AN APPROXIMATE CHARACTERIZATION OF VECTOR-VALUED BESOV SPACES DEFINED BY DIFFERENCES OF FRACTION ORDER

Abstract

In the work the approximate characterization of isotropic vector-valued Besov’s spaces defined by differences of fraction order is studied.

The characterization of Besov’s spaces by the best approximations has been studied in many works. In scalar case when the order of differences is integer, it is done in [1] (see also [2]). When the order of differences in fractional in different variants it is possible to find it in [3] and [4]. The vector-valued case for differences of integer order has been considered in [5].

Let $E$ be a Banach space. We’ll denote by $L^p(R^n; E)$ the space of strongly measurable on $R^n$ functions $f: R^n \to E$ for which the norm

$$\|f\|_{L^p(R^n; E)} = \|f\|_{p; E} = \|f\| = \left(\int_{R^n} \|f\|^p_E \, dx\right)^{1/p}$$

is finite.

The class of approximation will be entire functions of exponential type of degree $\sigma = (\sigma_1, \ldots, \sigma_n)$, i.e. $\sigma_1$ over $x_1$, $\sigma_2$ over $x_2$, and etc. $\sigma_n$ over $x_n$, $\sigma_i > 0$, $i = 1, 2, \ldots, n$ and entire functions of the exponential spherical type of degree $\sigma, \sigma > 0$ belonging for the real $x$ to $L^p(R^n; E)$ (it is denoted by $\mathcal{M}_{\sigma; p}(E)$ and $\mathcal{SM}_{\sigma; p}(E)$, respectively) (in detail about these classes see [1]).

The best approximation of the function $f \in L_p(R^n, E)$ is denoted by $E_\sigma(f)_{p; E}$:

$$E_\sigma(f)_{p; E} = E_\sigma(f) = \inf_g \|f - g\|_{p; E} \quad (g \in \mathcal{M}_{\sigma; p}(E) \text{ or } g \in \mathcal{SM}_{\sigma; p}(E)).$$

The difference of order $\rho > 0$ of the function $f \in L_p(R^n; E)$ at the point $x \in R^n$ with step $h \in R^n$ and with step $h \in R^1$ on $k$-th variable is defined by the formulae ([3]), respectively

$$\Delta^\rho(h) f(x) = \exp(\pi \rho \bar{e}) \sum_{j=0}^{\infty} A_j^{-1-\rho} f(x + jh), \quad (1)$$

$$\Delta_k^\rho(h) f(x) = \exp(\pi \rho e^k) \sum_{j=0}^{\infty} A_j^{-1-\rho} f(x + jhe^k), \quad (2)$$

where $e^k$ is a $k$-th ort vector, $A_j^{-1-\rho}$ are binomial coefficients and they are defined from the correlation $(1-t)^\rho = \sum_{j=0}^{\infty} A_j^{-1-\rho} t^j$. It is obvious that as $\rho > 0$, $\sum_{j=0}^{\infty} A_j^{-1-\rho} = 0$, $\sum_{j=0}^{\infty} A_j^{-1-\rho} t^j = 0,$
and if \( \rho > 0 \) is integer, then differences (1),(2) turn into ordinary differences of integer order, since \( A_j^{-1-\rho} = 0 \) as \( j \geq \rho + 1 \). It is known that
\[
\sum_{j=0}^{\infty} |A_j^{-1-\rho}| < \infty,
\]
(3)
\[
\|\Delta^\rho (h) f (\cdot)\| \leq C \|f\|,
\]
(4)
\[
\|f^{(k)}\| \leq \sigma^k \|g\|,
\]
(5)
\[
\|\Delta^\rho_k (h) g\| \leq C(\sigma h)^\rho \|g\|.
\]
(6)

**Definition 1.** Let \( h_0 > 0, \ r > 0, \ \rho > r - k > 0, \ k \in \mathbb{N}_0, \ \rho, \theta \in [1, \infty]. \) The class of the functions \( f \in L_p (\mathbb{R}^n, \mathcal{E}) \) with the finite norm
\[
\|f\|_{B^r_p(R^n, \mathcal{E})} = \|f\| + \|f\|_{B^r_p(R^n, \mathcal{E})} =
\]
\[
= \|f\| + \sum_{|h| = k} \left\{ \int_{0 < |h| < h_0} \left[ \frac{\|\Delta^\rho (h) D^s f\|}{|h|^{r-k}} \right]^{1/\theta} dh \right\}^{\theta}, \quad 1 \leq \theta < \infty
\]
(8)
is called isotropic vector Besov’s space \( B^r_p(R^n, \mathcal{E}) \).

At \( \theta = \infty \) the space \( B^r_p(R^n, \mathcal{E}) \) is denoted also by \( H^r_p(R^n, \mathcal{E}) \) and the norm is defined by the following way:
\[
\|f\|_{H^r_p(R^n, \mathcal{E})} = \|f\| + \sum_{|h| = k} \sup_{0 < |h| < h_0} \left\{ \frac{\|\Delta^\rho (h) f^{(s)} (\cdot)\|}{|h|^{r-k}} \right\}.
\]
(9)

**Definition 2.** Let \( h_0 > 0, \ r > 0, \ \rho_i > l_i - k_i > 0, \ k_i \in \mathbb{N}_0, \ i = 1, 2, \ldots, n, \ l = (l_1, l_2, \ldots, l_n) \in (0, \infty)^n, \ \rho, \theta \in [1, +\infty]. \) The class of the functions \( f \in L_p (\mathbb{R}^n, \mathcal{E}) \) with the finite norm
\[
\|f\|_{B^l_p(R^n, \mathcal{E})} = \|f\| + \|f\|_{B^l_p(R^n, \mathcal{E})} =
\]
\[
= \|f\| + \sum_{i=0}^{n} \left\{ \int_{0}^{h_0} \left[ \frac{\|A^r_i (h) D^{l_i}_i f (\cdot)\|}{h_l^{r-k_i}} \right]^{1/\theta} dh \right\}^{\theta}
\]
(10)
is called anisotropic vector Besov’s space \( B^l_p(R^n, \mathcal{E}) \).

At \( \theta = \infty \) the space \( B^l_p(R^n, \mathcal{E}) \) is denoted also by \( H^l_p(R^n, \mathcal{E}) \) and the norm is defined by the following way:
\[
\|f\|_{H^l_p(R^n, \mathcal{E})} = \|f\| + \|f\|_{H^l_p(R^n, \mathcal{E})} = \|f\| + \sum_{i=0}^{n} \sup_{0 < h < h_0} \left\{ \frac{\|\Delta^l_i (h) D^{l_i}_i f\|}{h_l^{r-k_i}} \right\}.
\]
(11)
Let \( g(\xi) \) be an even nonnegative function of one variable of exponential type 1, satisfying the condition ([1], p.182)

\[
\kappa_n \int_0^\infty g(\xi) \xi^{n-1} d\xi = \int_{\mathbb{R}^n} g(|u|) du = 1,
\]

where \( \kappa_n \) is the corresponding constant. Let’s take \( \nu > 0 \) and \( f \in L_\nu(\mathbb{R}^n, E) \) and consider the vector-valued function

\[
g_\nu(x) = \int_{\mathbb{R}^n} g(|y|) \{ - \exp(-\pi \nu i) \Delta^\nu (y/\nu) f(x) + f(x) \} dy.
\]

It is possible to reduce \( g_\nu(x) \) to the following form ([6]):

\[
g_\nu(x) = \int_{\mathbb{R}^n} K_\nu(x-t) f(t) dt,
\]

where \( K_\nu(y) = \sum_{j=1}^\infty g\left(\frac{\nu}{j} |y|\right) d_j \left(\frac{\nu}{j}\right)^n, \quad d_j = -A_j^{-1-\nu}. \)

**Lemma.** \( K_\nu(z) \) is an entire function of spherical type \( \nu \), i.e. \( K_\nu(z) \in SM_{\nu,1}(E) \), at that \( g_\nu(z) \in SM_{\nu,\nu}(E) \).

**Proof.** Since \( g(\xi) \) is even nonnegative function of exponential type 1 it follows that \( g(\nu |y|/j) \) is an entire function of spherical type \( \nu/j \) ([1]).

Therefore, \( g(\nu |y|/j) = \sum_{k=0}^\infty (g^{(2k)}(0)/(2k)!)(\nu/j)^{2k} |y|^{2k} \) and

\[
K_\nu(y) = \sum_{j=1}^\infty d_j \left(\frac{\nu}{j}\right)^n \sum_{k=0}^\infty \frac{g^{(2k)}(0)}{(2k)!} \left(\frac{\nu}{j}\right)^{2k} |y|^{2k} =
\]

\[
= \sum_{k=0}^\infty \frac{g^{(2k)}(0)}{(2k)!} \left( \sum_{j=1}^\infty d_j \left(\frac{\nu}{j}\right)^{n+2k} \right) |y|^{2k}.
\]

From (3) it follows that the series \( \sum_{j=1}^\infty d_j \) is convergent, therefore the series \( \sum_{j=1}^\infty d_j (\nu/j)^{n+2k} \) is absolutely convergent. The expression \( |y|^{2k} = (y_1^2 + \ldots + y_n^2)^k \) is a polynomial of degree 2k with natural coefficients. Then series (15) after removing the parenthesis in every its term will be absolutely convergent for any \( y = (y_1, \ldots, y_n) \) the power series by the degrees \( y_1, \ldots, y_n \), i.e. \( K_\nu(z) \) is entire function.

From (12) it follows that \( \sup_x |g(x)| = G < \infty \). Then

\[
\sup_x |K_\nu(x)| \leq \sum_{j=1}^\infty |d_j| \left(\frac{\nu}{j}\right)^n \sup_x g\left(\frac{\nu}{j}\right) x \leq CG,
\]
\[ |K_\nu(z)| \leq \sum_{j=1}^{\infty} d_j \left( \frac{\nu}{j} \right)^n g \left( \frac{\nu}{j} \right) |z| \leq \sum_{j=1}^{\infty} d_j \left( \frac{\nu}{j} \right)^n G \exp \left( \frac{\nu}{j} \right) |y| \leq \]
\[ \leq \sum_{j=1}^{\infty} |d_j| \left( \frac{\nu}{j} \right)^n \exp(\nu|y|) \leq CG\exp(\nu|y|). \]

So \( K_\nu(z) \in SM_{\nu,1}(E) \). From the correlation \( \sum_{j=1}^{\infty} A_j^{-1-\rho} = 1 \) and from (12) it follows that \((u = (\nu/j) y, \ dy = (j + \nu) du)\)
\[ \int_{R^n} K_\nu(y) dy \sum_{j=1}^{\infty} d_j \int_{R^n} g(|u|) du = \sum_{j=1}^{\infty} d_j = 1, \]
\[ \text{i.e. } K_\nu(y) \in L(R^n). \] From (14) and by theorem 2.6.2 from [1] (page 135) \( g_\nu(z) \in SM_{\nu,1}(E) \), Q.E.D.

**Theorem 1.** In order that the function \( f \in L_p(R^n, E) \) belong to the space \( B_{p,0}^l(R^n, E) \), \( 1 \leq p, \theta \leq \infty \), it is necessary and sufficient that the quantity \((a > 1)\)
\[ \left\{ \sum_{j=0}^{\infty} a^{i\nu\theta} E_{\alpha^j}(f) \right\}^{1/\theta}, \ 1 \leq \theta < \infty \] \[ \sup_i \left\{ a^{i\nu} E_{\alpha^j}(f) \right\}, \ \theta = \infty \] be finite. At that norm (8) ((9) as \( \theta = \infty \)) is equivalent to expression (16) ((17) as \( \theta = \infty \)).

**Proof.** From (12) and (13) it follows that
\[ g_\nu(x) - f(x) = \int_{R^n} g(|y|) (-\exp(-\pi\rho i)) \Delta^\rho \left( \frac{y}{\nu} \right) f(x) dy. \] (18)

Let us suppose now that the function \( f \) has on \( R^n \) generalized derivatives of order \( k \) belonging to \( L_p(R^n, E) \). Then taking in (18) \( k + \rho \) instead of \( \rho \) we have
\[ E_\nu(f) \leq \| f - g \| = \left\| \int_{R^n} g(|y|) (-\exp(-\pi(\rho k + i)) \Delta^{k+\rho} \left( \frac{y}{\nu} \right) f(\cdot) dy \right\| \leq \]
\[ \leq \int_{R^n} g(|y|) \left\| \Delta^{k+\rho} \left( \frac{y}{\nu} \right) f(\cdot) \right\| dy = \int_{R^n} g(|y|) \left\| \Delta^k \left( \frac{y}{\nu} \right) \Delta^\rho \left( \frac{y}{\nu} \right) f(\cdot) \right\| dy \leq \]
\[ \leq \int_{R^n} g(|y|) \left\| \Delta^k \left( \frac{y}{\nu} \right) \Delta^\rho \left( \frac{y}{\nu} \right) f(\cdot) \right\| dy \leq \]
\[ \leq \frac{c}{\nu^k} \sum_{|l|=k} \int_{R^n} g(|y|)|y|^k \left\| \Delta^\rho \left( \frac{y}{\nu} \right) f(\cdot) \right\| dy. \] (19)

Let us apply the generalized Minkowsky inequality, inequality (7) and change the order operations of differentiation and taking the fractional difference [3].
First of all we’ll consider the case $\theta = \infty$. Let $\|f\|_{b_p, \infty} < \infty$. From (19) it follows that

$$E_{\nu}(f) \leq \frac{c}{\nu^r} \sum_{|s|=k} \int_{R^n} g(|y|)|y|^k \left(\frac{y}{\nu}\right)^{-k} \|f\|_{b_p, \infty}(R^n, E) \, dy \leq \frac{c}{\nu^r} \int_0^\infty g(t) t^{n+r-1} \, dt \cdot \|f\|_{b_p, \infty}(R^n, E) \, dy.$$ 

The function $g$ is chosen such that the last integral be finite (see [1] or [6]). So, for arbitrary $\nu > 0$

$$E_{\nu}(f) \leq \frac{c}{\nu^r} \|f\|_{b_p, \infty}(R^n, E)$$

or, that is the same $E_{\nu}(f) \nu^r \leq c \|f\|_{b_p, \infty}(R^n, E)$. Taking $\nu = a^j$, where $a > 1$ we’ll obtain

$$\sup_{\nu} E_{a^j}(f) \leq c \|f\|_{b_p, \infty}(R^n, E) \cdot$$

Let’s prove the inverse inequality. Let $M = \sup_{\nu} E_{a^j}(f) < \infty$. Hence $E_{a^j}(f) \leq M/a^{jr}$. Let $g_{a^j}$ be entire function of exponential spherical type $a^j$ for which

$$\|f - g_{a^j}\| = E_{a^j}(f) \leq \frac{M}{a^{jr}}.$$ 

From this inequality it follows that the function $f$ is represented in the form of convergent in $L_p(R^n, E)$ series

$$f = \sum_{j=0}^{\infty} q_j,$$ 

where

$$q_0 = q_{a^0} = q_1, \quad \|q_0\| \leq \|q_{a^0} - f\| + \|f\| \leq \|f\| + M,$$

$$q_j = q_{a^j} - q_{a^{j-1}}, \quad \|q_j\| \leq \|q_{a^j} - f\| + \|f - q_{a^{j-1}}\| \leq \frac{M}{a^{jr}} + \frac{M}{a^{(j-1)r}} = \frac{M}{a^{jr}} (1 + a^r) \leq C \frac{\|f\| + M}{a^{3r}}.$$ 

By virtue of (5) and (21) the series $f^{(s)}(x) = \sum_{j=0}^{\infty} q_j^{(s)}(x)$ is convergent in $L_p(R^n, E)$, too. Taking into account (4),(6) ($|s| = k, \quad r - k > 0, \quad \rho - (r - k) > 0$)
we'll obtain

$$\| \Delta^\rho (h) f(s) \| \leq \sum_{j=0}^{N} \| \Delta^\rho (h) q_j^{(s)} (\cdot) \| + \sum_{j=N+1}^{\infty} \| \Delta^\rho (h) q_j^{(s)} (\cdot) \| \leq$$

$$\leq \sum_{j=0}^{N} c_1 (|h| a^j)^p \| q_j^{(s)} \| + \sum_{j=N+1}^{\infty} c_2 \| q_j^{(s)} \| \leq c_1 |h|^p \sum_{j=0}^{N} a^{j(p+k)} \| q_j \| +$$

$$+ c_2 \sum_{j=N+1}^{\infty} a^{jk} \| q_j^{(s)} \| \leq c_1 |h|^p \sum_{j=0}^{N} a^{j(p+k)} a^{-jr} (\| f \| + M) +$$

$$+ c_2 \sum_{j=N+1}^{\infty} a^{jk} a^{-jr} (\| f \| + M) =$$

$$= \left( \| f \|_{p,E} + M \right) \left[ c_1 |h|^p \sum_{j=0}^{N} a^{j(p-(r-k))} + c_2 \sum_{j=N+1}^{\infty} a^{jk-r} \right] =$$

$$= c \left( \| f \|_{p,E} + M \right) \left( |h|^{r-k} (a^N |h|)^{(p-(r-k))} + a^{1-(N+1)(r-k)} \right).$$

For $h \in \mathbb{R}^n$ we'll choose the natural number $N$ such that $a^{-(N+1)} < |h| < a^{-N}$. Then $a^N |h| < 1$, $(a^{-(N+1)} (r-k)) < |h|^{r-k}$. Hence

$$\| \Delta^\rho (h) f(s) (\cdot) \| \leq c \left( \| f \|_{p,E} + M \right) |h|^{r-k},$$

from here

$$\| f \|_{B^\rho_{\infty}(\mathbb{R}^n,E)} = \sup_{0 < |h| < h_0, |\alpha| = k} \sum_{\alpha < |h| < h_0, |\alpha| = k} |h|^{r-k} \frac{\| \Delta^\rho (h) f^{(s)} (\cdot) \|}{\| f \|_{p,E} + M} \leq c \left( \| f \|_{p,E} + M \right).$$

Consider the case $1 \leq \theta < \infty$. Let now $\| f \|_{B^\rho_{p,\theta}} < \infty$. The function $a^{jr \theta} (a > 1)$ increases on $j$, but the function $E_{a_j}^{\theta} (f)$ doesn't increase on $j$, therefore

$$\int_{j-1}^{j} a^{jr \theta} E_{a_j}^{\theta} (f) \, dl \geq \int_{j-1}^{j} a^{r \theta (j-1)} E_{a_j}^{\theta} (f) \, dl \geq a^{r \theta (j-1)} E_{a_j}^{\theta} (f)$$

or

$$a^{jr \theta} E_{a_j}^{\theta} (f) \leq a^{r \theta} \int_{j-1}^{j} a^{jr \theta} E_{a_j}^{\theta} (f) \, dl. \quad (23)$$
From here, taking into account (21) and (19) as \( \nu = a^j \) we’ll obtain

\[
\left( \sum_{j=0}^{\infty} a^j \theta E_{\alpha_j}^0 (f) \right)^{1/\theta} \leq \\
\leq \left( \sum_{j=0}^{\infty} a^j \theta \int_{\mathbb{R}^n} a^j \theta E_{\alpha_j}^0 (f) \, dl \right)^{1/\theta} \leq a^\nu \left( \int_{\mathbb{R}^n} a^j \theta E_{\alpha_j}^0 (f) \, dl \right)^{1/\theta} \\
\leq a^\nu \left( \int_{\mathbb{R}^n} g (|y|) \left( \frac{|y|}{a^j} \right)^k \Delta^\rho (\frac{y}{a^j}) f^{(k)} (\cdot) \, dy \right)^{\theta} \left( \int_{\mathbb{R}^n} \Delta^\rho (\frac{dy}{a^j}) f^{(k)} (\cdot) \, dy \right)^{1/\theta} \\
\leq a^\nu \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g (u) \left( \frac{|u|}{a^j} \right)^k \Delta^\rho (\frac{u\xi}{a^j}) f^{(k)} (\cdot) \, dx \right)^{\theta} \left( \int_{\mathbb{R}^n} \Delta^\rho (\frac{dx}{a^j}) f^{(k)} (\cdot) \, dx \right)^{1/\theta}
\]

(we have applied above the inequality \( \| \Delta^k (h) \Delta^\rho (h) f \| \leq |h|^k \| \Delta^\rho (h) f \| \), where \( f^{(k)} \) is a derivative of order \( k \) on the direction \( h \) passed to the polar coordinates \( y = (u, \xi) \), \( u = |y| \), \( du = u^{n-1} dud\xi \), then applied the Minkowsky inequality for integrals)

\[
\leq a^\nu \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g (u) \left( \frac{|u|}{a^j} \right)^k \Delta^\rho (\frac{u\xi}{a^j}) f^{(k)} (\cdot) \, dx \right)^{\theta} \left( \int_{\mathbb{R}^n} \Delta^\rho (\frac{dx}{a^j}) f^{(k)} (\cdot) \, dx \right)^{1/\theta}
\]

\[
\left( ua^{-j} = \nu, \, dv = -ua^{-j} \ln a dj, \, u/v = a^{-j}, \, dj = -dv/v \ln a \right)
\]

\[
\leq a^\nu \int_{0}^{\infty} \left( \int_{\mathbb{R}^n} g (u) \left( \frac{|u|}{a^j} \right)^k \Delta^\rho (\frac{u\xi}{a^j}) f^{(k)} (\cdot) \, dx \right)^{\theta} \left( \int_{\mathbb{R}^n} \Delta^\rho (\frac{dx}{a^j}) f^{(k)} (\cdot) \, dx \right)^{1/\theta} \frac{du}{v \ln a}
\]

\[
\leq a^\nu \int_{0}^{\infty} \left( \int_{\mathbb{R}^n} g (u) \left( \frac{|u|}{a^j} \right)^k \Delta^\rho (\frac{u\xi}{a^j}) f^{(k)} (\cdot) \, dx \right)^{\theta} \left( \int_{\mathbb{R}^n} \Delta^\rho (\frac{dx}{a^j}) f^{(k)} (\cdot) \, dx \right)^{1/\theta} \frac{du}{v \ln a}
\]

\[
\leq c \int_{0}^{\infty} \left( \int_{\mathbb{R}^n} |h|^{-\frac{\theta}{n+\theta}} \Delta^\rho (h) f^{(k)} (\cdot) \right)^{\theta} \frac{dh}{|h|^\frac{n-\theta}{n+\theta}} \left( \int_{\mathbb{R}^n} \Delta^\rho (h) f^{(k)} (\cdot) \right)^{1/\theta} dh \leq
\]

\[
\leq c \int_{0}^{\infty} \left( \int_{\mathbb{R}^n} |h|^{-\frac{\theta}{n+\theta}} \Delta^\rho (h) f^{(k)} (\cdot) \right)^{\theta} \frac{dh}{|h|^\frac{n-\theta}{n+\theta}} \left( \int_{\mathbb{R}^n} \Delta^\rho (h) f^{(k)} (\cdot) \right)^{1/\theta} dh \leq
\]

\[
\leq c \int_{0}^{\infty} \left( \int_{\mathbb{R}^n} |h|^{-\frac{\theta}{n+\theta}} \Delta^\rho (h) f^{(k)} (\cdot) \right)^{\theta} \frac{dh}{|h|^\frac{n-\theta}{n+\theta}} \left( \int_{\mathbb{R}^n} \Delta^\rho (h) f^{(k)} (\cdot) \right)^{1/\theta} dh \leq
\]
Let’s prove the inverse inequality. Let \( \left( \sum_{j=0}^{\infty} a^{j\theta} E_{a_j}^\theta (f) \right)^{1/\theta} = M < \infty \). From the finiteness of \( M \) it follows that \( E_{a_j} (f) \to 0 \) \( (j \to \infty) \), therefore it is possible to represent \( f \) in form (20), where

\[
\| q_j \| \leq \| f \| + E_{a_j} (f) , \quad \| q_j \| \leq \| g_{a_j} - f \| + \| f - g_{a_j-1} \| \leq 2E_{a_j-1} (f) \quad (j = 1, 2, \ldots),
\]

because \( E \) \( (f) \leq \| f \| \) and \( E_{a_j} \) doesn’t increase on \( j \). Therefore

\[
\left( \sum_{j=0}^{\infty} a^{j\theta} \| q_j \|^\theta \right)^{1/\theta} \leq \left\{ \| f \| + E_{a_j} (f) + \left( \sum_{j=1}^{\infty} a^{j\theta} E_{a_j} (f) \right)^{1/\theta} \right\} \leq 2a^{r} \left( \| f \|^\theta + \sum_{j=0}^{\infty} a^{j\theta} E_{a_j} (f) \right)^{1/\theta} \leq \frac{\| f \|_{p,E}}{2} + \left( \sum_{j=0}^{\infty} a^{j\theta} E_{a_j} (f) \right)^{1/\theta} = C ({\| f \| + M}).
\]

Further \( (t = a^{-r}, \ dt = -a^{-r} \ln a \ dt) \)

\[
\| f \|_{b_{p,\infty} (R^n,E)} = \sum_{|\xi|=k} \left\{ \int_{|h|<ho} \| \Delta^\rho (h) f (\xi) \|^{\theta} |h|^{-(r-k)\theta-\rho} \ dt \right\}^{1/\theta} \leq \sum_{|\xi|=k} \left\{ \int_{0}^{\infty} dt \int_{|\xi|=1} \| \Delta^\rho (t, \xi) f (\xi) \|^{\theta} t^{1-(r-k)\theta} d\xi \right\}^{1/\theta} \leq \sum_{|\xi|=k} \left\{ \int_{-\infty}^{\infty} a^{-r} \int_{|\xi|=1} \| \Delta^\rho (a^{-r} \xi) f (\xi) \|^{\theta} d\xi d\tau \right\}^{1/\theta} = \sum_{|\xi|=k} \left\{ \int_{0}^{N+1} \int_{|\xi|=1} \| \Delta^\rho (a^{-r} \xi) f (\xi) \|^{\theta} d\xi d\tau \right\}^{1/\theta}.
\]

Estimate the first integral in (25) using (22):

\[
\int_{0}^{\infty} a^{-r} \int_{|\xi|=1} \| \Delta^\rho (a^{-r} \xi) f (\xi) \|^{\theta} d\xi d\tau = \int_{N}^{N+1} \int_{|\xi|=1} \| \Delta^\rho (a^{-r} \xi) f (\xi) \|^{\theta} d\xi d\tau \leq
\]
Applying inequalities 5.6.19 and 5.6.20 from [1] (page 216) to \( I \) and \( II \), respectively, we’ll get:

\[
I = \sum_{N=0}^{\infty} a^{-N_{r-k}N_{\theta}} \left( \sum_{j=0}^{N} a^{j(r+k)} \|q_j\|^{\theta} \right) \leq c \sum_{j=0}^{\infty} a^{\theta j} \|q_j\|^{\theta},
\]

\[
II = \sum_{N=0}^{\infty} a^{-N_{r-k}N_{\theta}} \left( \sum_{j=N}^{\infty} a^{j(r+k)} \|q_j\|^{\theta} \right) \leq c \sum_{j=0}^{\infty} a^{\theta j} \|q_j\|^{\theta}.
\]

Using (4) it is possible to prove that (see [6]) the second integral

\[
\int_{-\infty}^{0} a^{\tau(r-k)\theta} \int_{|\xi|=1} \|\Delta^\theta (a^{-\tau \xi} f(s))\|^{\theta} d\xi d\tau \leq c \|f(s)\|^{\theta}.
\]

But (let’s recall of that \(|s|=k\)) applying the Holder inequality and (5):

\[
\|f(s)\|^{\theta} = \left( \sum_{j=0}^{\infty} a^j q_j(s) \right)^{\theta} \leq c \left( \sum_{j=0}^{\infty} a^j \|q_j\| \right)^{\theta} \leq c \left( \sum_{j=0}^{\infty} a^{-j(r-k)\theta} a^j \|q_j\| \right)^{\theta} \leq c \left( \sum_{j=0}^{\infty} a^{-j(r-k)\theta} \right)^{\theta} \left( \sum_{j=0}^{\infty} a^{j\theta} \|q_j\| \right)^{\theta} \leq c \sum_{j=0}^{\infty} a^{j\theta} \|q_j\|^{\theta}.
\]

So, taking into account (25) and (24) we have

\[
\|f\|_{b^r_{p,\infty}(R^n,E)} \leq c \sum_{|l|=k} \left( \sum_{j=0}^{\infty} a^{j\theta} \|q_j\|^\theta \right) \leq c (\|f\| + M).
\]

Similarly, when the order of finite differences is fractional, then with the help of our lemma and following the scheme of proof from [1] with corresponding changes it is possible to prove the following theorem for anisotropic Banach-valued functional Besov spaces.
Theorem 2. In order the function \( f \in L_p(\mathbb{R}^n,E) \) belong to the space \( B_{p,\theta}^j(\mathbb{R}^n,E) \), 1 \( \leq p, \theta \leq \infty \), it is necessary and sufficient that the quantity

\[
\left\{ \sum_{j=0}^{\infty} a^0 E_{\alpha^{j,1,\alpha^{j,2},\ldots,\alpha^{j,n}}}(f) \right\} 1 \leq \theta < \infty
\]

\[
\sup_i \left\{ a^i E_{\alpha^{j,1,\alpha^{j,2},\ldots,\alpha^{j,n}}}(f) \right\} \theta = \infty
\]

be finite. At that norm (10) ((11) as \( \theta = \infty \)) is equivalent to expression (26) ((27) as \( \theta = \infty \)).

References


[6]. Maharov I.K. The imbedding theorems for classes of functions constructed on base of averages containing the differences of fractional order. Trudy IMM AN Azerb., 1996, v.5(13), pp.51-59. (Russian)

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