y_{n-1} + \lambda Cy_{n-1} + \lambda^2 By_{n-1} + Cx_{n-1} + 2\lambda Bx_{n-1}, \\ y_1 = \lambda Cy_0 + \lambda^2 By_0 + Cx_0 + 2\lambda Bx_0 = (\lambda C + \lambda^2 B)y_0 + (C + 2\lambda B)x_0, \\ y_2 = \lambda Cy_1 + \lambda^2 By_1 + Cx_1 + 2\lambda Bx_1 = (\lambda C + \lambda^2 B)y_1 + (C + 2\lambda B)x_1 = (\lambda C + \lambda^2 B)^2 y_0 + \\ + 2(\lambda C + \lambda^2 B)(C + 2\lambda B)x_0, \\ y_3 = (\lambda C + \lambda^2 B)^3 y_0 + 3(\lambda C + \lambda^2 B)^2 (C + 2\lambda B)x_0, \\ \dots\dots\dots \\ y_{n-1} = (\lambda C + \lambda^2 B)^{n-1} y_0 + (n-1)(\lambda C + \lambda^2 B)^{n-2}(C + 2\lambda B)x_0. \end{cases} \tag{8}$$

After successive substitutions in the first equation from (8) all consequent ones, and grouping on x_0 and y_0 we'll obtain

$$y_0 = L(\lambda)y_0 + \frac{\partial}{\partial\lambda}L(\lambda)x_0.$$

So, y_0 is an adjoined vector to x_0.

Reasoning by analogy we can show that if $\widetilde{y}_0, \widetilde{y}_1, ..., \widetilde{y}_k$ is a Jordan chain, corresponding to the eigenvector $\widetilde{x}_0$, then we have that the coordinates of there elements form a Jordan chain of the sheaf (5).

Twofold completeness of a system of eigen adjoined elements of the sheaf (6) in the space H^n means $2n$-fold completeness of a system of eigen adjoined elements of the sheaf (5) in the space H.

Theorem 2 is proved.

Remark to theorem 2. Theorem 2 is also true in that case, if characteristic values lie inside arbitrary small angles, except their finite numbers the whose bisictrices are rays, outgoing from origin.

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Received January 19, 2001; Revised June 21, 2001.
Translated by Agayeva R.A.