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A HARNACK INEQUALITY FOR DEGENERATE PARABOLIC EQUATIONS OF THE SECOND ORDER IN NONDIVERGENCE FORM

Abstract

A class of non-uniformly degenerated parabolic equations of the second order of nondivergent structure with, generally, speaking, discontinuous coefficients is considered. For nonnegative solutions of these equations a Harnack inequality has been proved.

Let $\mathbf{R}_{n+1}$ be $(n+1)$-dimensional Euclidean space of points $(x,t)=(x_1,...,x_n,t)$, D be bounded domain in $\mathbf{R}_{n+1}$, ∂D and $\Gamma(D)$ its Euclidean and parabolic boundaries of D respectively, $(0,0)\in D$. Let's consider in D the following parabolic equation

$$Lu=\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i x_j}+\sum_{i,j=1}^{n}b_i(x,t)\frac{\partial u}{\partial x_i}+c(x,t)u-\frac{\partial u}{\partial t}=0 \qquad (1)$$

under assumption that $\|a_{ij}(x,t)\|$ is a real symmetric matrix, moreover for all $(x,t)\in D$ and for any n-dimensional vector ξ.

$$\mu\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2\le\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\le\mu^{-1}\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2 . \qquad (2)$$

Here $\mu\in(0,1]$ is constant, $\lambda_i(x,t)=\left(|x|_\alpha+\sqrt{|t|}\right)^{\alpha_i}$, $|x|_\alpha=\sum_{k=1}^{n}|x_k|^{\frac{2}{2+\alpha_k}}$, $-2<\alpha_i\le 2$; $i=1,...,n$.

Relative to the minor coefficients of equation (1) we shall assume that for all $(x,t)\in D$

$$|b_i(x,t)|\le b_0;\ i=1,...,;\ -b_0\le c(x,t)\le 0, \qquad (3)$$

where b_0 is some constant. The aim of the present paper is proof of a Harnack inequality for nonnegative solutions of equation (1).

We mean by solution of equation (1) its classical solution, i.e. function $u(x,t)\in C^{2,1}(D)\cap C(\overline{D})$ which turns (1) into identity.

Note that for nondivergent equations in the form of (1) principle part of which satisfy the Cordes condition, the analogous result has been established in R.Ya.Glagoleva's paper [1]. In the work of N.V.Krylov and M.V.Safonov [2] it has been shown that for the validity of a Harnack inequality the Cordes condition is unnecessary (see also [3-4]). In the case $b_i\equiv c\equiv 0$ and $\alpha_i\ge 0$ $(i=1,...,n)$ the above mentioned inequality has been proved in paper [5]. As to second order parabolic equations of divergent structure we note in this connection classical papers of J.Nash [6] and J.Moser [7] (see also [9]). Specially note that the approach being used in the present paper based on the statement called in E.M.Landis [9] monograph "The lemma on increasing of positive solutions".

Let's agree to some denotations. For n-dimensional vector x^0 and positive numbers R and k we shall denote ellipsoid $\left\{x:\sum_{i=1}^{n}\frac{\left(x_i-x_i^0\right)}{R^{\alpha_i}}<(kR)^2\right\}$ by $\mathcal{E}_{R;k}\left(x^0\right)$, ball

$\{x:|x-x^0|<R\}$ by $B_R(x^0)$. Further let's for $t^1<t^2$ $C_{R;k}^{t^1,t^2}(x^0)$ be cylinder $\mathcal{E}_{R;k}(x^0)\times(t^1,t^2)$. Notation $C(...)$ means that the positive constant C depends only on contents of brackets. We shall use the assertion proved in paper [10].

Let $\mathbf{C}^1(x^0,t^0)=C_{R;17}^{t^0-4bR^2,t^0}(x^0)$, $\mathbf{C}^2(x^0,t^0)=C_{R;1}^{t^0-bR^2,t^0}(x^0)$, $\mathbf{C}^3(x^0,t^0)=C_{R;9\frac{1}{2}}^{t^0-3bR^2,t^0-\frac{bR^2}{2}}(x^0)\setminus \overline{C}_{R;8\frac{1}{2}}^{t^0-\frac{5bR^2}{2},t^0-\frac{bR^2}{2}}(x^0)$, $\mathbf{C}^4(x^0,t^0)=C_{R;34}^{t^0-8bR^2,t^0}(x^0)$, where exact value of constant $b(\alpha,\mu,n)\in(0,1)$ be defined in [10]. Here and later on $\alpha=(\alpha_1,...,\alpha_n)$. If $(x^0,t^0)=(0,0)$, then we shall denote the set $\mathbf{C}^i(0,0)$ simply by $\mathbf{C}^i$; $i=1,...,4$.

Let $G\subset\mathbf{R}_{n+1}$ be some bounded domain. Function $u(x,t)\in C^{2,1}(G)\cap C(\overline{G})$ is called $\mathcal{L}$-subparabolic ($\mathcal{L}$-superparabolic) in G if $\mathcal{L}u(x,t)\ge 0$ $(\mathcal{L}u(x,t)\le 0)$ for $(x,t)\in G$.

Lemma 1. ([10]) *Let $\mathbf{C}^1(x^0,t^0)\subset\mathbf{C}^4$ and domain G which intersects $\mathbf{C}^2(x^0,t^0)$ and has limiting points on $\Gamma(\mathbf{C}^1(x^0,t^0))$ be situated in $\mathbf{C}^1(x^0,t^0)$. Let positive $\mathcal{L}$-subparabolic function $u(x,t)$ vanishing in $\Gamma(G)\cap\mathbf{C}^1(x^0,t^0)$ be defined in G. Then there exists such $R_0(\alpha,\mu,n,b_0)$ that if $R\le R_0$*

$$mes(\mathbf{C}^3(x^0,t^0)\setminus G)\ge a\ \ mes(\mathbf{C}^3(x^0,t^0)),\ a>0, \tag{4}$$

and relative to the coefficients of operator $\mathcal{L}$ conditions (2)-(3) are satisfied, then

$$\sup_{(x,t)\in G} u(x,t)\ge(1+\eta(\alpha,\mu,n,a))\sup_{(x,t)\in G\cap\mathbf{C}^2(x^0,t^0)} u(x,t).$$

Lemma 2. *Let $(x^0,t^0)\in\mathbf{C}^1$ and relative to domain G and $\mathcal{L}$-subparabolic function $u(x,t)$ all the conditions of the previous lemma except of (4) be satisfied. Then for any $K>0$ there exists such $\delta(\alpha,\mu,n,K)$ that if $R\le R_0$ and*

$$mes(G)\le\delta\, mes(\mathbf{C}^1(x^0,t^0)), \tag{5}$$

then

$$\sup_{(x,t)\in G} u(x,t)\ge K\sup_{(x,t)\in G\cap\mathbf{C}^2(x^0,t^0)} u(x,t).$$

Proof. Let's constant η of the previous lemma corresponds to $a=\frac{1}{2}$ and p be the least natural number for which $(1+\eta)^p\ge K$. Suppose

$$\delta=\frac{4^n\left[5(19)^n-4(17)^n\right]}{2(17)^{2n+1}p^{n+1}}.$$

Divide the difference $\mathbf{C}^1(x^0,t^0)\setminus\mathbf{C}^2(x^0,t^0)$ by parabolic boundaries Γ_i of cylinder

$$\mathbf{C}_i=\mathcal{E}_{R;1+\frac{16i}{p}}(x^0)\times\left(t^0-bR^2\left(1+\frac{3i}{p}\right),t^0\right);\ i=0,1,...,p-1$$

into p parts. It's clear that Γ_0 coincides with $\Gamma(\mathbf{C}^2(x^0,t^0))$. Denote for $i=0,1,...,p-1$ $\sup\limits_{(x,t)\in G\cap\Gamma_i} u(x,t)$ by M_i and let $u(x,t)$ reach its value M_i at the point $(x^i,t^i)\in\Gamma_i$. It's easy to see that $\mathbf{C}^1(x^i,t^i)\subset\mathbf{C}^4$. Let's consider cylinders

$$B_1^{(i)} = C_{R;\frac{8}{p}}^{t^i - \frac{32}{17p}bR^2, t^i}\left(x^i\right),\ B_2^{(i)} = C_{R;\frac{8}{17p}}^{t^i - \frac{8}{17p}bR^2, t^i}\left(x^i\right)$$

and set

$$B_3^{(i)} = C_{R;\frac{76}{17p}}^{t^i - \frac{24}{17p}bR^2, t^i - \frac{4}{17p}bR^2}\left(x^i\right) \setminus \overline{C}_{R;\frac{4}{p}}^{t^i - \frac{20}{17p}bR^2, t^i - \frac{4}{17p}bR^2}\left(x^i\right);$$

$i = 0,1,\ldots,p-1$. Assuming $\mathbf{C}_p = \mathbf{C}^1\left(x^0,t^0\right)$ we obtain that $B_1^{(i)} \subset \mathbf{C}_{i+1}$ for $i = 0,1,\ldots,p-1$. We have

$$mes\left(B_3^{(i)} \setminus G\right) \geq mes\left(B_3^{(i)}\right) - mes(G);\ i = 0,1,\ldots,p-1. \tag{6}$$

On the other hand for $i = 0,1,\ldots,p-1$

$$mes\left(B_3^{(i)}\right) = \frac{4^{n+1}\Omega_n}{(17p)^{n+1}} bR^{n+2}\prod_{k=1}^{n} R^{\frac{\alpha k}{2}}\left[5(19)^n - 4(17)^n\right], \tag{7}$$

where Ω_n is volume of n-dimensional unit ball. Besides, according to (5)

$$mes(G) \leq \delta\ mes\left(\mathbf{C}^1\left(x^0,t^0\right)\right) = \delta 4\Omega_n (17)^n bR^{n+2}\prod_{k=1}^{n} R^{\frac{\alpha k}{2}}. \tag{8}$$

Using (7)-(8) in (6) and taking into account the choice of δ we conclude

$$mes\left(B_3^{(i)} \setminus G\right) \geq \frac{1}{2} mes\left(B_3^{(i)}\right);\ i = 0,1,\ldots,p-1.$$

Whence according to lemma 1 it follows that

$$M_{i+1} \geq (1+\eta)M_i;\ i = 0,1,\ldots,p-1,$$

where $M_p = \sup\limits_{(x,t)\in G} u(x,t)$. Thus,

$$M_p \geq (1+\eta)^p M_0,$$

and the lemma is proved.

Let now G be an arbitrary domain situated in $\mathbf{C}^1\left(x^0,t^0\right)$, where $\left(x^0,t^0\right) \in \mathbf{C}^1$, and $R \leq R_0$. We denote by $\mathcal{A}(G)$ the set of all $\mathcal{L}$-superparabolic in G functions and denote by $\mathcal{A}^+(G)$ the set of all nonnegative $\mathcal{L}$- superparabolic in G functions. Let for $\beta \in [0,1]$

$$\mathcal{A}_\beta^R\left(x^0,t^0\right) = \mathcal{A}^+\left(\mathbf{C}^1\left(x^0,t^0\right)\right) \cap \left\{u : mes\left(\mathbf{C}^1\left(x^0,t^0\right) \cap \left[(x,t) : u(x,t) \geq 1\right]\right) \geq \beta\, mes\left(\mathbf{C}^1\left(x^0,t^0\right)\right)\right\};$$

$$\gamma_\beta^R\left(x^0,t^0\right) = \inf\left\{u\left(x,t^0\right) : x \in \mathcal{E}_{R;\frac{1}{2}}\left(x^0\right), u \in \mathcal{A}_\beta^R\left(x^0,t^0\right)\right\};$$

$$\gamma_\beta^R = \inf_{\left(x^0,t^0\right)\in\mathbf{C}^1} \gamma_\beta^R;\left(x^0,t^0\right);\ \gamma(\beta) = \varliminf_{R\to 0} \gamma_\beta^R.$$

It's easy to see that $0 \leq \gamma(\beta) \leq 1$ and function $\gamma(\beta)$ doesn't decrease by β. It can be shown that the function $\gamma(\beta)$ is continuous on $[0,1]$.

Lemma 3. *Let* $u(x,t) \in \mathcal{A}^+\left(\mathbf{C}^1\left(x^0,t^0\right)\right)$, $\left(x^0,t^0\right) \in \mathbf{C}^1$, $R \leq R_0$. *If there exist* $\beta \in [0,1]$ *and* $\varepsilon > 0$ *such that*

$$mes\left\{\mathbf{C}^1\left(x^0,t^0\right) \cap \left[(x,t) : u(x,t) \geq \varepsilon\right]\right\} \geq \beta\, mes\left(\mathbf{C}^1\left(x^0,t^0\right)\right),$$

then $u\left(x,t^0\right) \geq \varepsilon\gamma(\beta)$ *for* $x \in \mathcal{E}_{R;\frac{1}{2}}\left(x^0\right)$.

The statement of lemma is follows from the definition of function $\gamma(\beta)$.

Lemma 4. *Let* $-u(x,t)\in \mathcal{A}\left(\mathbf{C}^1\left(x^0,t^0\right)\right)$, $\left(x^0,t^0\right)\in \mathbf{C}^1$, $R\le R_0$. *If there exist* $\beta\in[0,1]$ *and* $v>0$ *such that* $u\left(x^0,t^0\right)\ge v$ *and*

$$mes\left\{\mathbf{C}^1\left(x^0,t^0\right)\cap\left[(x,t):u(x,t)\le\frac{v}{2}\right]\right\}\ge\beta\, mes\left(\mathbf{C}^1\left(x^0,t^0\right)\right),$$

then

$$\sup_{(x,t)\in C^1\left(x^0,t^0\right)}u(x,t)\ge\frac{v}{2}\left(1+\frac{1}{1-\gamma(\beta)}\right). \tag{9}$$

Proof. Suppose that (9) isn't satisfied. Then there exists $\varepsilon_1>0$ such that if $\omega(x,t)=\frac{2u(x,t)}{v}-1$, then

$$\sup_{(x,t)\in C^1\left(x^0,t^0\right)}\omega(x,t)=\frac{1}{1-\gamma(\beta)+\varepsilon_1}=a_1.$$

Let $z(x,t)=1-\frac{\omega(x,t)}{a_1}$. Since $c(x,t)\le 0$, then $z(x,t)\in\mathcal{A}^+\left(\mathbf{C}^1\left(x^0,t^0\right)\right)$. Moreover, if $u(x,t)\le\frac{v}{2}$, then $z(x,t)\ge 1$. Applying lemma 3 for $\varepsilon=1$ we obtain $z\left(x^0,t^0\right)\ge\gamma(\beta)$. On the other hand by the condition $\omega\left(x^0,t^0\right)\ge 1$, therefore

$$1-\frac{1}{a_1}\ge 1-\frac{\omega\left(x^0,t^0\right)}{a_1}\ge\gamma(\beta),,$$

i.e. $a_1\ge\frac{1}{1-\gamma(\beta)}$ which is impossible. The lemma is proved.

Theorem 1. *The following limiting equality holds*

$$\lim_{\beta\to 1-0}\gamma(\beta)=1.$$

Proof. At first let's rephrase the statement of lemma 2. Let $u(x,t)\in\mathcal{A}^+(G)$, $u\big|_{\Gamma(G)\cap\mathbf{C}^1\left(x^0,t^0\right)}=1$. Then for any $K>0$ there exists $\delta(\alpha,\mu,n,K)$ such that if $R\le R^0(\alpha,\mu,n,K,b_0)$ and condition (5) is satisfied, then

$$\inf_{(x,t)\in G\cap\mathbf{C}^2\left(x^0,t^0\right)}u(x,t)\ge 1-\frac{1}{K}. \tag{10}$$

In fact, let $G^1=\{(x,t):u(x,t)<1\}$, $v(x,t)=1-u(x,t)-C_1\left(t-t^0+4bR^2\right)$, where positive constant C_1 will be chosen later. We have

$$Lv=c(x,t)-\mathcal{L}\omega(x,t)+C_1+C_1c(x,t)\left(t-t^0+4bR^2\right)\ge C_1\left(1-4bb_0R^2\right)-b_0.$$

Let's subordinate R^0 to the condition $R^0\le\frac{1}{2\sqrt{2bb_0}}$ and choose $C_1=2b_0$. Then function $v(x,t)$ will be $\mathcal{L}$-subparabolic in G'. At first suppose that $G'\cap\mathbf{C}^2\left(x^0,t^0\right)\ne\varnothing$. Two cases are possible: i) $\sup_{(x,t)\in G'\cap\mathbf{C}^2\left(x^0,t^0\right)}v(x,t)>0$, ii) $\sup_{(x,t)\in G'\cap\mathbf{C}^2\left(x^0,t^0\right)}v(x,t)\le 0$.

Let's case i) occurs. Then according to lemma 2 if δ corresponds to the constant $2K$, then

$$1-\inf_{(x,t)\in G'} u(x,t)\geq 2K\left(1-\inf_{(x,t)\in G'\cap \mathbf{C}^2(x^0,t^0)} u(x,t)-4C_1bR^2\right),$$

i.e.

$$\inf_{(x,t)\in G'\cap \mathbf{C}^2(x^0,t^0)} u(x,t)\geq \frac{2K-1}{2K}-8C_1bR^2 .$$

Let's subordinate R^0 to the additional condition $R^0\leq \dfrac{1}{4K\sqrt{2bb_0}}$.

Then

$$\inf_{(x,t)\in G'\cap \mathbf{C}^2(x^0,t^0)} u(x,t)\geq \frac{2K-1}{2K}-\frac{1}{2K}=1-\frac{1}{K} . \qquad (11)$$

If the alternative ii) occurs, then

$$\inf_{(x,t)\in G'\cap \mathbf{C}^2(x^0,t^0)} u(x,t)\geq 1-2C_1bR^2 . \qquad (12)$$

Let's subordinate $R^0\leq \dfrac{1}{2K\sqrt{2Kbb_0}}$ to the third condition. Then from (12) we again obtain estimation (11). Let's now fix $R^0=\min\left\{R_0,\dfrac{1}{2\sqrt{2bb_0}},\dfrac{1}{4K\sqrt{2bb_0}},\dfrac{1}{2\sqrt{2Kbb_0}}\right\}$.

Then (10) follows from (11) since $u(x,t)\geq 1$ for $(x,t)\in G\setminus G'$. If $G'\cap \mathbf{C}^2(x^0,t^0)=\varnothing$, then $u(x,t)\geq 1$ for $(x,t)\in G\cap \mathbf{C}^2(x^0,t^0)$. Thus inequality (10) is proved.

Let's return to proving of the theorem. Suppose that its statement doesn't occur. Let's fix arbitrary $\varepsilon_2\in(0,1)$. Then there exists $a\in(0,1)$ such that $\gamma(\beta)<1-a$ for $\beta\in(1-\varepsilon_2,1)$. Assume in (10) $K=\dfrac{4}{a}$ and choose corresponding δ and R^0. Let $\varepsilon_3=\min\{\varepsilon_2,\delta\}$. By definition of function $\gamma(\beta)$ there exists $R_1\leq R^0$ such that $\gamma^{R_1}(\beta)<1-\dfrac{a}{2}$ for $\beta\in(1-\varepsilon_3,1)$. Let's fix arbitrary $\beta_0\in(1-\varepsilon_3,1)$. Then there exists point $(x^0,t^0)\in \mathbf{C}^1$ (at $R=R_1$), function $u(x,t)\in \mathcal{A}_{\beta_0}^{R_1}(x^0,t^0)$ and point $x^1\in \mathcal{E}_{R_1;\frac{1}{2}}(x^0)$ such that

$$u(x^1,t^0)<1-\frac{a}{4} . \qquad (13)$$

Let $D'=\{(x,t):(x,t)\in \mathbf{C}^1(x^0,t^0), u(x,t)<1\}$. According to the definition of class $\mathcal{A}_{\beta_0}^{R_1}(x^0,t^0)$ we have

$$mes(D')<(1-\beta_0)mes(\mathbf{C}^1(x^0,t^0))\leq \delta\, mes(\mathbf{C}^1(x^0,t^0)).$$

Then according to (10)

$$\inf_{(x,t)\in D'\cap \mathbf{C}^2(x^0,t^0)} u(x,t)\geq 1-\frac{a}{4} .$$

Taking into account that $u(x,t)\geq 1$ for $(x,t)\in \mathbf{C}^2(x^0,t^0)\setminus D'$ we conclude

$$\inf_{(x,t)\in \mathbf{C}^2(x^0,t^0)} u(x,t)\geq 1-\frac{a}{4} ,$$

and, particularly,

$$u\left(x^1,t^0\right)\ge 1-\frac{a}{4}.$$

The last inequality contradicts (13). The theorem is proved.

Lemma 5. *Let* $R\le R_0$, $\sigma\in(0,1]$, $H_j\in\left[\frac{b}{4},b\right]$; $j=1,2$; $x^0\in\mathcal{E}_{R;4}(0)$, $-H_1^{-1}R^2\le\tau\le$ $\le -H_1R^2$, $2H_2R^{1+\frac{\alpha_i}{2}}\le 4H_2^{-1}R^{1+\frac{\alpha_i}{2}}$, $x_i^1+H_2R^{1+\frac{\alpha_i}{2}}\le x_i^0\le x_i^2-H_2R^{1+\frac{\alpha_i}{2}}$; $i=1,\dots,n$; $\mathcal{H}=\{(x,t)$: $x_i^1<x_i<x_i^2$; $i=1,\dots,n$; $\tau-2H_1R^2<t<\tau\}$.

If $u(x,t)\in\mathcal{A}^+(\mathcal{H})$, $u\left(x,\tau-2H_1R^2\right)\ge 1$ for $x\in\overline{\mathcal{E}}_{R;\sigma}\left(x^0\right)$, then there exists such $m(\alpha,\mu,n,b,H_1,H_2)$ that $u(x,\tau)\ge\sigma^m$ for $x_i^1+H_0R^{1+\frac{\alpha_i}{2}}\le x_i\le x_i^2-H_0R^{1+\frac{\alpha_i}{2}}$; $i=1,\dots,n$; $H_0=\min\{H_1,H_2\}$.

Proof. Without loss of generality we'll assume that $x^0=0$ and $2\sigma^2<H_0^2$. Let's fix point $\left(x^*,\tau\right)\in\overline{\mathcal{H}}$ such that

$$x_i^1+H_0R^{1+\frac{\alpha_i}{2}}\le x_i^*\le x_i^2-H_0R^{1+\frac{\alpha_i}{2}};\ i=1,\dots,n.$$

Let's denote $\xi=\dfrac{H_0^2}{4H_1}$, $y=\dfrac{x^*}{2H_1R^2}$. We consider set

$$S=\left\{(x,t):\sum_{i=1}^{n}\frac{\left[x_i-\left(t-\tau+2H_1R^2\right)y_i\right]^2}{R^{\alpha_i}}<\xi\left(t-\tau+2H_1R^2\right)+\sigma^2R^2;\ \ \tau-2H_1R^2<t<\tau\right\}.$$

It's easy to see that set S is entirely situated in oblique cylinder

$$S_1=\left\{(x,t):\sum_{i=1}^{n}\frac{\left[x_i-\left(t-\tau+2H_1R^2\right)y_i\right]^2}{R^{\alpha_i}}<2\xi H_1R^2+\sigma^2R^2;\ \ \tau-2H_1R^2<t<\tau\right\}.$$

On the other hand on the lower base of S_1, i.e. at $t=\tau-2H_1R^2$,

$$\sum_{i=1}^{n}\frac{x_i^2}{R^{\alpha_i}}<\frac{H_0^2}{4H_1}2H_1R^2+\sigma^2R^2=\frac{H_0^2}{2}R^2+\sigma^2R^2<H_0^2R^2.$$

Thus, for point $\left(x,\tau-2H_1R^2\right)$ of the lower base of S_1 the inequalities $|x_i|<H_0R^{1+\frac{\alpha_i}{2}}$; $i=1,\dots,n$ occur. Taking into account that $x_i^1\le -H_2R^{1+\frac{\alpha_i}{2}}$, $x_i^2\ge H_2R^{1+\frac{\alpha_i}{2}}$, we conclude $x_i^1<x_i<x_i^2$; $i=\overline{1,n}$. By that we have shown that the lower base S_1 is situated in $\overline{\mathcal{H}}$.

Further for the points of upper base of S_1, i.e. if $t=\tau$

$$\sum_{i=1}^{n}\frac{\left(x_i-2H_1R^2y_i\right)}{R^{\alpha_i}}=\sum_{i=1}^{n}\frac{\left(x_i-x_i^*\right)^2}{R^{\alpha_i}}<H_0^2R^2.$$

Thus, for the noted points $\left|x_i-x_i^*\right|<H_0R^{1+\frac{\alpha_i}{2}}$, i.e. $x_i<x_i^*+H_0R^{1+\frac{\alpha_i}{2}}\le x_i^2$ and $x_i>x_i^*-H_0R^{1+\frac{\alpha_i}{2}}\ge x_i^1$; $i=1,\dots,n$. Thus, upper base of S_1 is also situated in $\overline{\mathcal{H}}$. It follows from convexity of $\mathcal{H}$ that both the oblique cylinder and set S_1 are situated in $\overline{\mathcal{H}}$.

Note that parabolic boundary S is the sum of sets $\Gamma_1(S)$ and $\Gamma_2(S)$, where

$$\Gamma_1(S)=\left\{(x,t):\sum_{i=1}^{n}\frac{\left[x_i-\left(t-\tau+2H_1R^2\right)y_i\right]^2}{R^{\alpha_i}}=\xi\left(t-\tau+2H_1R^2\right)+\sigma^2R^2;\ \tau-2H_1R^2\le t\le\tau\right\},$$

$$\Gamma_2(S)=\left\{(x,t):\sum_{i=1}^{n}\frac{x^2}{R^{\alpha_i}}=\sigma^2R^2;\ t=\tau-2H_1R^2\right\}.$$

Let's introduce functions for $(x,t)\in\overline{S}$

$$z(x,t)=\frac{x-\left(t-\tau+2H_1R^2\right)y}{\sqrt{\xi\left(t-\tau+2?H_1R^2\right)+\sigma^2R^2}};\ r(x,t)=\sum_{i=1}^{n}\frac{\left[x_i-\left(t-\tau+2H_1R^2\right)y_i\right]^2}{R^{\alpha_i}}\Big/$$

$$\Big/\left[\xi\left(t-\tau+2H_1R^2\right)+\sigma^2R^2\right];\ \varphi(x,t)=\frac{\left[1-r(x,t)\right]^2}{\left[\xi\left(t-\tau+2H_1R^2\right)+\sigma^2R^2\right]^d},$$

where the positive constant d will be chosen later. It's easy to see that $0\le r(x,t)\le 1$ for $(x,t)\in\overline{S}$, at that $r\big|_{\Gamma_1(S)}=1$.

We have

$$L\varphi=\left[\xi\left(t-\tau+2H_1R^2\right)+\sigma^2R^2\right]^{-d-1}\left\{8\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{z_iz_j}{R^{\alpha_i+\alpha_j}}+(1-r)^2\xi d+\right.$$

$$+2(r-1)\left[2\sum_{i=1}^{n}\frac{a_{ii}(x,t)}{R^{\alpha_i}}+2\sum_{i=1}^{n}b_i(x,t)\frac{\left(x_i-\left(t-\tau+2H_1R^2\right)y_i\right)}{R^{\alpha_i}}+\right.$$

$$\left.\left.+\frac{r-1}{2}c(x,t)\left(\xi\left(t-\tau+2H_1R^2\right)+\sigma^2R^2\right)+r-2\sum_{i=1}^{n}\frac{y_i\left(x_i-\left(t-\tau+2H_1R^2\right)y_i\right)}{R^{\alpha_i}}\right]\right\}. \quad (14)$$

From condition (2) we obtain

$$\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{z_iz_j}{R^{\alpha_i+\alpha_j}}\ge\mu\sum_{i=1}^{n}\frac{\lambda_i(x,t)}{R^{\alpha_i}}\frac{z_i^2}{R^{\alpha_i}};\quad \sum_{i=1}^{n}\frac{a_{ii}(x,t)}{R^{\alpha_i}}\le\mu^{-1}\sum_{i=1}^{n}\frac{\lambda_i(x,t)}{R^{\alpha_i}}. \quad (15)$$

On the other hand for $(x,t)\in S$

$$|x|_\alpha\le C_2(\alpha,\mu,n,H_1,H_2)R,\ C_3(\alpha,\mu,n,H_1,H_2)R^2\le|t|\le C_4(\alpha,\mu,n,H_1,H_2)R^2.$$

Thus,

$$C_5(\alpha,\mu,n,H_1,H_2)R^{\alpha_i}\le\lambda_i(x,t)\le C_6(\alpha,\mu,n,H_1,H_2)R^{\alpha_i};\ i=1,\dots,n. \quad (16)$$

Using (16) in (15) we conclude

$$\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{z_iz_j}{R^{\alpha_i+\alpha_j}}\ge C_7(\alpha,\mu,n,H_1,H_2)r;\ \sum_{i=1}^{n}\frac{a_{ii}(x,t)}{R^{\alpha_i}}\le C_8(\alpha,\mu,n,H_1,H_2). \quad (17)$$

We have subject to condition (3)

$$\left|\sum_{i=1}^{n}b_i(x,t)\frac{\left(x_i-\left(t-\tau+2H_1R^2\right)y_i\right)}{R^{\alpha_i}}\right|\le b_0\left(\sum_{i=1}^{n}\frac{\left(x_i-\left(t-\tau+2H_1R^2\right)y_i\right)^2}{R^{\alpha_i}}\sum_{i=1}^{n}R^{-\alpha_i}\right)^{\frac{1}{2}}\le$$

$$\le b_0\left[\xi\left(t-\tau+2H_1R^2\right)+\sigma^2R^2\sum_{i=1}^{n}R^{-\alpha_i}\right]^{\frac{1}{2}}\le b_0C_9(\alpha,\mu,n,H_1,H_2)\sum_{i=1}^{n}R^{2-\alpha_i}\le$$

$$\le b_0C_{10}(\alpha,\mu,n,H_1,H_2), \quad (18)$$

and analogously

$$\left|\frac{r-1}{2}c(x,t)\left(\xi\left(t-\tau+2H_1R^2\right)+\sigma^2R^2\right)\right|\le b_0C_{11}(\alpha,\mu,n,H_1,H_2). \quad (19)$$

Finally,

$$\left|\sum_{i=1}^{n}\frac{y_i\left(x_i-\left(t-\tau+2H_1R^2\right)y_i\right)}{R^{\alpha_i}}\right|\le\left(\sum_{i=1}^{n}\frac{\left(x_i-\left(t-\tau+2H_1R^2\right)y_i\right)^2}{R^{\alpha_i}}\sum_{i=1}^{n}\frac{y_i^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}\le$$

$$\le\left[\left(\xi\left(t-\tau+2H_1R^2\right)+\sigma^2R^2\right)\frac{1}{4H_1^2R^4}\sum_{i=1}^{n}\frac{\left(x_i^*\right)^2}{R^{\alpha_i}}\right]^{\frac{1}{2}}\le C_{12}(\alpha,\mu,n,H_1,H_2). \quad (20)$$

Using (18)-(20) in (14) we obtain

$$\mathcal{L}\varphi\ge\left[\xi\left(t\text{-}\tau+2H_1R^2\right)+\sigma^2R^2\right]^{-d-1}\times$$
$$\times\left\{8C_7r+(1-r)^2\xi d-2(1-r)\left(2C_8+2b_0C_{10}+b_0C_{11}+2C_{12}\right)\right\}.$$

Whence there exist $r_0(\alpha,\mu,n,b_0,H_1,H_2)<1$ sufficiently closed to unit such that

$$8C_7r\ge2(1-r)\left(2C_8+2b_0C_{10}+b_0C_{11}+2C_{12}\right),$$

provided if $r_0\le r\le1$. If $0\le r<r_0$ then there exists sufficiently large $d(\alpha,\mu,n,b_0,H_1,H_2)$ such that

$$(1-r)^2\xi d\ge2(1-r)\left(2C_8+2b_0C_{10}+b_0C_{11}+2C_{12}\right).$$

Let's fix this d. Then function $\varphi(x,t)$ is $\mathcal{L}$-subparabolic in S. Let now $u(x,t)\in\mathcal{A}^+(\mathcal{H})$. Consider auxiliary function $\omega(x,t)=u(x,t)-\sigma^{2d}R^{2d}\varphi(x,t)$. It's clear that $\omega(x,t)\in\mathcal{A}(S)$. Besides $\omega\big|_{\Gamma_1(S)}\ge0$, since $\varphi\big|_{\Gamma_1(S)}=0$. On the other hand

$$\omega\big|_{\Gamma_2(S)}\ge1-\sigma^{2d}R^{2d}\varphi\big|_{\Gamma_2(S)}=1-(1-r)^2\ge0.$$

By the maximum principle $\omega(x,t)\ge0$ for $(x,t)\in\overline{S}$. In particular, at point $\left(x^*,\tau\right)$, where $r=0$ we obtain

$$u\left(x^*,\tau\right)\ge\frac{\sigma^{2d}R^{2d}}{\left(\xi2H_1R^2+\sigma^2R^2\right)^d}\ge\frac{\sigma^{2d}R^{2d}}{\left(H_0^2R^2\right)^d}\ge\sigma^{2d}.$$

Now it's sufficient to choose $m=2d$, and the lemma is proved.

Remark. It's clear from proof that the largest value of m is reached at $H_1=\frac{b}{4},\ H_2=b$.

Let's denote by $\Delta(D)$ set $\partial D\setminus\Gamma(D)$.

Theorem 2. *Let $u(x,t)$ is nonnegative solution of equation (1) in domain D, moreover, relative to the coefficients of the operator $\mathcal{L}$ conditions (2)-(3) be satisfied. Then if $\overline{\mathbf{C}}^{\mathbf{1}}\subset D\cup\Delta(D)$ and $R\le R_0$, then*

$$u\left(0,bR^2\right)\le C_{13}(\alpha,\mu,n,b_0)\inf_{x\in\mathcal{E}_{R;\frac{1}{4}}(0)}u\left(x,-\frac{bR^2}{2}\right). \quad (21)$$

Proof. Let number m from the previous lemma corresponds to $H_1=\frac{b}{4},H_2=b$. Let's fix this m and according to theorem 1 we'll find such $\beta\in(0,1)$ that

$$\frac{1}{2}\left(1+\frac{1}{1-\gamma(1-\beta)}\right)>2^m . \qquad (22)$$

Suppose for $r\in(0,1)$

$v(r)=u(0,-bR^2)(1-r)^{-m}$; $Q(r)=\{(x,t): x\in\bar{\mathcal{E}}_{R;r}(0), bR^2(1+r^2)\le t\le -bR^2\}$; $g(r)=\max\limits_{(x,t)\in Q(r)} u(x,t)$

Further let r_1 be the greatest root of equation $g(r)=v(r)$. It's easy to see that $g(0)=v(0)$, $\lim\limits_{r\to 1-0} v(r)=\infty$ and function $g(r)$ is continuous and bounded for $r\in[0,1]$. Therefore number r_1 exists and $r_1<1$. Let $(x^*,t^*)\in Q(r_1), g(r_1)=v(r_1)=u(x^*,t^*)$, $F=\left\{(x,t): x\in\mathcal{E}_{R;\frac{1-r_1}{2}}(x^*), t^*-\frac{b(1-r_0^2)}{4}R^2<t<t^*\right\}$. For $(x,t)\in\bar{F}$ we have

$$\left(\sum_{i=1}^{n}\frac{x_i^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}\le\left(\sum_{i=1}^{n}\frac{(x_i-x_i^*)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{n}\frac{(x_i^*)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}\le\frac{1-r_1}{2}R+r_1R=\frac{1+r_1}{2}R .$$

On the other hand

$$1+r_1^2+\frac{1-r_1^2}{4}<1+\frac{(1+r_1)^2}{4},$$

therefore $\bar{F}\subset Q\left(\frac{1+r_1}{2}\right)$ and for $(x,t)\in\bar{F}$ by virtue of (22)

$$u(x,t)\le u(0,-bR^2)\left(1-\frac{1+r_1}{2}\right)^{-m}=2^m v(r_1)<\frac{v(r_1)}{2}\left(1+\frac{1}{1-\gamma(1-\beta)}\right). \qquad (23)$$

If we'll suppose now tat

$$mes\left\{F\cap\left[(x,t): u(x,t)\le\frac{v(r_1)}{2}\right]\right\}\ge(1-\beta)mes(F),$$

then from equality $u(x^*,t^*)=v(r_1)$ and lemma 4 the following inequality follows

$$\sup_{(x,t)\in F} u(x,t)\ge\frac{v(r_1)}{2}\left(1+\frac{1}{1-\gamma(1-\beta)}\right).$$

The last inequality is impossible by virtue of (23). We used the fact that $u(x,t)$ is solution of equation (1), i.e. $-u(x,t)\in\mathcal{A}(F)$. Thus,

$$mes\left\{F\cap\left[(x,t): u(x,t)\le\frac{v(r_1)}{2}\right]\right\}<(1-\beta)mes(F),$$

i.e.

$$mes\left\{F\cap\left[(x,t): u(x,t)\le\frac{v(r_1)}{2}\right]\right\}\ge\beta\, mes(F). \qquad (24)$$

Now we use lemma 5. Two cases are possible: $r_1>\frac{1}{3}$ and $r_1\in\left(0,\frac{1}{3}\right]$. Let the first case take place.

Suppose

$$x=\frac{9r_1-1}{8r_1}x^*,\ \tau=-\frac{bR^2}{2},-\frac{bR^2}{2}-2H_1R^2=t^*, H_2=b\frac{9r_1-1}{8}.$$

It's easy to see that $\frac{b}{4} \le H_1 < b, \frac{b}{4} \le H_2 \le b$. Now if $\sigma = \frac{1-r_1}{8}$, then $\mathcal{E}_{R;\sigma}(x_0) \subset \mathcal{E}_{R;\frac{1-r_1}{4}}(x^*)$. In fact, let $x \in \mathcal{E}_{R;\sigma}(x^0)$, then

$$\sum_{i=1}^{n} \frac{(x_i - x_i^0)^2}{R^{\alpha_i}} < \frac{R^2(1-r_1)^2}{64},$$

therefore

$$\left(\sum_{i=1}^{n} \frac{(x_i - x_i^*)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{n} \frac{(x_i - x_i^0)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} \frac{(x_i^0 - x_i^*)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}} \le \frac{R(1-r_1)}{8} +$$

$$+ \frac{1-r_1}{8r_1}\left(\sum_{i=1}^{n} \frac{(x_i^*)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}} < \frac{R(1-r_1)}{8} + \frac{1-r_1}{8r_1} r_1 R = \frac{R(1-r_1)}{4}.$$

Suppose $x_i^1 = -2H_2 R^{1+\frac{\alpha_i}{2}}, x_i^2 = 2H_2 R^{1+\frac{\alpha_i}{2}}; i = 1,\dots,n$. Then from lemmas 3 and 5 subject to (24) it follows that for $x \in \mathcal{E}_{R;\frac{1}{4}}(0)$

$$u\left(x, -\frac{bR^2}{2}\right) \ge \left(\frac{1-r_1}{8}\right)^m \frac{v(r_1)}{2}\gamma(\beta) = \left(\frac{1-r_1}{8}\right)^m \frac{1}{2} u(0,-bR^2)(1-r_1)^{-m}\gamma(\beta) =$$

$$= 2^{-3m-1}\gamma(\beta)u(0,-bR^2). \qquad (25)$$

Now let $r_1 \in \left(0, \frac{1`}{3}\right]$ and τ, σ and H_1 have the same meaning as above. We suppose

$$x^0 = \frac{7r_1+1}{8r_1}x^*, \overline{H}_2 = b, x_i^1 = -\left(\frac{8r_1}{7r_1+1}+1\right)R^{1+\frac{\alpha_i}{2}}, x_i^2 = \left(\frac{8r_1}{7r_1+1}+1\right)R^{1+\frac{\alpha_i}{2}}; i = 1,\dots,n.$$

Then taking into account lemmas 3, 5, inequality (24) and the fact that $H_0 \ge \frac{b}{4}$ we obtain estimation (25). Hence required inequality (21) is proved with $C_{13} = \frac{2^{3m+1}}{\gamma(\beta)}$.

Corollary. *If conditions of theorem 2 are fulfilled then the following estimate occurs*

$$u(0,-bR^2) \le C_{14}(\alpha, \mu, n, b_0) \inf_{x \in \mathcal{E}_{R;\frac{1}{4}}(0)} u\left(x, -\frac{bR^2}{4}\right).$$

Lemma 6. *Let conditions of theorem 2 are satisfied. Then if* $\overline{x} \in \partial\mathcal{E}_{R;\frac{1}{4}}(0)$, *then*

$$u\left(\overline{x}, -\frac{bR^2}{2}\right) \le C_{15}(\alpha, \mu, n, b_0) \inf_{\theta \in (0,1)} u\left(\overline{x}, -(1-\theta)\frac{bR^2}{4}\right).$$

Proof. Let's fix arbitrary point $\overline{x} \in \partial\mathcal{E}_{R;\frac{1}{4}}(0)$. It's easy to see that if $\overline{x} \in \partial\mathcal{E}_{R;\frac{1}{4}}(0)$, $x \in \mathcal{E}_{R;\frac{1}{8}}(\overline{x})$, , then $x \notin \mathcal{E}_{R;\frac{1}{8}}(0)$. Consider cylinder $\mathbf{C}^5 = \mathcal{E}_{R;\frac{1}{8}}(\overline{x}) \times (-2bR^2, 0)$. Let's make transformation of variables $y_i = R^{-1-\frac{\alpha_i}{2}} x_i; i = 1,\dots,n; \tau = R^{-2}t$. Then cylinder

$\widetilde{\mathbf{C}}^5 = B_{\frac{1}{8}}(\bar{y}) \times (-2b,0)$ will be image of $\mathbf{C}^5$, where $\bar{y}$ is the image of point $\bar{x}$. It's clear that $\bar{y} \in \partial B_{\frac{1}{4}}(0)$.

Let $\widetilde{u}_R(y,\tau)$ be image of function $u(x,t)$. Then equation (1) in variables (y,τ) will take on the form

$$\widetilde{\mathcal{L}}_R \widetilde{u} = \sum_{i,j=1}^{n} a_{ij}^R(y,\tau) \frac{\partial^2 \widetilde{u}_R}{\partial y_i \partial y_j} + \sum_{i=1}^{n} b_i^R(y,\tau) \frac{\partial \widetilde{u}_R}{\partial y_i} + c^R(y,\tau)\widetilde{u}_R - \frac{\partial \widetilde{u}_R}{\partial \tau} = 0,$$

where

$$a_{ij}^R(y,\tau) = R^{-\frac{(\alpha_i+\alpha_j)}{2}} a_{ij}\left(R^{1+\frac{\alpha_1}{2}} y_1, \dots, R^{1+\frac{\alpha_n}{2}} y_n\right), b_i^R(y,\tau) = R^{1-\frac{\alpha_i}{2}} b_i\left(R^{1+\frac{\alpha_1}{2}} y_1, \dots, R^{1+\frac{\alpha_n}{2}} y_n\right),$$

$$b_i^R(y,\tau) = R^{1-\frac{\alpha_i}{2}} b_i\left(R^{1+\frac{\alpha_1}{2}} y_1, \dots, R^{1+\frac{\alpha_n}{2}} y_n\right), c^R(y,\tau) = R^2 c\left(R^{1+\frac{\alpha_1}{2}} y_1, \dots, R^{1+\frac{\alpha_n}{2}} y_n\right); i,j = 1,\dots,n.$$

For $(y,\tau) \in \widetilde{\mathbf{C}}^5$ (i.e. for $(x,t) \in \mathbf{C}^5$) and arbitrary n-dimensional vector ξ according to condition (2) we have

$$\mu \sum_{i=1}^{n} \frac{\lambda_i(x,t)}{R^{\alpha_i}} \xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}^R(y,\tau)\xi_i \xi_j \le \mu^{-1} \sum_{i=1}^{n} \frac{\lambda_i(x,t)}{R^{\alpha_i}} \xi_i^2. \tag{26}$$

But for $x \notin \mathcal{E}_{R;\frac{1}{8}}(0)$, $\sum_{i=1}^{n} \frac{x_i^2}{R^{\alpha_i}} \ge \frac{R^2}{64}$ is satisfied. Hence, the existence of such $i_0, 1 \le i_0 \le n$ that $\left|x_{i_0}\right| \ge \frac{R^{1+\frac{\alpha_{i_0}}{2}}}{8\sqrt{n}}$ follows. Thus, it's shown that $|x|_\alpha \ge C_{16}(\alpha,n)R$. On the other hand since $x \in \mathcal{E}_{R;\frac{3}{8}}(0)$ then $\sum_{i=1}^{n} \frac{x_i^2}{R^{\alpha_i}} \le \frac{9R^2}{64}$. Thus, $|x_i| < \frac{3R^{1+\frac{\alpha_i}{2}}}{8}$, for $i = 1,\dots,n$. Whence $|x|_\alpha \le C_{17}(\alpha,n)R$. If we'll take into account that $|t| \le 2bR^2$, then

$$C_{18}(\alpha,\mu,n)R^{\alpha_i} \le \lambda_i(x,t) \le C_{19}(\alpha,\mu,n)R^{\alpha_i}; \; i = 1,\dots,n. \tag{27}$$

Using (27) in (26) we obtain

$$\mu_1|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}^R(y,\tau)\xi_i\xi_j \le \mu_1^{-1}|\xi|^2,$$

where constant $\mu_1 \in (0,1]$ depends only on α,μ,n. Besides modules of coefficients $b_i^R(y,\tau)$; $i = 1,\dots,n$ and $c^R(y,\tau)$ are bounded module by constant dependent only on α and b_0 and also $c^R(y,\tau) \le 0$. Then by N.V.Krylov-M.V.Safonov [2] theorem for uniformly parabolic equations we conclude

$$\widetilde{u}^R\left(\bar{y}, -\frac{b}{2}\right) \le C_{20}(\alpha,\mu,n,b_0) \inf_{\theta \in (0,1)} \widetilde{u}^R\left(\bar{y}, -(1-\theta)\frac{b}{4}\right).$$

Now it's sufficient to return to the variables (x,t), and the lemma is proved.

Let $\mathbf{C}^6 = \mathcal{E}_{R;\frac{1}{4}}(0) \times \left(-\frac{bR^2}{4}, 0\right)$.

Corollary. *Let conditions of theorem 2 be satisfied. Then*

$$u(0,-bR^2)\le C_{21}(\alpha,\mu,n,b_0)\inf_{(x,t)\in\mathbf{C}^6} u(x,t).$$

In fact, let $S(\mathbf{C}^6)$ be the lateral surface of cylinder $\mathbf{C}^6$ and $(\bar{x},\bar{t})$ be point of $S(\mathbf{C}^6)$, where $u(\bar{x},\bar{t})=\inf_{(x,t)\in S(\mathbf{C}^6)} u(x,t)$.

According to theorem 2

$$u(0,-bR^2)\le C_{13}\inf_{x\in\mathcal{E}_{R;\frac{1}{4}}(0)} u\left(x,-\frac{bR^2}{2}\right)\le C_{13}u\left(\bar{x},-\frac{bR^2}{2}\right).$$

Applying lemma 6 we obtain

$$u(0,-bR^2)\le C_{13}C_{15}\inf_{(x,t)\in S(\mathbf{C}^6)} u(x,t), \tag{28}$$

On the other hand according to corollary to theorem 2

$$u(0,-bR^2)\le C_{14}\inf_{(x,t)\in P(\mathbf{C}^6)} u(x,t) \tag{29}$$

holds, where $P(\mathbf{C}^6)$ is lower base of cylinder $\mathbf{C}^6$.

It follows from (28)-(29) that

$$u(0,-bR^2)\le C_{22}\inf_{(x,t)\in\Gamma(\mathbf{C}^6)} u(x,t),$$

where $C_{22}=\max\{C_{13}C_{15},C_{14}\}$. Now it's sufficient to apply the maximum principle, and the corollary is proved.

Let $\mathbf{C}^7=\mathcal{E}_{R;\frac{1}{4}}(0)\times\left(-2bR^2,-\frac{7bR^2}{4}\right)$.

Theorem 3. *Let $u(x,t)$ be non-negative solution of equation (1) in D, moreover, relative to the coefficients of operator $\mathcal{L}$ conditions (2)-(3) be satisfied. At that time if $\mathbf{C}^1\subset D\cup\Delta(D)$ and $R\le R_0$, then*

$$\sup_{(x,t)\in\mathbf{C}^7} u(x,t)\le C_{23}(\alpha,\mu,n,b_0)\inf_{(x,t)\in\mathbf{C}^6} u(x,t). \tag{30}$$

Proof. Let's consider cylinders $\mathbf{C}^8=\mathcal{E}_{R;1}(0)\times(-3bR^2,-bR^2)$ and $\mathbf{C}^9=\mathcal{E}_{R;\frac{1}{4}}(0)\times\left(-\frac{5bR^2}{4},-bR^2\right)$. Let's make the same coordinate transformation as in proof of lemma 6. Then cylinders $\widetilde{\mathbf{C}}^7=B_{\frac{1}{4}}(0)\times\left(-2b,-\frac{7b}{4}\right)$, $\widetilde{\mathbf{C}}^8=B_1(0)\times(-3b,-b)$ and $\widetilde{\mathbf{C}}^9=B_{\frac{1}{4}}(0)\times\left(-\frac{5b}{4},-b\right)$ will be images of $\mathbf{C}^7,\mathbf{C}^8$ and $\mathbf{C}^9$ respectively. Operating by the same way as in proof of lemma 6 we can show that image $\widetilde{u}^R(y,\tau)$ of function $u(x,t)$ satisfies in $\widetilde{\mathbf{C}}^8$ uniformly parabolic equation of the form (1), moreover, its parabolicity constant depends only on α,μ and n, minor coefficients are bounded on modulus by constant dependent only on α,μ,n and b_0. Besides image of coefficient $c(x,t)$ is non-positive. According to Harnack inequality for the second order uniformly parabolic equations of nondivergent structure (see, e.g. [4]) we have

$$\sup_{(y,\tau)\in\widetilde{\mathbf{C}}^7}\widetilde{u}^R(y,\tau)\le C_{24}(\alpha,\mu,n,b_0)\inf_{(y,\tau)\in\widetilde{\mathbf{C}}^9}\widetilde{u}^R(y,\tau)\le C_{24}\widetilde{u}^R(0,-b).$$

Returning to variables (x,t) we obtain

$$\sup_{(y,\tau)\in C^7} u(x,t) \le C_{24} u\left(0,-bR^2\right).$$

Now in order to complete the proof of (30) it's sufficient to apply the corollary to lemma 6. The theorem is proved.

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