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ON THE BEST APPROXIMATION OF COMPLEX VARIABLE FUNCTION IN INFINITE DOMAINS

Abstract

In the present paper the best approximation of the complex variable function $f(z)$ is studied depending on the constructive character of function.

Let $\{\lambda_k\}$ be some increasing sequence of positive numbers, and $f(z)$ be analytic in the half-plane $\operatorname{Re} z > a$.

Assume that $f(z)$ in this half-plane is representable by the Dirichlet series in the form

$$f(z) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k z}, \tag{1}$$

where

$$a_k e^{-\lambda_k x} = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} f(x+iy)\, e^{i\lambda_k y} dy, \quad x > a. \tag{2}$$

We'll denote by $H(\lambda_k)$ the class of function $f(z)$ representableby the Dirichlet series in form (1) on some half-plane.

As is well known at $\lim\limits_{k\to\infty} \frac{\ln k}{\lambda_k} = 0$ the abcissa of simple and absolute convergence of series (1) coincide and is calculated by the formula

$$a = \overline{\lim_{k\to\infty}} \frac{\ln|a_k|}{\lambda_k} = 0 \tag{3}$$

(see [1]).

Denote by $Q_n(z)$ a polynomial of the form

$$Q_n(z) = \sum_{k=1}^{n} c_k e^{-\lambda_k z}. \tag{4}$$

Denote by D_n the set of polynomials of form (4) at the fixed n.

Assume that the function $f(z)$ is analytical in half-plane $\operatorname{Re} z > 0$, is continuous and bounded on $\operatorname{Re} z \geq 0$.

Denote by G_0 the domain $\operatorname{Re} z > 0$, and by $\overline{G}_0$ the$\operatorname{Re} z \geq 0$.

We'll call the value

$$E_n\left(f; \overline{G}_0\right) = \inf_{Q_n \in D_n} \left\{ \max_{z\in\overline{G}_0} |f(z) - Q_n(z)| \right\}. \tag{5}$$

the best polynomial approximation of the function $f(z)$ on $\overline{G}_0$.

Theorem 1. *If $f(z)$ is analytical in G_0, and $f^{\,p}(z)$ it exists on $\overline{G}_0$ and bounded then*

$$E_n\left(f; \bar{G}_0\right) \leq \frac{M_p}{\lambda_n^p}.$$

Proof.

$$E_n\left(f;\overline{G}_0\right)\le\sum_{k=n+1}^{\infty}|a_k|_{\max}\left|e^{-\lambda_k z}\right|=\sum_{k=n+1}^{\infty}\frac{1}{\lambda_k^p}|a_k|\,\lambda_k^p\max_{z\in\overline{G}_0}\left|e^{-\lambda_k z}\right|\le$$

$$\le\frac{1}{\lambda_n^p}\sum_{k=n+1}^{\infty}|a_k|\,\lambda_k^p\max_{z\in\overline{G}_0}\left|e^{-\lambda_k z}\right|=\frac{1}{\lambda_n^p}\sum_{k=n+1}^{\infty}|a_k|\,\lambda_k^p\le\frac{M_p}{\lambda_n^p}.$$

The theorem is proved.

Now let's prove the following lemma.

Lemma. *If $f(z)\in H(\lambda_k)$ then the function $f_h(z)=\frac{1}{h}\int\limits_z^{z+h}f(t)\,dt$ is also belongs to this class where $h>0$.*

Proof.

$$f_h(z)=\frac{1}{h}\int\limits_z^{z+h}f(t)\,dt=\frac{1}{h}\int\limits_0^{h}f(z+t)\,dt.$$

But

$$f(z+t)=\sum_{k=1}^{\infty}e^{-\lambda_k(z+t)}. \tag{6}$$

Series (6) converges uniformly in G_0.

Therefore we can integrate it term by term:

$$f_h(z)=\frac{1}{h}\int\limits_0^{h}f(z+t)\,dt=\frac{1}{h}\sum_{k=1}^{\infty}a_k\int\limits_0^{h}e^{-\lambda_k(z+t)}dt=$$

$$=-\frac{1}{h}\sum_{k=1}^{\infty}\frac{a_k}{\lambda_k}\left(e^{-\lambda_k(z+t)}-e^{-\lambda_k z}\right)=-\frac{1}{h}\sum_{k=1}^{\infty}\frac{a_k}{\lambda_k}e^{-\lambda_k z}\left(e^{-\lambda_k t}-1\right). \tag{7}$$

The last series is convergent since

$$\left|\frac{a_k}{\lambda_k}e^{-\lambda_k z}\left(e^{-\lambda_k t}-1\right)\right|\le\frac{2}{\lambda_1}\left|a_k e^{-\lambda_k z}\right|. \tag{8}$$

Proceeding from inequality (8) series (7) is convergent in G_0, and this means that $f_h(z)\in H(\lambda_k)$.

Consider the difference $f(z)-f_h(z)$:

$$f(z)-f_h(z)=\frac{1}{h}\int\limits_0^{h}f(z)\,dt-\frac{1}{h}\int\limits_0^{h}f(z+t)\,dt=\frac{1}{h}\int\limits_0^{h}\left[f(z)\,dt-f(z+t)\right]dt.$$

Then

$$\max_{z\in G_0}|f(z)-f_h(z)|\le\max_{0\le z\le h}|f(z)-f(z+t)|=\omega(f;h), \tag{9}$$

where $\omega(f;h)$ is module of continuity of the function $f(z)$.

Prove the following theorem.

Theorem 2. *If* $f(z)$ *is continuous and bounded on* $\overline{G}_0$ *then*

$$E_n\left(f;\overline{G}_0\right) \leq C\omega\left(f;1/\lambda_n\right).$$

Proof. By the lemma $f_h(z) \in H(\lambda_k)$. For given $h>0$ we take such a polynomial $Q_n(z)$ that at sufficiently large n the

$$\max_{z\in\overline{G}_0}\left|f_h(z)-Q_n(z)\right| \leq M\omega\left(f;h\right).$$

be fulfilled.

For this as $Q_n(z)$ we can take n-th partial sum of the Dirichlet series corresponding to the function $f_h(z)$. Then

$$\max_{z\in\overline{G}_0}\left|f(z)-Q_n(z)\right| \leq \max_{z\in\overline{G}_0}\left|f(z)-f_h(z)\right| + \max_{z\in\overline{G}_0}\left|f_h(z)-Q_n(z)\right| \ \leq C\omega\left(f;h\right).$$

Taking $h=\lambda_n^{-1}$ we'll obtain

$$\max_{z\in\overline{G}_0}\left|f(z)-Q_n(z)\right| \ \leq C\omega\left(f;1/\lambda_n\right)$$

or

$$E_n\left(f;\overline{G}_0\right) \leq C\omega\left(f;1/\lambda_n\right).$$

Theorem 3. *If* $f(z)$ *is analytical in* G_0 *and there exists the continuous and bounded on* $\overline{G}_0$ *derivative* $f^{(p)}(z)$ *then*

$$E_n\left(f;\overline{G}_0\right) \leq C\cdot\frac{1}{\lambda_n^p}\omega\left(f^{(p)};1/\lambda_n\right).$$

where C *is a constant.*

In particular, the next theorem will be obtained from this theorem.

Theorem 4. *If* $f(z)$ *is analytical in* G_0 *and there exists continuous and bounded on* $\overline{G}_0$ *derivative* $f^{(p)}(z)$ *belonging to the Lipchitz class* α, $0<\alpha\leq 1$ *then*

$$E_n\left(f;G_0\right) \leq \frac{C}{\lambda_n^{p+\alpha}}.$$

References

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