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TWO-WEIGHTED INEQUALITIES OF WEAK TYPE FOR SOME ANISOTROPIC INTEGRAL OPERATOR ON THE DOMAINS IN $\mathbf{R}^n$

Abstract

In this paper two-weighted inequalities of weak type are proved for integral operator, generated on the basis of Ilyin-Besov integral representation .

Suppose that R^n is n-dimensional Euclidean space of the points $x = (x_1, ..., x_n)$, $x = (x', x'')$, $x' \in R^k$, $x'' \in R^{n-k}$, $R_0^0 = R^n \setminus \{0\}$, $a = (a_1, ..., a_n)$, $a_i > 0$, $i = 1, ..., n$, $\rho(x) = \sum\limits_{i=1}^{n} |x_i|^{1/a_i}$, $S^{n-1} = \{x : x \in R^n;\ \rho(x) = 1\}$.

Let

$$\Omega_k = \left\{x : x' \in R^k, \varphi_i(x') < x_i < \infty\ \ (i = k+1, ..., n)\right\}, \tag{1}$$

$$k = 1, ..., n-1,\ \ \Omega_0 = \left\{x : x \in R^n, x_i^{(0)} < x_i < \infty,\ \ i = 1, ..., n\right\},$$

$$\Gamma_k = \left\{x : x' \in R^k,\ \ x'' = \overline{\varphi}(x')\right\},\quad k = 1, ..., n-1,$$

where the vector function $\overline{\varphi}(x') = \left(\varphi_{k+1}(x'), ..., \varphi_n(x')\right)$, $k = 1, ..., n-1$ satisfies the anisotropic Hölder condition:

$$\rho\left(\overline{\varphi}(x') - \overline{\varphi}(y')\right) \le M\rho\left(x' - y'\right),\quad \forall x', y' \in R^k,$$

$\rho(x, \Gamma_k) = \inf\limits_{y \in \Gamma_k} \rho(x - y)$, $k = 1, ..., n-1$. In the case $k = 0$, $\rho(x, \Gamma_0) = \rho\left(x - x^{(0)}\right)$, $x^{(0)} = \left(x_1^{(0)}, ..., x_n^{(0)}\right)$ - is a fixed point in R^n. At $x^{(0)} = (0, ..., 0)$, $\Omega_0 = R^n_{++}$.

Let ω be a positive, measurable function given in R^n. Denote by $L_{p,\omega}(\Omega_k)$ the set of all measurable functions $f(x)$, $x \in \Omega_k$ with the finite norm

$$\|f\|_{L_{p,\omega}(\Omega_k)} = \left(\int\limits_{\Omega_k} |f(x)|^p \omega(x)\, dx\right)^{1/p},\qquad 1 \le p < \infty.$$

Let $b = (b_1, ..., b_n)$, $c = (c_1, ..., c_n)$, $0 < b_i < c_i < \infty$, $i = 1, ..., n$.

The set

$$R(1/a) = \left\{y : y_i > 0,\ b_i h < y_i^{1/a_i} < c_i h\quad (i = 1, ..., n),\ \ 0 < h < \infty\right\}$$

is called $1/a-$ horn.

Lemma 1 [1]. *The domain Ω_k, $k = 0, 1, ..., n-1$ satisfies the $1/a-$ horn condition, i.e. there exists the horn $R(1/a)$ such that the arithmetical sum*

$$\Omega_k + R(1/a) = \Omega_k.$$

Suppose that

$$\pi_k(x)=\rho\left(x''-\overline{\varphi}(x')\right)=\sum_{i=k+1}^{n}|x_i-\varphi_i(x')|^{1/a_i},\quad k=1,...,n-1;$$
$$\pi_0(x)=\rho\left(x-x^0\right).$$

Lemma 2 [1]. *Suppose that Ω_k is of the form (1). Then $\rho(x,\Gamma_k)$ is equivalent to $\pi_k(x)$ for all $x\in\Omega_k$, more exactly,*

$$\exists C_0>0,\quad \forall x\in\Omega_k,\quad C_0\pi_k(x)\le\rho(x,\Gamma_k)\le\pi_k(x).$$

Suppose, that K_α is real function given in R_0^n such that $suppK_\alpha\subset R(1/a)$ and
a) at $0<\alpha<|a|$, $K_\alpha(x)=\rho(x)^{\alpha-|a|}$, $x\in suppK_\alpha$;
b) at $\alpha=0$

$$K_0(t^a x)=t^{-|a|}K_0(x),\ \int_{S_k}K_0(x)\sum_{i=1}^{n}a_i x_i^2 d\sigma(x)=0,$$

$$S_k=S^{n-1}\cap\Omega_k,\ k=\overline{0,n-1}$$

and there exists a constant $C>0$ such that

$$|K_0(x-y)-K_0(x)|\le C\omega\left(\frac{\rho(y)}{\rho(x)}\right)\rho(x)^{-|a|}\quad at\quad \rho(x)>2\rho(y),$$

where C doesn't depend on x,y, the function $\omega:[0,1]\to R_+$ is increasing, $\omega(0)=0$, $\omega(2s)\le C_1\omega(s)$, $C_1\ge1$ for any $s>0$ and $\int_0^1\omega(t)\frac{dt}{t}<\infty$.

Consider the integral operator $K_\alpha:f\to K_\alpha f$, where

$$K_\alpha f(x)=\int_{R(1/a)}K_\alpha(y)f(x+y)\,dy.$$

They say that $\nu>0$ belongs to $A_p(\Omega_k)$, $k=0,1,..,n-1$ if

$$\frac{1}{|B|}\int_{B\cap\Omega_k}\nu(x)\,dx\le C\operatorname*{ess\,inf}_{x\in B\cap\Omega_k}\nu(x),$$

where C doesn't depend on all balls $B\subset R^n$.

It is true

Theorem 1: (Weak variant of Hardy weight inequality). *Let $q\ge1$, $u(t)$ and $v(t)$ be positive functions on $(0,\infty)$:*

1) for validity of the inequality

$$\left(\int_0^\infty u(t)\left|\int_0^t\varphi(\tau)\,d\tau\right|^q dt\right)^{1/q}<C_1\int_0^\infty v(t)\,|\varphi(t)|\,dt \tag{2}$$

with the constant C_1 *not depending on* φ, *it is sufficient the fulfillment of the conditions*

$$\sup_{t>0}\left(\int_0^\infty u(\tau)\,d\tau\right)^{1/q} ess\sup_{r\in(0,2t)}\frac{1}{v(t)}<\infty;$$

2) for validity of the inequality

$$\left(\int_0^\infty u(t)\left|\int_t^\infty \varphi(\tau)\,d\tau\right|^q dt\right)^{1/q}\le C_2\int_0^\infty v(t)\,|\varphi(t)|\,dt \qquad (3)$$

with the constant C_2 *not depending on* φ, *it is sufficient the fulfillment of the condition*

$$\sup_{t>0}\left(\int_0^t u(\tau)\,d\tau\right)^{1/q} ess\sup_{r\in(t/2,\infty)}\frac{1}{v(t)}<\infty.$$

Theorem 1 was established in papers V.M.Kokilashvili, A.Meskhi [2] at $q=1$ and A.Meskhi [3] at $q>1$.

Theorem 2. *Let* $0\le\alpha<|a|$, $\frac{1}{q}=1-\frac{\alpha}{|a|}$. *At* $\alpha=0$ *suppose that the kernel of anisotropic singular integral operator (ASIO) satisfies the condition b). If* $\omega\in A_1(R^n)$, $k=0,1,...,n-1$, *then there exists a positive constant* C *such that for any* $f\in L_{1,\omega}(R^n)$ *it holds the following inequality:*

$$\int\limits_{\left\{x:\left|K_\alpha\left(f\omega^{\frac{\alpha}{|a|}}\right)(x)>\lambda\right|\right\}}\omega(x)\,dx\le\frac{C}{\lambda^q}\left(\int_{R^n}|f(x)|\,\omega(x)\,dx\right)^q.$$

If furthermore $\alpha=0$, *then for ASIO it holds the weak type inequality (1.1).*

Theorem 2 at $0<\alpha<|a|$ in isotropic case was proved in [4], and at anisotropic in- [5]. At $\alpha=0$ in isotropic case theorem 2 was proved in [6], and at anisotropic-in [7].

Lemma 3 ([2], [3]). *Let* $0\le\alpha<|a|$, $1\le p<\frac{|a|}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{|a|}$, $\beta\ge1$, $\varphi\in A_{1+\frac{q}{p'}}(R^n)$ *be a radial function and let* u *and* u_1 *be the positive decreasing functions defined on* $(0,\infty)$. *Suppose that* $\omega=u\varphi$, $\omega_1=u_1\varphi$ *and weight pair* (ω,ω_1) *satisfies the following condition:*

$$\sup_{t>0}\left(\int_0^{t/2}\omega_1(\tau)\,\tau^{|a|-1}d\tau\right)^{\frac{p}{q}}\left(\int_t^\infty\left(\varphi^{-\frac{\alpha p}{|a|}}(\tau)\,\omega(\tau)\right)^{1-p'}\tau^{-1-\frac{|a|p'}{q}}d\tau\right)^{p-1}<\infty.$$

Then there exists a positive constant C *such that for any* $t>0$ *it holds the following inequality:*

$$u_1^{\frac{p}{q}}\left(\frac{t}{\beta}\right)\le Cu(t)\ .$$

Lemma 3 at $1 \le p < q < \infty$ has been proved in [3], and at $1 \le p = q < \infty$ in [2]. It holds

Theorem 3. *Let* $0 \le \alpha < |a|$, $\frac{1}{q} = 1 - \frac{\alpha}{|a|}$, $v \in A_1(\Omega_k)$ *be positive radial function depending on* $\rho(x,\Gamma_k)$, u *and* u_1 *are positive monotone functions defined on* $(0,\infty)$ $k = 0,1,...,n$.

Suppose that the weight pair of radial functions $(\omega_1(\rho(x,\Gamma_k)), \omega(\rho(x,\Gamma_k)))$ *satisfies the conditions 1) or 2);*

1) the weight functions ω *and* $\omega_1 = u_1 v$ *satisfy the following condition:*

$$\forall\gamma \ge 1 \ \ \exists C > 0, \ \ \forall t \in (0,\infty), u_1(\gamma t)^{1/q} \le C\frac{\omega(t)}{v(t)},$$

where u_1 *is increasing function on* $(0,\infty)$.

2) the weight functions $\omega = uv$ *and* $\omega_1 = u_1 v$ *satisfy the following condition:*

$$\sup_{t>0}\left(\int_0^t \omega_1(\tau)\,\tau^{|a''|-1}d\tau\right)^{1/q} \underset{\tau\in\left(\frac{t}{2},\infty\right)}{ess\sup}\frac{1}{\omega(\tau)\,v^{-\alpha/|a|}(\tau)\,\tau^{|a''|/q}} < \infty \ ,$$

where u, u_1 *are decreasing functions on* $(0,\infty)$, $v(\rho(x,\Gamma_k))\,\tilde{}\,v(\pi_k(x))$.

Then it holds the following inequality:

$$\int\limits_{\left\{x:\left|K_\alpha\left(fv^{\frac{\alpha}{|a|}}\right)(x)\right|>\lambda\right\}} \omega_1(\rho(x,\Gamma_k))\,dx \le \frac{C}{\lambda^q}\left(\int\limits_{\Omega_k}|f(x)|\,\omega(\rho(x,\Gamma_k))\,dx\right)^q. \qquad (4)$$

It furthermore $\alpha = 0$ then the inequality (4) is true when $q = 1$, i.e. for ASIO holds the weak (1,1) inequality.

Proof of theorem 3. Let $f \in L_{p,\omega(\rho(x,\Gamma_k))}(\Omega_k)$ and suppose that the weight pair (ω_1,ω) satisfies the condition 1).

It is sufficient to prove the theorem those increasing functions for which the following representation holds:

$$u_1(t) = u_1(0) + \int_0^t \psi(\tau)\,d\tau,$$

where $u_1(0) = \lim\limits_{t\to+0} u_1(t)$ and $\psi(t) \ge 0$, $t \in (0,\infty)$ (see [8]).

We have

$$\int\limits_{\left\{x:\left|K_\alpha\left(fv^{\frac{\alpha}{|a|}}\right)(x)\right|>\lambda\right\}} \omega_1(\rho(x,\Gamma_k))\,dx = \int\limits_{\left\{x:\left|K_\alpha\left(fv^{\frac{\alpha}{|a|}}\right)(x)\right|>\lambda\right\}} v(x)\,u_1(0)\,dx +$$

$$+ \int\limits_{\left\{x:\left|K_\alpha\left(fv^{\frac{\alpha}{|a|}}\right)(x)\right|>\lambda\right\}} v(x)\left(\int_0^{\rho(x,\Gamma_k)}\psi(t)\,dt\right)dx = D_1 + D_2.$$

If $u_1(0)=0$ then $D_1=0$ and if $u_1(0)\neq 0$ then by theorem 2 and the condition 1) we have

$$D_1=u_1(0)\int\limits_{\left\{x:\left|K_\alpha\left(fv^{\frac{\alpha}{|a|}}\right)(x)\right|>\lambda\right\}} v(x)\,dx\le \frac{C_1u_1(0)}{\lambda^q}\left(\int\limits_{\Omega_k}|f(x)|v(x)\,dx\right)^q\le$$

$$\le\frac{C_1}{\lambda^q}\left(\int\limits_{\Omega_k}|f(x)|\,v(x)\,u_1^{\frac{1}{q}}(\rho(x,\Gamma_k))\,dx\right)^q\le\frac{C_2}{\lambda^q}\left(\int\limits_{\Omega_k}|f(x)|^p\,\omega(\rho(x,\Gamma_k))\,dx\right)^q.$$

Now let's estimate D_2. It is easy to prove that at $x\in\Omega_k,\ y\in R(l)\ \pi_k(x+y)>\pi_k(x),\ k=0,1,...,n-1$. Allowing for that out of the horn $R(1/a)$ $K_\alpha(x)$ is equal to zero, then by virtue of the condition 1), theorem 2 and lemma 2 we have

$$D_2=\int\limits_{\left\{x:\left|K_\alpha\left(fv^{\frac{\alpha}{|a|}}\right)(x)\right|>\lambda\right\}} v(x)\left(\int\limits_0^{\rho(x,\Gamma_k)}\psi(t)\,dt\right)dx=$$

$$=\int\limits_0^\infty\psi(t)\left(\int\limits_{\rho(x,\Gamma_k)>t}\chi\left\{x:\left|K_\alpha\left(fv^{\frac{\alpha}{|a|}}\right)(x)\right|>\lambda\right\}v(x)\,dx\right)dt\le\int\limits_0^\infty\psi(t)\times$$

$$\times\left(\int\limits_{\pi_k(x)>t}\chi\left\{x:\left|\int\limits_{\pi_k(y)>\pi_k(x)}K_\alpha(y-x)\,f(y)\,v^{\frac{\alpha}{|a|}}(y)\,dy\right|>\lambda\right\}v(x)\,dx\right)dt=$$

$$=\int\limits_0^\infty\psi(t)\left(\int\limits_{\pi_k(x)>t}\chi\left\{x:\left|\int\limits_{\pi_k(y)>t}K_\alpha(y-x)\,f(y)\,v^{\frac{\alpha}{|a|}}(y)\,dy\right|>\lambda\right\}v(x)\,dx\right)dt\le$$

$$\le\left[\int\limits_0^\infty\psi(t)\left(\int\limits_{\Omega_k}\chi\left\{x:\left|\int\limits_{\pi_k(y)>\tau}K_\alpha(y-x)\,f(y)\,v^{\frac{\alpha}{|a|}}(y)\,dy\right|>\lambda\right\}\times\right.\right.$$

$$\left.\left.\times v(x)\,dx\right)dt\right]\le\frac{C_3}{\lambda^q}\int\limits_0^\infty\psi(t)\left(\int\limits_{\pi_k(x)>t}|f(x)|v(x)\,dx\right)^q dt\le$$

$$\le\frac{C_3}{\lambda^q}\left(\int\limits_{\Omega_k}|f(x)|\,v(x)\left(\int\limits_0^{\pi_k(x)}\psi(t)\,dt\right)^{1/q}dx\right)^q\le$$

$$\le\frac{C_3}{\lambda^q}\left(\int\limits_{\Omega_k}|f(x)|\,v(x)\,u_1^{1/q}\left(\frac{1}{C_0}(x,\Gamma_k)\right)dx\right)^q\le\frac{C_3}{\lambda^q}\left(\int\limits_{\Omega_k}|f(x)|\,\omega(\rho(x,\Gamma_k))\,dx\right)^q.$$

Combining the estimations for D_1 and D_2 we'll obtain (4).

Suppose that the weight pair (ω_1, ω) satisfies the condition 2). It is sufficient to prove theorem for those decreasing functions for which the following representation holds

$$u_1(t) = u_1(\infty) + \int_t^{\infty} \psi(\tau)\, d\tau,$$

where $u_1(\infty) = \lim\limits_{t\to+\infty} u_1(t)$ and $\psi(t) \geq 0,\ t \in (0,\infty)$.

We have

$$\int\limits_{\left\{x:\left|K_\alpha\left(fv^{\frac{\alpha}{|a|}}\right)(x)\right|>\lambda\right\}} \omega_1(x)\left(\rho(x,\Gamma_k)\right) dx =$$

$$= \int\limits_{\left\{x:\left|K_\alpha\left(fv^{\frac{\alpha}{|a|}}\right)(x)\right|>\lambda\right\}} v(x)\, u_1(\infty)\, dx +$$

$$+ \int\limits_{\left\{x:\left|K_\alpha\left(fv^{\frac{\alpha}{|a|}}\right)(x)\right|>\lambda\right\}} v(x)\left(\int\limits_{\rho(x,\Gamma_k)}^{\infty} \psi(t)\, dt\right) dx = B_1 + B_2\ .$$

If $u_1(\infty) = 0$ then $B_1 = 0$ and if $u_1(\infty) \neq 0$ then by theorem 2 and by lemma 3 (at $p = 1$) we have:

$$B_1 = u_1(\infty) \int\limits_{\left\{x:\left|K_\alpha\left(fv^{\frac{\alpha}{|a|}}\right)(x)\right|>\lambda\right\}} v(x)\, dx \leq$$

$$\leq C_1 \tfrac{u_1(\infty)}{\lambda^q}\left(\int\limits_{\Omega_k} |f(x)|\, v(x)\, dx\right)^q \leq \tfrac{C_1}{\lambda^q}\left(\int\limits_{\Omega_k} |f(x)|\, v(x)\, u_1^{\frac{1}{q}}\left(\rho(x,\Gamma_k)\right) dx\right)^q \leq$$

$$\leq \tfrac{C_2}{\lambda^q}\left(\int\limits_{\Omega_k} |f(x)|\, v(x)\, u\left(\rho(x,\Gamma_k)\right) dx\right)^q = \tfrac{C_2}{\lambda^q}\left(\int\limits_{\Omega} |f(x)|\, \omega\left(\rho(x,\Gamma_k)\right) dx\right)^q .$$

Let estimate B_2.

$$B_2 = \int\limits_0^{\infty} \psi(t)\left(\int\limits_{\rho(x,\Gamma_k)<t} \chi_{\left\{z:\left|K_\alpha\left(fv^{\frac{\alpha}{|a|}}\right)(z)\right|>\lambda\right\}}(x)\, v(x)\, dx\right) dt \leq \int\limits_0^{\infty} \psi(t) \times$$

$$\times \left(\int\limits_{\rho(x,\Gamma_k)<t} \chi\left\{x : \left|\int\limits_{\rho(y,\Gamma_k)>2c_0 t} K_\alpha(y-x)\, f(y)\, v^{\frac{\alpha}{|a|}}(y)\, dy\right| > \tfrac{\lambda}{2}\right\} v(x)\, dx\right) dt +$$

$$+ \int\limits_0^{\infty} \psi(t)\left(\int\limits_{\rho(x,\Gamma_k)<t} \chi\left\{x : \left|\int\limits_{\rho(y,\Gamma_k)\leq 2c_0 t} K_\alpha(y-x)\, f(y)\, v^{\frac{\alpha}{|a|}}(y)\, dy\right| > \tfrac{\lambda}{2}\right\} v(x)\, dx\right) dt =$$

$$= B_{21} + B_{22}.$$

Again using theorem 2, lemma 3 and generalized Minkowsky inequality with the exponent $q > 1$ we have

$$B_{22} \leq \frac{C_3}{\lambda^q} \int_0^\infty \psi(t) \left(\int_{\Omega_k} |f(x)| \chi_{\{z:\rho(z,\Gamma_k)\leq 2c_0 t\}}(x) v(x) \, dx \right)^q dt \leq$$

$$\leq \frac{C_3}{\lambda^q} \left(\int_{\Omega_k} |f(x)| v(x) \left(\int_0^{\frac{\rho(x,\Gamma_k)}{2c_0}} \psi(t) \, dt \right)^{\frac{1}{q}} dx \right)^q \leq$$

$$\leq \frac{C_3}{\lambda^q} \left(\int_{\Omega_k} |f(x)| v(x) u_1^{\frac{1}{q}} \left(\frac{\rho(x,\Gamma_k)}{2c_0} \right) dx \right)^q \leq$$

$$\leq C_4 \frac{1}{\lambda^q} \left(\int_{\Omega_k} |f(x)| v(x) u(\rho(x,\Gamma_k)) \, dx \right)^q = C_4 \frac{1}{\lambda^q} \left(\int_{\Omega_k} |f(x)| \omega(\rho(x,\Gamma_k)) \, dx \right)^q .$$

Estimate now B_{21}. If $\rho(x) < t, \ \rho(y) > 2c_0 t$ then $\rho(y-x) \geq \frac{1}{c_0}\rho(y) - \rho(x) > \frac{1}{c_0}\rho(y) - \frac{1}{2c_0}\rho(y) = \frac{1}{2c_0}\rho(y)$, i.e. $\rho(y-x) > \frac{1}{2c_0}\rho(y)$. Performing substitution $\xi'' = x'' - \bar{\varphi}(x'), \ \eta'' = y'' - \bar{\varphi}(y')$ and then redenoting $\xi' = x', \ \eta' = y'$ we'll obtain:

$$B_{21} \leq \frac{C_5}{\lambda^q} \int_0^\infty \psi(t) \left(\int_{\rho(x,\Gamma_k)<t} \left| \int_{\rho(y,\Gamma_k)>2t} \frac{f(y) v^{\frac{\alpha}{|a|}}(y)}{\rho(y-x)^{|a|-\alpha}} dy \right|^q v(x) \, dx \right) dt \leq$$

$$\leq \frac{C_6}{\lambda^q} \int_0^\infty \psi(t) \left(\int_{\pi_k(x)<t/C_0} v(\pi_k(x)) \, dx \right) \left(\int_{\pi_k(y)>2t} \frac{|f(y)| v^{\frac{\alpha}{|a|}}(y)}{\rho(y)^{|a|-\alpha}} dy \right)^q dt =$$

$$= \frac{C_6}{\lambda^q} \int_0^\infty \psi(t) \left(\int_{\rho(\xi'')<t/C_0} v(\rho(\xi'')) \, d\xi'' \right) \times$$

$$\times \left(\int_{R^k} dy' \int_{\rho(\eta'')>2t} \frac{|f(y',\eta''+\bar{\varphi}(y'))| v^{\frac{\alpha}{|a|}}(y',\eta''+\bar{\varphi}(y'))}{\rho(y,\eta''+\bar{\varphi}(y'))^{|a|-\alpha}} d\eta'' \right)^q dt =$$

$$= \frac{C_6}{\lambda^q} \int_0^\infty \psi(t) \left(\int_{R^k} d\xi' \int_{\rho(\xi'')<t/C_0} v(\rho(\xi'')) \, d\xi'' \right) \times$$

$$\times\left(\int\limits_{R^k} d\eta' \int\limits_{\rho(\eta'')>2t} \frac{\left|f\left(\eta',\eta''+\bar{\varphi}\left(\eta'\right)\right)\right| v^{\frac{\alpha}{|a|}}\left(\eta',\eta''+\bar{\varphi}\left(\eta'\right)\right)}{\rho\left(\eta',\eta''+\bar{\varphi}\left(\eta'\right)\right)^{|a|-\alpha}} d\eta''\right)^q dt =$$

$$=\frac{C_7}{\lambda^q}\int\limits_0^\infty \psi\left(\frac{s}{2}\right)\left(\int\limits_0^{\frac{s}{2}} v\left(\tau\right)\tau^{|a''|-1}d\tau\right)\left(\int\limits_{s/c}^\infty v^{\frac{\alpha}{|a|}}\left(\eta',\delta^{a''}\zeta''+\bar{\varphi}\left(\eta'\right)\right)\delta^{|a''|-1}\times\right.$$

$$\left.\times\left(\int\limits_{S_{++}^{n-k-1}}\left|f\left(\eta',\delta^{a''}\zeta''+\bar{\varphi}\left(\eta'\right)\right)\right| d\sigma\left(\bar{y}\right)d\delta\right)d\delta\right)^q ds.$$

Besides the following estimation holds:

$$\int\limits_0^t \psi\left(\frac{s}{2}\right)\left(\int\limits_0^{s/2} v\left(\tau\right)\tau^{|a|-1}d\tau\right)ds \le \int\limits_0^{t/2}\psi\left(s\right)\left(\int\limits_0^{s} v\left(\tau\right)\tau^{|a|-1}d\tau\right)ds =$$

$$=\int\limits_0^{t/2} v\left(\tau\right)\tau^{|a|-1}\left(\int\limits_0^{t/2}\psi\left(s\right)ds\right)d\tau \le \int\limits_0^{t/2} v\left(\tau\right)u_1\left(\tau\right)\tau^{|a|-1}d\tau = \int\limits_0^{t/2}\omega_1\left(\tau\right)\tau^{|a|-1}d\tau.$$

Therefore we have:

$$\sup_{t>0}\left(\int\limits_0^t \psi\left(\frac{s}{2}\right)\left(\int\limits_0^{s/2} v\left(\tau\right)\tau^{|a|-1}d\tau\right)ds\right)^{1/q} \operatorname*{ess\,sup}_{\tau\in\left(\frac{t}{2},\infty\right)}\frac{1}{\omega\left(\tau\right)v^{-\alpha/|a|}\left(\tau\right)\tau^{|a|/q}} \le$$

$$\le \sup_{t>0}\left(\int\limits_0^t \omega_1\left(\tau\right)\tau^{|a|-1}d\tau\right)^{1/q} \operatorname*{ess\,sup}_{\tau\in\left(\frac{t}{2},\infty\right)}\frac{1}{\omega\left(\tau\right)v^{-\alpha/|a|}\left(\tau\right)\tau^{|a|/q}} < \infty\ .$$

Allowing for this estimation and the second part of theorem 2 we'll get

$$B_{21}\le\frac{C_8}{\lambda_q}\left(\int\limits_0^\infty \omega\left(t\right)v^{-\frac{\alpha}{|a|}}\left(t\right)t^{\frac{|a''|}{q}}v^{\frac{\alpha}{|a|}}\left(t\right)t^{\alpha-1}\left(\int\limits_{S_{++}^{n-k-1}}\left|f\left(t^a\bar{y}\right)d\sigma\left(\bar{y}\right)\right|\right)dt\right)^q =$$

$$=\frac{C_8}{\lambda^q}\left(\int\limits_0^\infty t^{|a''|-1}\omega\left(t\right)\left(\int\limits_{S_{++}^{n-k-1}}\left|f\left(t^a\bar{y}\right)d\sigma\left(\bar{y}\right)\right|\right)dt\right)^q =$$

$$=\frac{C_8}{\lambda^q}\left(\int\limits_{\Omega_k}\left|f\left(x\right)\right|\omega\left(\rho\left(x,\Gamma_k\right)\right)dx\right)^q .$$

Theorem is proved.

At $\varphi = 1$ the following corollaries follow from theorem 3.

Corollary 1. *Let* $1 < q < \infty$, $\alpha = |a|\left(1 - \frac{1}{q}\right)$, $\beta > 0$. *Then for the operator* $f \to K_\alpha f$ *it holds the following inequality:*

$$\int\limits_{\{x:|K_\alpha f(x)|>\lambda\}} e^{q\rho(x,\Gamma_k)^\beta} dx \leq \frac{C}{\lambda^q}\left(\int\limits_{\Omega_k} |f(x)|\, \omega\left(\rho\left(x,\Gamma_k\right)\right)^\beta dx\right)^q$$

If $\alpha = 0$ then in this case for ASIO it holds the weak (1.1) inequality.

Corollary 2. *Let* $1 < q < \infty$, $\alpha = |a|\left(1 - \frac{1}{q}\right)$. *Suppose that* $\omega(x)$ *is increasing radial function and* $\omega_1(x)$ is *arbitrary radial function. Then for the operator* $f \to K_\alpha f$ *the following inequality holds:*

$$\int\limits_{\{x:|K_\alpha f(x)|>\lambda\}} \omega\left(\rho\left(x,\Gamma_k\right)\right)^{q\beta} dx \leq \frac{C}{\lambda^q}\left(\int\limits_{\Omega_k} |f(x)|\, \omega\left(\rho\left(x,\Gamma_k\right)\right)^\beta dx\right)^q .$$

If $\alpha = 0$ then in this case for ASIO it holds the weak (1.1) inequality.

The sufficient conditions for general radial weights providing the validity of two weight inequality of weak type it given in the following theorem

Theorem 4. *Let* $0 < \alpha < |a|$, $\dfrac{1}{q} = 1 - \dfrac{\alpha}{|a|}$, $\omega\left(\rho\left(x,\Gamma_k\right)\right)$ *and* $\omega_1\left(\rho\left(x,\Gamma_k\right)\right)$ *are redial functions on* $(0,\infty)$, $\omega\left(\rho\left(x,\Gamma_k\right)\right)$ *equivalent to* $\omega\left(\pi_k(x)\right)$, $\omega_1\left(\rho\left(x,\Gamma_k\right)\right)$ *equivalent to* $\omega_1\left(\pi_k(x)\right)$ *and the following conditions are fulfilled:*

1) $\exists C > 0,\ \ \forall t > 0,\ \ \left(\sup\limits_{\frac{t}{c_0}<\tau\leq 8c_0 t} \omega_1(\tau)\right)^{\frac{1}{q}} \leq C \sup\limits_{\frac{t}{c_0}<\tau\leq 8c_0 t} \omega(\tau)$,

2) $\sup\limits_{t>0}\left(\int\limits_t^\infty \dfrac{\omega_1(\tau)}{\tau} d\tau\right)^{\frac{1}{q}} ess \sup\limits_{\tau\in(0,2t)} \dfrac{1}{\omega(\tau)} < \infty,$

3) $\sup\limits_{t>0}\left(\int\limits_0^t \omega_1(\tau)\, \tau^{|a''|-1} d\tau\right)^{\frac{1}{q}} ess \sup\limits_{\tau\in\left(\frac{t}{2},\infty\right)} \dfrac{1}{\omega(\tau)\, \tau^{|a''|/q}} < \infty.$

Then it holds the following inequality

$$\int\limits_{\{x:|K_\alpha f(x)|>\lambda\}} \omega_1\left(\rho\left(x,\Gamma_k\right)\right) dx \leq C\left(\frac{1}{\lambda}\int\limits_{\Omega_k} |f(x)|\, \omega\left(\rho\left(x,\Gamma_k\right)\right) dx\right)^q . \qquad (5)$$

If furthermore $\alpha = 0$ then the inequality (5) is true at $q = 1$ i.e. for ASIO it holds the weak (1,1) inequality.

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Received October 21, 2002; Revised March 10, 2003.
Translated by Mamedova V.I.