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ON ASYMPTOTICS OF SOLUTION OF BOUNDARY VALUE PROBLEM FOR ONE CLASS OF EQUATIONS OF ARBITRARY ODD ORDER

Abstract

Boundary value problem is considered for one class of equations of arbitrary odd order containing small parameter at higher derivatives in rectangular domain. Using iteration process we construct asymptotics with respect to small parameter to any accuracy and estimate residue member.

By studying numerous real phenomena where there are irregular transition from one physical characteristics to other ones, one has to investigate singular perturbed problems. Such problems attracted attention of many mathematicians. Works of Vishik M.I. and Lusternik L.A. (see [1], [2]) are significant development of theory of singular perturbed problems. Most of investigated problems refer to classical equations.

There are no so many works devoted to asymptotics of solutions of boundary value problems for non-classical differential equations. Here we should note the papers [3]-[6].

In the present paper in rectangular domain $D = \{(t,x) \mid \ 0 < t < T, \ \ 0 < x < 1\}$ the following boundary value problem is considered for one class of equations of arbitrary odd order

$$L_\varepsilon u \equiv (-1)^m \varepsilon^{2m} \frac{\partial^{2m+1} u}{\partial t^{2m+1}} + \varepsilon^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + au = f(t,x), \tag{1}$$

$$u|_{t=0} = \left.\frac{\partial u}{\partial t}\right|_{t=0} = ... = \left.\frac{\partial^m u}{\partial t^m}\right|_{t=0} = 0 \ , \tag{2}$$

$$\left.\frac{\partial^{m+1} u}{\partial t^{m+1}}\right|_{t=T} = \left.\frac{\partial^{m+2} u}{\partial t^{m+2}}\right|_{t=T} = ... = \left.\frac{\partial^{2m} u}{\partial t^{2m}}\right|_{t=T} = 0 \ , \tag{3}$$

$$u|_{x=0} = u|_{x=1} = 0 \ , \quad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=1} = 0 \ , \tag{4}$$

where $\varepsilon > 0$ is a small parameter, $a > 0$ is a constant, $f(t,x)$ is a given sufficiently smooth function.

It is known that for every fixed ε there exists a unique solution of problem (1)-(4) (see [7]). The aim of our work is a construction of asymptotics of solution of the posed problem with respect to small parameter.

To construct asymptotics we realize iteration processes. In the first iteration process we find approximate solution of equation (1) in the form

$$W = W_0 + \varepsilon^2 W_1 + \varepsilon^4 W_2 + ... + \varepsilon^{2n} W_n. \tag{5}$$

Putting (5) into (1) and equating the terms with the similar powers of ε, we obtain

$$\frac{\partial W_0}{\partial t} - \frac{\partial^2 W_0}{\partial x^2} + aW_0 = f(t,x), \tag{6}$$

$$\frac{\partial W_j}{\partial t} - \frac{\partial^2 W_j}{\partial x^2} + aW_j = -\frac{\partial^4 W_{j-1}}{\partial x^4}; \quad j = 1, 2, ..., m-1, \tag{7}$$

$$\begin{gathered} \frac{\partial W_k}{\partial t} - \frac{\partial^2 W_k}{\partial x^2} + aW_k = \\ = -\frac{\partial^4 W_{k-1}}{\partial x^4} + (-1)^{m+1} \frac{\partial^{2m+1} W_{k-m}}{\partial t^{2m+1}}; \quad k = m, m+1, ..., n. \end{gathered} \tag{8}$$

Equations (6), (7), (8) differ from each other only by right-hand parts. We solve these equations under the following boundary conditions, respectively

$$W_j|_{t=0} = 0, \tag{9}$$

$$W_j|_{x=0} = W_j|_{x=1} = 0; \quad j = 0, 1, ..., n. \tag{10}$$

Problem (6), (9), (10) at $j = 0$ will be called degenerate problem corresponding to problem (1)-(4).

It is obvious that if $f(t,x)$ is a sufficiently smooth function and all its even order derivatives with respect to x vanish at $x = 0$ and $x = 1$, then degenerate problem has sufficiently smooth solution, moreover, all even order derivatives of $W_0(t,x)$ with respect to x will vanish at $x = 0$ and $x = 1$.

Having known the function W_0, we define the function W_1 from (7), (9), (10) at $j = 1$. Continuing this process, we define all functions W_i contained in expansion (5). It should be noted that functions W_i satisfy also the conditions

$$\left.\frac{\partial^2 W_i}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 W_i}{\partial x^2}\right|_{x=0} = 0; \quad i = 0, 1, ..., n. \tag{11}$$

Therefore further on we will not need to construct functions of boundary layer type close to boundaries $x = 0$ and $x = 1$.

Thus, approximate solution W of equation (1) which satisfies the following boundary conditions is constructed

$$W|_{t=0} = 0, \tag{12}$$

$$W|_{x=0} = W|_{x=1} = 0\ , \qquad \left.\frac{\partial^2 W}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 W}{\partial x^2}\right|_{x=1} = 0\ . \tag{13}$$

The function W satisfies only the first of conditions (2) and does not satisfy, generally speaking, the rest of the boundary conditions for $t=0$. Therefore performing the second iteration process, we add the function to the function W

$$V = \varepsilon\left(V_0 + \varepsilon V_1 + ... + \varepsilon^{2n+m-1} V_{2n+m-1}\right) \tag{14}$$

of boundary layer type close to the boundary $t=0$ so that the obtained sum $W+V$ satisfy the rest m conditions $t=0$:

$$\left.\frac{\partial}{\partial t}(W+V)\right|_{t=0} = \left.\frac{\partial^2}{\partial t^2}(W+V)\right|_{t=0} = ... = \left.\frac{\partial^m}{\partial t^m}(W+V)\right|_{t=0} = 0. \tag{15}$$

To construct functions $V_j;\quad j=0,1,...,2n+m-1$, we write the second decomposition of operator L_ε close to boundary $t=0$, to this end we make change of variables $t=\varepsilon\tau,\ x=x$. New expansion of the operator L_ε close to boundary $t=0$ has the form

$$L_{\varepsilon,1} \equiv \varepsilon^{-1}\left\{(-1)^m \frac{\partial^{2m+1}}{\partial \tau^{2m+1}} + \frac{\partial}{\partial \tau} + \varepsilon\left(-\frac{\partial^2}{\partial x^2} + a\right) + \varepsilon^3 \frac{\partial^4}{\partial x^4}\right\}\ .$$

Putting expression (14) for V into equation

$$L_{\varepsilon,1} V = 0$$

and comparing the terms with similar powers of ε, we obtain

$$(-1)^m \frac{\partial^{2m+1} V_0}{\partial \tau^{2m+1}} + \frac{\partial V_0}{\partial \tau} = 0\ , \tag{16}$$

$$(-1)^m \frac{\partial^{2m+1} V_j}{\partial \tau^{2m+1}} + \frac{\partial V_j}{\partial \tau} = \frac{\partial^2 V_{j-1}}{\partial x^2} - aV_{j-1};\quad j=1,2, \tag{17}$$

$$\begin{gathered}(-1)^m \frac{\partial^{2m+1} V_k}{\partial \tau^{2m+1}} + \frac{\partial V_k}{\partial \tau} = \\ = \tfrac{\partial^2 V_{k-1}}{\partial x^2} - aV_{k-1} - \tfrac{\partial^4 V_{k-3}}{\partial x^4};\quad k=3,4,...,2n+m-1.\end{gathered} \tag{18}$$

Putting expressions of W and V from (5) and (14) respectively, into (15) and comparing the coefficients at same powers of ε, we find boundary conditions, under which equations (16), (17), (18) are solved.

For example, for equation (16), boundary conditions will have the following form

$$\left.\frac{\partial V_0}{\partial \tau}\right|_{\tau=0} = -\left.\frac{\partial W_0}{\partial t}\right|_{t=0},\quad \left.\frac{\partial^2 V_0}{\partial \tau^2}\right|_{\tau=0} = 0, ...,\quad \left.\frac{\partial^m V_0}{\partial \tau^m}\right|_{\tau=0} = 0. \tag{19}$$

Consequently, V_0 is a solution of boundary layer type problem (16), (19).

Characteristic equation corresponding to ordinary differential equation (16), has m different roots with a negative real part. It should be noted that this fact provides the regularity of degeneration of problem (1)-(4) for $t = 0$.

Solution of boundary layer type problem (16), (19) has the form

$$V_0 = -\frac{\partial W_0 (0, x)}{\partial t} \left(a_{01} e^{\lambda_1 \tau} + a_{02} e^{\lambda_2 \tau} + ... + a_{0m} e^{\lambda_m \tau} \right) ,$$

where $a_{01}, a_{02}, ..., a_{0m}$ are the known numbers, $\lambda_1, \lambda_2, ..., \lambda_{\bar{m}}$ are those roots of equation $(-1)^m \lambda^{2m} + 1 = 0$ for which $\operatorname{Re} \lambda_i < 0$; $i = 1, 2, ..., m$.

The following lemma is valid

Lemma 1. *Functions V_s; $s = 1, 2, ..., 2n + m - 1$, defined as solutions of boundary layer type equations (17), (18) and satisfying boundary conditions (15), have the following representation*

$$V_s = \sum_{i=1}^{m} \left[a_{s0}^{(i)} (x) + a_{s1}^{(i)} (x) \tau + ... + a_{ss}^{(i)} (x) \tau^s \right] e^{\lambda_i \tau},$$

where $a_{sj}^{(i)} (x)$; $i = 1, 2, ..., m$; $j = 1, 2, ..., s$ are expressed by means of values of functions $W_0 (t, x)$, $W_1 (t, x), ..., W_{s-m+1} (t, x)$ for $t = 0$ and their derivatives for $t = 0$ (functions W_i with negative indices are assumed to be equal to zero identically).

For the functions V_s to satisfy also the condition

$$V_s|_{t=0} = 0; \quad s = 0, 1, .., 2n + m - 1, \tag{20}$$

we correct them in the following way

$$V_0 = -\frac{\partial W_0 (0, x)}{\partial t} \left[a_{01} \left(e^{\lambda_1 \tau} - 1 \right) + a_{02} \left(e^{\lambda_2 \tau} - 1 \right) + ... + a_{0m} \left(e^{\lambda_m \tau} - 1 \right) \right], \tag{21}$$

$$V_S = \sum_{i=1}^{m} \left[a_{s0}^{(i)} (x) + a_{s1}^{(i)} (x) \tau + ... + a_{ss}^{(i)} (x) \tau^s \right] \left(e^{\lambda_i \tau} - 1 \right). \tag{22}$$

It is obvious that the retouched functions will also satisfy those equations and boundary conditions to which satisfied the previous functions V_s.

Let us multiply functions V_s by smoothing function and denote the obtained new functions again by V_s; $s = 0, 1, ..., 2n + m - 1$.

Following (11), (12), (13), (14), (15), (20), (21), (22) we obtain that the constructed sum $W + V$ satisfies the following conditions

$$(W + V)|_{t=0} = \frac{\partial}{\partial t} (W + V)|_{t=0} = ... = \frac{\partial^m}{\partial t^m} (W + V)|_{t=0} = 0, \tag{23}$$

$$(W+V)\left|_{x=0}=(W+V)\right|_{x=1}=0,$$
$$\frac{\partial^2}{\partial x^2}(W+V)\left|_{x=0}=\frac{\partial^2}{\partial x^2}(W+V)\right|_{x=1}=0. \tag{24}$$

The sum $W+V$, generally speaking, does not satisfy boundary conditions (3) at $t=T$. Therefore we add the function of boundary layer type close to boundary $t=T$

$$\eta=\varepsilon^{m+1}\left(\eta_0+\varepsilon\eta_1+...+\varepsilon^{2n+m-1}\eta_{2n+m-1}\right) \tag{25}$$

to the sum $W+V$ such that the obtained sum $W+V+\eta$ satisfy the following boundary conditions

$$\frac{\partial^{m+1}}{\partial t^{m+1}}(W+V+\eta)\left|_{t=T}=\frac{\partial^{m+2}}{\partial t^{m+2}}(W+V+\eta)\right|_{t=T}=...$$
$$...=\frac{\partial^{2m}}{\partial t^{2m}}(W+V+\eta)|_{t=T}=0. \tag{26}$$

We define function η as approximate solution of the equation

$$L_{\varepsilon,2}\eta\equiv\varepsilon^{-1}\left\{-\left[(-1)^m\frac{\partial^{2m+1}\eta}{\partial y^{2m+1}}+\frac{\partial\eta}{\partial y}\right]+\varepsilon\left(-\frac{\partial^2\eta}{\partial x^2}+a\eta\right)+\varepsilon^3\frac{\partial^4\eta}{\partial x^4}\right\}=0,$$

where $y=\dfrac{T-t}{\varepsilon}$.

We do not stop on construction of functions η_s; $s=0,1,...,2n+m-1$. We just note that the following lemma similar to lemma 1 is valid.

Lemma 2. *The functions η_s; $s=0,1,...,2n+m-1$ defined as solutions of boundary layer type of corresponding problems have the following representation*

$$\eta_0=\frac{\partial^{m+1}W_0(T,x)}{\partial t^{m+1}}\left[\frac{b_{01}}{\lambda_1^m}e^{\lambda_1 y}+\frac{b_{02}}{\lambda_2^m}e^{\lambda_2 y}+...+\frac{b_{0m}}{\lambda_m^m}e^{\lambda_m y}\right], \tag{27}$$

$$\eta_s=\sum_{i=1}^m\left[b_{s0}^{(i)}(x)+b_{s1}^{(i)}(x)\,y+...+b_{ss}^{(i)}(x)\,y^s\right]e^{\lambda_i y}, \tag{28}$$

where $b_{sj}^{(i)}(x)$; $i=1,2,...,m$; $j=1,2,...,s$ are expressed by means of values of functions $W_0(t,x)$, $W_1(t,x),...,W_{s-m+1}(t,x)$ and values of their derivatives at $t=T$.

Let us multiply functions η_s by smoothing functions and for obtained new functions we leave the previous denotation.

Since functions η_s equal to zero identically for $t=0$, then (23) implies that the sum $W+V+\eta$ will satisfy also the following conditions

$$(W+V+\eta)\left|_{t=0}=\frac{\partial}{\partial t}(W+V+\eta)\right|_{t=0}=...=\frac{\partial^m}{\partial t^m}(W+V+\eta)|_{t=0}=0. \tag{29}$$

From (10), (11), (27), (28) we obtain that

$$\eta_s|_{x=0} = \eta_s|_{x=1} = 0, \quad \left.\frac{\partial^2 \eta_s}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 \eta_s}{\partial x^2}\right|_{x=1} = 0; \quad s = 0, 1, ..., 2n + m - 1.$$

This fact and (25) imply that the sum $W + V + \eta$ satisfies the conditions

$$\begin{gathered} (W + V + \eta)|_{x=0} = (W + V + \eta)|_{x=1} = 0, \\ \frac{\partial^2}{\partial x^2}(W + V + \eta)|_{x=0} = \frac{\partial^2}{\partial x^2}(W + V + \eta)|_{x=1} = 0\ . \end{gathered} \tag{30}$$

If we denote the constructed approximate solution of problem (1)-(4) by $\tilde{u}$, i.e.

$$W + V + \eta = \tilde{u},$$

and denote the difference of exact and approximate solutions by z, i.e.

$$u - \tilde{u} = z, \tag{31}$$

then we obtain the following asymptotic representation for solution of the problem (1)-(4)

$$u = \sum_{i=0}^{n} \varepsilon^i W_i + \sum_{j=0}^{2n+m-1} \varepsilon^{j+1} V_j + \sum_{j=0}^{2n+m-1} \varepsilon^{j+m+1} \eta_j + z, \tag{32}$$

where z is a residual member.

Acting on the both sides of equality (31) by the corresponding decomposition of operator L_ε and taking into account equations obtained from iteration processes, we have:

$$L_\varepsilon z = \varepsilon^{2n+2} H \tag{33}$$

where $H(t, x, \varepsilon)$ is a bounded function on D.

Following (31), (2), (3), (4), (29), (26), (30) we have that function z satisfies the homogeneous boundary conditions

$$z|_{t=0} = \left.\frac{\partial z}{\partial t}\right|_{t=0} = ... = \left.\frac{\partial^m z}{\partial t^m}\right|_{t=0} = 0, \tag{34}$$

$$\left.\frac{\partial^{m+1} z}{\partial t^{m+1}}\right|_{t=T} = \left.\frac{\partial^{m+2} z}{\partial t^{m+2}}\right|_{t=T} = ... = \left.\frac{\partial^{2m} z}{\partial t^{2m}}\right|_{t=T} = 0, \tag{35}$$

$$z|_{x=0} = z|_{x=1} = 0, \quad \left.\frac{\partial^2 z}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 z}{\partial x^2}\right|_{x=1} = 0. \tag{36}$$

The following lemma is valid.

Lemma 3. *For solution of problem (33)-(36) the following estimate holds*

$$\varepsilon^{2m}\left\|\left.\frac{\partial^m z}{\partial t^m}\right|_{t=T}\right\|^2_{L_2(0;1)}+\varepsilon^2\left\|\frac{\partial^2 z}{\partial x^2}\right\|^2_{L_2(D)}+$$
$$+\left\|\frac{\partial z}{\partial x}\right\|^2_{L_2(D)}+c_1\left\|z\right\|^2_{L_2(D)}\leq c_2\varepsilon^{2n+2}, \tag{37}$$

where $c_7>0,\ c_2>0$ are constants independent of ε.

To prove lemma 3 we multiply the both sides of equaeion (33) by z and if we integrate by parts over the domain D the obtained expressions, then subject to conditions (34)-(36) we obtain estimate (37).

The following assertion follows from the above mentioned.

Theorem. *Let $f(t,x)$ be a given function on D which has continuous derivatives with respect to t to the $2m(n+1)$-st order inclusively and with respect to x to the $5m(2n+1)$-st order inclusively, and all its even order derivatives with respect to x vanish at $x=0$ and $x=1$. Then for solution of problem (1)-(4) asymptotic reprelentation (32) holds, where functions W_i are defined by the first iteration process, V_j dre fucntions of boundary layer type close to boundary $t=0$,* and *η_j - close to $t=T$, which are defined by the corresponding iteration processes, z is a residual member, moreover, for it, estimate (33) holds.*

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