2002 VOL.XVII(XXV)

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ON DETERMINATION OF EXTREMALS IN DOMAIN DIFFERENT FROM RECTANGULAR

Abstract

The simple method of calculation of the best approximation is established and the extremal function on approximation of two-variable function by sums of one-variable functions in constructed in a domain different from rectangular.

After my articles of thirty year's prescription [1,2] it became known that there exists a class Π of multivariable functions for which the basic intractable functions for which the basic intractable problem of approximation theory, namely exact calculation of value of the best approximation and construction of extremal - best approximating function were solved. During these years in many articles (see. for example: references in [3]) these results were generalized to the various classes of multi-variable functions, but to fall outside the approximation domain different from rectangular (or parallelepiped) with sides parallel to coordinate axes wasn't succeeded.

It's failed to spoil the approximation domain with preservation of calculation formula of the best approximation especially the construction formula of the best approximating function.

In the present paper it's proved that the formula allowing to calculate a value of the best approximation by the sums of the type $\varphi(x)+\psi(y)$ by using some points of approximation domain continues to operate in sufficiently spoiled domain – in a rectangular with removed piece of interior of positive measure and the best approximating function is constructed in such a domain.

Let the function $f = f(x,y)$ be determined on the set $Q \subset I^2$, $I=[0,1]$. Consider the best approximation of the function f by the function of the type $\varphi(x)+\psi(y)$

$$E_f = \inf_{\varphi,\psi} \sup_{(x,y)\in Q} \left| f(x,y) - \varphi(x) - \psi(y) \right|, \tag{1}$$

where the functions φ and ψ are determined on the projections Q on the axis Ox and Oy relatively.

We determine the function

$$g = g(x,y) = f(x,y) - f(x,0) - f(0,y) + f(0,0)$$

and let the equation

$$g(1,y) = \frac{1}{2} g(1,1) \tag{2}$$

have the solution $y=\bar{y}$.

We'll call the set Q as $\mathcal{D}$-set, if the boundaries of the square I^2 and segment $(x,\bar{y})$ belong to Q. Denote by $\Pi(Q)$ a set of the functions $f = f(x,y)$ satisfying the inequality

$$f(x'',y'') - f(x'',y') - f(x',y'') + f(x',y') \geq 0$$

for arbitrary $x'' \geq x'$, $y'' \geq y'$; (x'',y''), (x'',y'), (x',y''), $(x',y') \in Q$.

Theorem. *Let $f \in \Pi(Q)$, where Q is $\mathcal{D}$-set. Then the function*

$$\varphi_0(x)+\psi_0(y)=f(x,\bar{y})+\frac{1}{2}[f(1,y)+f(0,y)]-$$
$$-f(0,\bar{y})-\frac{1}{4}[f(1,1)+f(1,0)-f(0,1)-f(0,0)]$$

is an extremal function in the approximation (1) and the best approximation is equal to

$$E_f=\frac{1}{4}[f(1,1)+f(0,0)-f(1,0)-f(0,1)].$$

Proof. Denote

$$L(f)=\frac{1}{4}[f(1,1)+f(0,0)-f(1,0)-f(0,1)].$$

The functional L is linear and is an annihilator of functions of the type $\varphi(x)+\psi(y)$, therefore

$$L(f)=L(f-\varphi(x)+\psi(y)).$$

Then

$$|L(f)|\le\|f-\varphi-\psi\|_{C(Q)}.$$

The left hand side of the last inequality doesn't depend on $\varphi(x)+\psi(y)$, that allows to write

$$|L(f)|\le\inf_{\varphi,\psi}\|f-\varphi-\psi\|_{C(Q)}=E[f,\varphi+\psi,Q]. \tag{3}$$

We'll show that there exists the function $\varphi_0(x)+\psi_0(y)$ for which

$$\|f-\varphi-\psi\|=L(f).$$

By that it's proved that $\varphi_0(x)+\psi_0(y)$ is the best approximating function in the considered approximation.

According to construction of the function $g(x,y)$

$$g(x,0)=g(0,y)=0 \quad \forall(x,0),(0,y)\in Q.$$

Besides

$$L(f)=L(g)=\frac{1}{4}g(1,1).$$

The functions f and g are distinguished to the function of the type $\varphi(x)+\psi(y)$, so $E_f=E_g$.

Statement 1. The function $g(1,y)$ increases. Indeed for arbitrary $y''\ge y'$

$$g(1,y'')-g(1,y')=f(1,y'')-f(0,y'')-f(1,y')+f(0,y')\ge 0,$$

by virtue of $f\in\Pi(Q)$.

We determine the functions

$$\varphi_2(x)=g(x,\bar{y});$$
$$\psi_2(y)=\frac{1}{2}\left[g(1,y)-\frac{1}{2}g(1,1)\right];$$
$$F(x,y)=g(x,y)-\varphi_2(x)-\psi(y).$$

According to determination the function $F(x,y)$ also differs from the function f in the form of $\varphi(x)+\psi(y)$ therefore

$$f\in\Pi(Q)\Rightarrow\begin{cases}g\in\Pi(Q),\\ F\in\Pi(Q).\end{cases}$$

Statement 2. The function F increases with respect to x when $y \ge \bar{y}$ and decreases with respect to x when $y \le \bar{y}$.

Proof. Let $x'' \ge x'$. We have

$$F(x'',y)-F(x',y)=g(x'',y)-\varphi_2(x'')-g(x',y)+\varphi_2(x')=$$
$$=g(x'',y)-g(x'',\bar{y})-g(x',y)+g(x',\bar{y}).$$

Then by virtue of $g \in \Pi(Q)$ we obtain

$$F(x'',y)-F(x',y)\begin{cases}\ge 0, & \text{if } y \ge \bar{y}\\ \le 0, & \text{if } y \le \bar{y}.\end{cases}$$

Statement 2 is proved.

We calculate $\max\limits_{(x,y)\in Q} F(x,y)$. It's clear that

$$\max_{x,y} F(x,y)=\max\left[\max_{\substack{x\\ y\ge\bar{y}}} F(x,y),\ \max_{\substack{x\\ y\le\bar{y}}} F(x,y)\right]. \tag{4}$$

Further

$$\max_{\substack{(x,y)\in Q\\ (y\ge\bar{y})}} F(x,y)=\max_{\substack{x\\ (\text{when } y\ge\bar{y})\\ (x,y)\in Q}}\ \max_{\substack{y\ge\bar{y}\\ (x,y)\in Q}} F(x,y)=\max_{y\ge\bar{y}} F(1,y)=$$
$$=\max_{y\ge\bar{y}}\left[g(1,y)-\varphi_2(1)-\psi_2(y)\right]=\max_{y\ge\bar{y}}\left[g(1,y)-g(1,\bar{y})-\psi_2(y)\right]=$$
$$\max_{y\ge\bar{y}}\left\{g(1,y)-\frac{1}{2}g(1,1)-\frac{1}{2}\left[g(1,y)-\frac{1}{2}g(1,1)\right]\right\}=\max_{y\ge\bar{y}}\left\{\frac{1}{2}g(1,y)-\frac{1}{4}g(1,1)\right\}.$$

Taking into account that by virtue of statement 1 the function $g(1,y)$ increases, we continue the calculation

$$=\frac{1}{2}g(1,1)-\frac{1}{4}g(1,1)=\frac{1}{4}g(1,1).$$

Further

$$\max_{\substack{(x,y)\in Q\\ y\le\bar{y}\\ (x,y)\in Q}} F(x,y)=\max_{\substack{x\\ (\text{when } y\le\bar{y})\\ (x,y)\in Q}}\ \max_{\substack{y\le\bar{y}\\ (x,y)\in Q}} F(x,y), \tag{5}$$

allowing for statement 2 we continue the calculation

$$=\max_{y\le\bar{y}} F(0,y)=\max_{0\le y\le\bar{y}}\left[g(0,y)-\varphi_2(0)-\psi_2(y)\right]=\max_{0\le y\le\bar{y}}\left\{0-g(0,\bar{y})-\frac{1}{2}\left[g(1,y)-\frac{1}{2}g(1,1)\right]\right\}=$$
$$=\max_{y\le\bar{y}}\left\{-\frac{1}{2}g(1,y)+\frac{1}{4}g(1,1)\right\}=-\frac{1}{2}g(1,0)+\frac{1}{4}g(1,1)=\frac{1}{4}g(1,1). \tag{6}$$

Substituting the expressions from (5) and (6) into the equality (4) we obtain

$$\max_{(x,y)\in Q} F(x,y)=\frac{1}{4}g(1,1). \tag{7}$$

We continue

$$\min_{(x,y)\in Q} F(x,y)=\min\left[\min_{\substack{x,y\ge\bar{y}\\ (x,y)\in Q}} F(x,y),\ \min_{\substack{x,y\le\bar{y}\\ (x,y)\in Q}} F(x,y)\right]; \tag{8}$$

$$\min_{\substack{x,y\ge\bar{y}\\ (x,y)\in Q}} F(x,y)=\min_{\substack{x\\ (\text{when } y\ge\bar{y})\\ (x,y)\in Q}}\ \min_{\substack{1\ge y\ge\bar{y}\\ (x,y)\in Q}} F(x,y)=$$

$$= F(0,y) = \min_{\substack{1 \ge y \ge \bar{y} \\ (x,y)\in Q}} \left\{ -\frac{1}{2} g(1,y) + \frac{1}{4} g(1,1) \right\}$$

and since $g(1,y)$ is an increasing function

$$= -\frac{1}{2} g(1,1) + \frac{1}{4} g(1,1) = -\frac{1}{4} g(1,1); \tag{9}$$

$$\min_{\substack{x,y \le \bar{y} \\ (x,y)\in Q}} F(x,y) = \min_{\substack{0 \le y \le \bar{y} \\ (x,y)\in Q}} \min_{\substack{x \\ (when\ y \le \bar{y}) \\ (x,y)\in Q}} F(x,y) =$$

$$= \min_{0 \le y \le \bar{y}} F(1,y) = \min_{0 \le y \le \bar{y}} \left\{ \frac{1}{2} g(1,y) - \frac{1}{4} g(1,1) \right\} =$$

$$= \frac{1}{2} g(1,0) - \frac{1}{4} g(1,1) = -\frac{1}{4} g(1,1). \tag{10}$$

By virtue of (9) and (10) from (8) we obtain

$$\min_{(x,y)\in Q} F(x,y) = -\frac{1}{4} g(1,1). \tag{11}$$

Since

$$\max_{x,y} |F(x,y)| = \max\left\{ \max_{x,y} F(x,y),\ -\min_{x,y} F(x,y) \right\}. \tag{12}$$

Then by virtue of (7) and (11) we'll have

$$\max_{(x,y)\in Q} |F(x,y)| = \frac{1}{4} g(1,1)$$

or

$$\|F\| = \|g - \varphi_2 - \psi_2\| = \frac{1}{4} g(1,1). \tag{13}$$

By using the determination of the function $g(x,y)$ and equality $L(f) = L(g)$ in (13) we obtain

$$\frac{1}{4} g(1,1) = L(g) = E[f, \varphi + \psi, Q]$$

hence

$$E[f, \varphi + \psi, Q] = \|g - \varphi_2 - \psi_2\|_{C(Q)} = \frac{1}{4} g(1,1)$$

that allows to confirm that the function $\varphi_2(x) + \psi_2(y)$ is the best approximation for the function $g(x,y)$ and $\frac{1}{4} g(1,1)$ is the best approximation of this approximation.

But

$$L(g) = L(f) = \frac{1}{4} [f(1,1) + f(0,0) - f(1,0) - f(0,1)].$$

Consequently, the last value is also the value of the best approximation and for the function $f(x,y)$.

Using the determination of the function φ_2, ψ_2 we have

$$g - \varphi_2 - \psi_2 = f(x,y) - f(x,0) + f(0,0) - \varphi_2(x) - \psi_2(y)$$

hence it follows that the function

$$f(x,0) + f(0,y) + \varphi_2(x) + \psi_2(y) \overset{df}{=} \varphi_0(x) + \psi_0(y)$$

is the best approximation for the function $f(x,y)$.

Using the expressions of the functions $\varphi_2(x)$ and $\psi_2(x)$ we express the function $\varphi_0(x)+\psi_0(y)$ by the approximated function

$$\varphi_0(x)+\psi_0(y)=f(x,y)+f(0,y)-f(0,0)+g(x,\bar{y})+\frac{1}{2}\left[g(1,y)-\frac{1}{2}g(1,1)\right]=$$
$$=f(x,0)+f(0,y)-f(0,0)+f(x,\bar{y})-f(x,0)-f(0,\bar{y})+f(0,0)+$$
$$+\frac{1}{2}\{f(1,y)-f(1,0)-f(0,y)-f(0,0)-[f(1,1)-f(1,0)-f(0,1)+f(0,0)]\}.$$

After reduction of similar members we obtain

$$\varphi_0(x)+\psi_0(y)=f(x,\bar{y})+\frac{1}{2}[f(1,y)+f(0,y)]-f(0,\bar{y})-\frac{1}{4}[f(1,1)+f(1,0)-f(0,1)+f(0,0)].$$

The theorem is proved.

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Received November 27, 2001; Revised May 17, 2002.
Translated by Mirzoyeva K.S.