Irina V. MAYO OVA

TO ESTIMATES OF L_q-TRACES OF FUNCTIONS AND THEIR DERIVATIVES ON BOUNDARY m-DIMENSIONAL SURFACE

Abstract

In this paper we investigate the properties of $L_q(\Gamma_m)-$ *traces of functions from new spaces*

$$W_p^{\langle r\rangle}(G;s)$$

on boundary m *-dimensional surface* $\Gamma_m \in \Pi^1$. *The integral inequality of the embedding theorem type*

$$D^{\nu}: W_p^{\langle r\rangle}(G;s) \hookrightarrow L_q(\Gamma_m)$$

is proved.

We consider the space

$$W_p^{\langle r\rangle}(G;s) \tag{0.1}$$

of the functions $f = f(x)$ of the points $x = (x_1;...;x_s) \in E_n$ $(1 \le s \le n)$ of many variables $x_k = (x_{k,1},...,x_{k,n_k}) \in E_{n_k}$ $(k = 1,2,..., s)$, defined in the domain $G \subset E_n = E_{n_1} \times ... \times E_n$, $(n = n_1 + \ ... \ + n_s)$ satisfying the condition of "σ - semi-horn" .

Space (0.1) is determined as a closure of set of sufficient smooth and finite functions in E_n by the norm

$$\|f\|_{W_p^{\langle r\rangle}(G;s)} = \sum_{i=(i_1,...,i_s)\in Q} \left\|D^{r^i} f\right\|_{L_p(G)} < \infty, \tag{0.2}$$

where the sum is taken in all possible vectors $i = (i_1,...,i_s) \in Q$ with the coordinates $i_k \in \{0,1,...,n_k\}.(k = 1,2,...,s)$.

It is assumed that the vector $r = (r_1;...;r_s)$ with coordinates-vectors $r_k = (r_{k,1},...,r_{k,n_k})$ $(k = 1,2,...,s)$ is "integer and positive", i.e., $r_{k,j} > 0$ $(j = 1,2,...,n_k)$ are integer at all $k = 1,2,...,s$, and for each $i = (i_1,...,i_s) \in Q$ the vector $r^i = \left(r_1^{i_1};...;\ r_s^{i_s}\right)$ has coordinates and vectors

$$r_k^{i_k} = \begin{cases} (0,...,0,r_{k,i_k},0,...,0) & \text{at } \ i_k \ne 0, \\ (0,...,0) & \text{at } \ i_k = 0 \end{cases}$$

at all $k = 1,2,...s$..

Space (0.1) in case of $s = 1$ coincides with the known S.L.Sobolev space $W_p^{\langle r\rangle}(G)$, and the case $s = n$ with the known S.M.Nikolskii space $S_p^{\langle r\rangle}W(G)$ of differentiable functions with dominant mixed derivative.

In this paper we define a class of surfaces Γ_m $(1 \le m \le n-1)$ and find the conditions for which there exist L_q-traces of functions and their derivatives on Γ_m. At that it is proved that

$$D^\nu f|_{\Gamma_m} \in L_q(\Gamma_m), \tag{0.3}$$

and the integral inequalities

$$\left\| D^\nu f|_{\Gamma_m} \right\|_{L_q(\Gamma_m)} \le \|f\|_{W_p^{\langle r\rangle}(G;s)} \tag{0.4}$$

are valid, where it is assumed that $1 < \quad \le q < \infty$.

1. Basic definitions and necessary notation.

1.1. Let $H = (H_1, ..., H_s)$, $H_k > 0$ $(k = 1, 2, ..., s)$, and the vector

$$\delta = (\delta_1; ...; \delta_s) \tag{1.1}$$

with the coordinate-vectors $\delta_k = (\delta_{k,1}, ..., \delta_{k,n_k})$ $(k = 1, 2, ..., s)$ be such that

$$\delta_{k,j} = +1 \text{ or } \delta_{k,j} = -1 \quad (j = 1, 2, ..., n_k) \tag{1.2}$$

at all $k = 1, 2, ..., s$..

Let the vector

$$\sigma = (\sigma_1; ...; \sigma_s) \tag{1.3}$$

with the coordinate-vectors

$$\sigma_k = (\sigma_{k,1}, ..., \sigma_{k,n_k}) \quad (k = 1, 2, ..., s) \tag{1.4}$$

be "positive", i.e.

$$\sigma_{k,j} > 0 \quad (j = 1, 2, ..., n_k) \quad (k = 1, 2, ..., s). \tag{1.5}$$

Denote by

$$\mathcal{R}_\delta(\sigma; H) =$$

$$= \bigcup_{\substack{0 < v_k \le H_k \\ (k=1,2,..,s)}} \left\{ y \in E_n; \quad C_k \le \frac{y_{k,j}\delta_{k,j}}{v_k^{\sigma_{k,j}}} \le C_k^* \quad (j = 1, 2, ..., n_k) \right\}, \tag{1.6}$$

where C_k, C_k^* $(k = 1, 2, ...s)$ are fixed positive constants, it is denoted a "σ-semi-horn" with vertex in origin of coordinates. Then

$$x + \mathcal{R}_\delta(\sigma; H) \tag{1.7}$$

is "σ-semi-horn" with vertex at the point $x \in E_n$.

Definition 1. *The subdomain* $\Omega \subset G$ *is called a subdomain satisfying the condition of "σ - semi-horn" if*

$$x + \mathcal{R}_\delta(\sigma; H) \subset G \tag{1.8}$$

at all $x \in \Omega$.

Definition 2. *The domain*

$$G \subset E_n \tag{1.9}$$

is called a domain satisfying the condition of "σ - semi-horn" if there exists a set of subdomains

$$G_1, G_2, ..., G_K \subset G \tag{1.10}$$

satisfying the condition of "σ - semi-horn" and covering the domain G, i.e. such that

$$G = \bigcup_{\mu=1}^{K} G_\mu. \tag{1.11}$$

The class of domains $G \subset E_n$ satisfying the condition of "σ - semi-horn" is denoted by

$$C(\sigma; H). \tag{1.12}$$

1.2. Let

$$m = m_1 + ... + m_\alpha < n = n_1 + ... + n_s \quad (1 \le \alpha \le s \le n), \tag{1.13}$$

at that

$$m_k \le n_k \qquad (k = 1, 2, ..., \alpha). \tag{1.14}$$

Let at each $k \in \{1, 2, ..., \alpha\}$

$$\begin{cases} x_{k,1} = x_{k,1}; ...;\ x_{k,m_k} = x_{k,m_k}, \\ x_{k,\beta} = \varphi_{k,\beta}(x_{k,1}, ..., x_{k,m_k}) \quad (\beta = m_k + 1, ..., n_k) \end{cases} \tag{1.15}$$

at that we assume that the functions

$$\varphi_{k,\beta} = \varphi_{k,\beta}(x_{k,1}, ..., x_{k,m_k}) \quad (\beta = m_k + 1, ..., n_k) \tag{1.16}$$

have continuous partial derivatives in some m_k-dimensional domain $\Omega_{m_k} \subset E_{m_k}$, moreover we assume that

$$\left| \frac{\partial}{\partial x_{k,j}} \varphi_{k,\beta}(x_{k,1}, ..., x_{k,m_k}) \right| \le M_k \tag{1.17}$$

at all $j = 1, 2, ..., m_k$ and $\beta = m_k + 1, ..., n_k$.

Let

$$x_k^* = \left(x_{k,1}, ..., x_{k,m_k}, \varphi_{k,m_k+1}(x_{k,1}, ..., x_{k,m_k}), ..., \varphi_{k,n_k}(x_{k,1}, ..., x_{k,m_k}) \right) \tag{1.18}$$

at all $k = 1, 2, ..., \alpha$..

Let further

$$\begin{cases} x_1^* = x_1^*, \\ .... \\ x_\alpha^* = x_\alpha^*, \\ x_{\alpha+1}^* = \psi_{\alpha+1}(x_1^*; ...; x_\alpha^*), \\ .... \\ x_s^* = \psi_s(x_1^*; ...; x_\alpha^*) \end{cases} \tag{1.19}$$

where the vector-functions

$$\psi_k = \psi_k\left(x_1^*; ...; x_\alpha^*\right) \quad (k = \alpha + 1, ..., s) \tag{1.20}$$

with the coordinate functions

$$\psi_{k,j} = \psi_{k,j}\left(x_1^*; ...; x_\alpha^*\right) \quad (j = 1, 2, ..., n_k) \tag{1.21}$$

and having continuous partial derivatives such that

$$\left|\frac{\partial}{\partial x_{\gamma,\beta}}\psi_{k,j}\left(x_1^*; ...; x_\alpha^*\right)\right| \le M_{k,j} \tag{1.22}$$

in some domain

$$\Omega_m = \Omega_{m_1} \times \ ... \ \times \Omega_{m_\alpha} \subset E_m = E_{m_1} \times \ ... \ \times E_{m_\alpha} \tag{1.23}$$

at all $\gamma = 1, 2, ..., \alpha$; $\beta = 1, 2, ..., m_\gamma$; $j = 1, 2, ..., n_k$; $k = \alpha + 1, ..., s$.

It follows from the cited reasons that the set of the points

$$x^* = \left(x_1^*, ..., x_\alpha^*, \ \psi_{\alpha+1}\left(x_1^*, ..., x_\alpha^*\right); \ ...; \ \psi_s\left(x_1^*; ...; x_\alpha^*\right)\right) =$$

$$= P\left(x_1^*, ..., x_\alpha^*\right) = T\left(x'\right)$$

at

$$\left(x_{1,1}, ..., x_{1,m_1}; \ ...; x_{\alpha,1}, ..., x_{\alpha,m_\alpha}\right) = \left(x_1', ..., x_\alpha'\right) = x' \ \in \Omega_m \subset E_m \tag{1.24}$$

it is defined the surface

$$\Gamma_m \subset \overline{G}. \tag{1.25}$$

The class of surfaces Γ_m from (1.13)-(1.26) is denoted by Π^1.

2. Basic result.

The basic result is given in the form of the following theorem.

Theorem. *1) Let $f \in W_p^{\langle r \rangle}(G; s)$, where $1 < \quad \le q < \infty$..*

2) Let the domain $G \in C(\sigma; H)$.

3) Let $\Gamma_m \in \Pi^1$ at $\Gamma_m \subset \overline{G}$.

4) Let the "positive nonnegative" vector $\nu = (\nu_1; ...; \nu_s)$ with the coordinates-vectors $\nu_k = (\nu_{k,1}, ..., \nu_{k,n_k})$ $(k = 1, 2, ..., s)$ be such that

$$\varkappa_{k,i_k} = r_{k,i_k}\sigma_{k,i_k} - (\nu_k, \sigma_k) - \frac{1}{\ }|\sigma_k| + \frac{1}{q}\left|\sigma_k'\right| \ge 0$$

$(i_k = 1, 2, ..., n_k)$ at all $k = 1, 2, ..., \alpha$;

$$\varkappa_{k,i_k} = r_{k,i_k}\sigma_{k,i_k} - (\nu_k, \sigma_k) - \frac{1}{\ }|\sigma_k| > 0 \qquad (i_k = 1, 2, ..., n_k)$$

at all $k = \alpha + 1, ..., s$, where $|\sigma_k| = \sigma_{k,1} + ... + \sigma_{k,n_k}$; $|\sigma_k'| = \sigma_{k,1} + ... + \sigma_{k,m_k}$; $(\nu_k, \sigma_k) = \sum_{j=1}^{n_k}\nu_{k,j}\sigma_{k,j}$.

Then the function $D^{\nu}f(x)$ on the surface Γ_m has L_q-traces

$$D^{\nu}f|_{\Gamma_m} \in L_q(\Gamma_m),$$

at that the inequalities

$$\left\| D^{\nu}f|_{\Gamma_m} \right\|_{L_q(\Gamma_m)} \le C \|f\|_{W_p^{\langle r\rangle}(G;s)}$$

are valid, the constant C is independent of the function $f = f(x)$.

3. Scheme of proof of the theorem.

The proof of the theorem is conducted by the method of integral representations on the basis of a new integral representation of functions led in [1].

3.1. We can take the function

$$f \in W_p^{\langle r\rangle}(G;s) \tag{3.1}$$

sufficiently smooth in E_n, consequently at each point $x \in E_n$ the identity

$$D^{\nu}f(x) = \sum_{i=(i_1,\dots i_s)\in Q} A_{i,\delta}f(x) \tag{3.2}$$

holds.

Here the integral operators

$$A_{i,\delta}f(x) = c_i \left(\prod_{k\in e_s\backslash e^i} H_k^{-\beta_{k,0}} \right) \int_{\vec{0}}^{\vec{H}} \prod_{k\in e^i} \frac{d\upsilon_k}{\upsilon_k^{1+\beta_{k,i_k}}} \times$$
$$\times \int_{E_n} D^{r^i} f(x+y)\, \Phi_{i,\delta}(\dots)\, dy \tag{3.3}$$

at each $i = (i_1,\dots,i_s) \in Q$, at that $e^i = s\ u \quad i,\ e_s = \{1,2,\dots,s\}$..

In integral operators (3.3) the numbers

$$\beta_{k,0} = |\sigma_k| + (\nu_k,\sigma_k) \qquad k \in e_s\backslash e^i, \tag{3.4}$$

$$\beta_{k,i_k} = |\sigma_k| + (\nu_k,\sigma_k) - r_{k,i_k}\sigma_{k,i_k} \tag{3.5}$$

at each $k \in e^i$, and at $i = (i_1,\dots,i_s) \in Q$.

The kernels in integral operators (3.3) are sufficiently smooth and finite functions, and are defined by the equalities

$$\Phi_{i,\delta}(\dots) = \prod_{k\in e^i} \Phi_{k,\delta_k,i_k}\left(\frac{y_k}{\upsilon_k^{\sigma_k}}\right) \prod_{k\in e_s/e^i} \Phi_{k,\delta_k,0}\left(\frac{y_k}{H_k^{\sigma_k}}\right), \tag{3.6}$$

and the vectors

$$\begin{cases} \dfrac{y_k}{\upsilon_k^{\sigma_k}} = \left(\dfrac{y_{k,1}}{\upsilon_k^{\sigma_{k,1}}}, ..., \dfrac{y_{k,n_k}}{\upsilon_k^{\sigma_{k,n_k}}}\right), \\ \dfrac{y_k}{H_k^{\sigma_k}} = \left(\dfrac{y_{k,1}}{H_k^{\sigma_{k,1}}}, ..., \dfrac{y_{k,n_k}}{H_k^{\sigma_{k,n_k}}}\right) \end{cases} \tag{3.7}$$

for corresponding $k \in e_s$, at that the support

$$su \quad \Phi_{k,i_k,\delta_k}(y_k) \tag{3.8}$$

belongs to the set

$$\{y_k \in E_{n_k};\ 0 < y_{k,j}\delta_{k,j} \leq 1 \quad (j = 1, 2, ..., n_k)\} \tag{3.9}$$

for corresponding $k \in e_s$.

3.2. Let "nonnegative vector"

$$h = (h_1; ...; h_s) \tag{3.10}$$

with the coordinates-vectors $h_k = (h_{k,1}, ..., h_{k,n_k})$ $(k = 1, 2, ..., s)$ define the surface

$$\Gamma_m + h = \{x^* + h;\ x^* \in \Gamma_m\}, \tag{3.11}$$

i.e., this set of the points

$$x^* + h = T(x') + h$$

at all $x' = (x'_1; ...; x'_\alpha) \in \Omega_m \subset E_m$, at that we choose a system of coordinates and vector (3.10) such that

$$\Gamma_m + h \subset G. \tag{3.13}$$

Note that

$$\left\| D^\nu f|_{\Gamma_m + h} \right\|_{L_q(\Gamma_m + h)} \leq C \left\{ \int\limits_{\Omega_m} \left| D^\nu f\left(T(x') + h\right)\right|^q dx' \right\}^{\frac{1}{q}} \tag{3.14}$$

at $1 < q < \infty$.

3.3.

coincides with surface (3.11).

We define a set of auxiliary functions coincident on corresponding parts of the surface $\Gamma_m + h$, with contraction $D^\nu f|_{\Gamma_m+h}$ on this part of surface.

This set of auxiliary functions is defined by the equalities

$$f_{\nu;\Gamma_m+h}\left(T\left(x'\right)+h\right) = \sum_{i=(i_1,\dots i_s)\in Q} A^*_{i,\delta^\mu} f\left(T\left(x'\right)+h\right) \tag{3.18}$$

at all $\mu = 1, 2, ..., K$.

Here

$$A^*_{i,\delta^\mu} f\left(T\left(x'\right)+h\right) = c_i \left(\prod_{k\in e_s\setminus e^i} H_k^{-\beta_{k,0}}\right) \int\limits_{\vec{0}}^{\vec{H}} \prod_{k\in e^i} \frac{d\upsilon_k}{\upsilon_k^{1+\beta_{k,i_k}}} \times$$
$$\times \int\limits_{E_n} \chi\left(G_\mu + R_{\delta^\mu}\right) D^{r^i} f\left(T\left(x'\right)+h\right) \Phi_{i,\delta^\mu}\left(\dots\right) dy, \tag{3.19}$$

at that all the notation from (3.3) are remained, and the function

$$\chi = \chi\left(G_\mu + R_{\delta^\mu}\right) \tag{3.20}$$

is a characteristic function of the set $G_\mu + R_{\delta^\mu}$ for corresponding $\mu = 1, 2, ..., K$.

It follows from inequality (3.14) and equality (3.18) that

$$\left\| D^\nu f|_{\Gamma_m+h}\right\|_{L_q(\Gamma_m+h)} \leq$$
$$\leq C \sum_{\mu=1}^{K} \sum_{i=(i_1,\dots i_s)\in Q} \left\| A^*_{i,\delta^\mu} f\left(T\left(\cdot\right)+h\right)\right\|_{L_q(E_m)} \tag{3.21}$$

consequently, the proof of the theorem is led to the estimations

$$\left\| A^*_{i,\delta^\mu} f\left(T\left(\cdot\right)+h\right)\right\|_{L_q(E_m)} \leq C \left(\prod_{k=1}^{s} H_k^{\varkappa_{k,i_k}}\right) \left\| D^{r^i} f\right\|_{L_q(G_\mu + R_{\delta^\mu})} \tag{3.22}$$

of integral operators (3.3) uniformly with respect to vector (3.10).

eferences

[1]. Maksudov F.G., Jabrayilov A.J. *Method of integral representations in theory of spaces.* Baki, "Elm", 2000, I v., 200p. (Russian)

[2]. Besov O.V., Il'in V.P., Nikolskii S.M. *Integral representations of functions and embedding theorems.* Moscow, "Nauka", 1975, 480p. (Russian)

[3]. Sobolev S.L. *Some applications of functional analysis in mathematical physics.* L. Pub. House LSU, 1950., 255p. (Russian)

[4]. Nikolskii S.M. *Approxiamtion of multi-variable functions and embedding theorems.* Moscow, "Nauka", 1977, 455p. (Russian)

[5]. Jabrayilov A.J. *To theory of "Embedding theorem"*. Trudy MIAN SSSR, 1967, v.89, pp. 80-118. (Russian)

Irina V. Mayorova
Azerbaijan University of Architecture and Construction
5, A.Sultanova str., AZ1073, Baku, Azerbaijan.
Tel.: (99412) 431-97-67 (apt.)

Received February 16, 2004; Revised July 12, 2004.
Translated by Mirzoyeva K.S.