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ON THE CHARACTERIZATION OF ZEROS AND FIXED POINT OF MAPPINGS

Abstract

In the paper, using the optimization problems the zeros and fixed points of mappings are investigated.

Let X be a Banach space, $M \subset X$, $f : X \to X$ and $\varphi_\alpha(x) = \|x - f(x)\|^\alpha$. It is clear that if x_0 the fixed point of the function f on the set M, then $\min\{\varphi_\alpha(x) : x \in M\} = \varphi_\alpha(x_0) = 0$, where $\alpha > 0$, and x_0 is a global minimum of the function $\varphi_\alpha(x)$ in the space X and therefore $0 \in \partial\varphi_\alpha(x_0)$.

In the paper it is studied a problem when the point of the minimum of the function φ_α on the set M will be a fixed point of the function f on the set M.

Denote $B^* = \{x^* \in X^* : \|x^*\| \le 1\}$, $g(x) = \|x - f(x)\|$, $g_1(x) = \|x - f(x)\|^2$, $B = \{x \in R^n : \|x\| \le 1\}$.

Lemma 1. *If $f : R^n \to R^n$ satisfies the Lipschitz condition near x_0,* $\det(I - A) \neq 0$ *for $A \in \partial f(x_0)$ and $0 \in \partial g(x_0)$ (or $0 \in \partial g_1(x_0)$), then $f(x_0) = x_0$.*

Proof. By theorem 2.6.6 [1] we get

$$\partial g(x_0) \subset \begin{cases} c\bar{o}\{x^*(1 - \partial f(x_0)) : x^* \in B\}, \text{ for } x_0 - f(x_0) = 0, \\ \\ x^*(I - \partial f(x_0)) : x^* \in R^n, \|x^*\| = 1, \langle x^*, x_0 - f(x_0)\rangle = \|x_0 - f(x_0)\|, \\ \text{ for } x_0 - f(x_0) \neq 0. \end{cases}$$

Since each element of the set $I - \partial f(x_0)$ is a non-degenerate matrix, then $0 \notin \partial g(x_0)$ for $x_0 - f(x_0) \neq 0$. Therefore, if $0 \in \partial g(x_0)$, we get $f(x_0) = x_0$.

The lemma is proved.

It follows from the lemma 1 that if $\bar{x} \in M$ is a minimum of the function g (or g_1) on the set M, f satisfies the Lipschitz condition near $\bar{x}$, $\det(I - A) \neq 0$ for $A \in \partial f(\bar{x})$ and $0 \in \partial g(\bar{x})$ (or $0 \in \partial g_1(\bar{x})$), then $f(\bar{x}) = \bar{x}$.

Assume (see [1]) $T_M(x) = \{v \in X : \forall x_i \in M,\ x_i \to x, \forall t_i \downarrow 0,\ \exists v_i \in X,\ v_i \to v$ that $x_i + t_i v_i \in M\}$, $N_M(x) = \{x^* \in X^* : \langle x^*, v\rangle \le 0\ \forall v \in T_M(x)\}$. Note that, if M is a convex set, then $T_M(x) = cl\left(\underset{h>0}{\cup} \frac{1}{h}(M - x)\right)$.

Theorem 1. *Let $M \subset R^n$ be a closed set, $f : R^n \to R^n$ be a Lipschitz function with a constant L, where $L \in (0,1)$, $f(x) \in x + T_M(x)$ for any $x \in M$. Then there exists a point $\bar{x} \in M$, such that $f(\bar{x}) = \bar{x}$.*

Proof. Assume $g(x) = \|x - f(x)\|$ and let $\bar{y} \in M$. By the condition $\|f(x) - f(\bar{y})\| \le L\|x - \bar{y}\|$. Therefore, $\|f(x)\| \le \|f(\bar{y})\| + L(\|x\| + \|\bar{y}\|)$. Then $g(x) \ge \|x\| - \|f(x)\| \ge (1 - L)\|x\| - \|f(\bar{y})\| - L\|\bar{y}\|$. It is easily verified that the set $\{x \in R^n : g(x) \le \alpha\}$ is compact. Then, by definition g is lower semi-compact. Therefore, by theorem 1.1 [2] the function g attains minimum on the set M at some point $\bar{x}$. Then by the corollary of supposition 2.4.3 and by supposition 2.9.8 [1] we have

$$0 \in \partial\left(g(x) + \delta_M(x)\right)_{x=\bar{x}} \subset \partial g(\bar{x}) + N_M(\bar{x}),$$

where $\delta_M(x) = \begin{cases} 0, x \in M, \\ +\infty, x \notin M. \end{cases}$ We get from theorem 2.6.6 [1]

$$\partial g(\bar{x}) \subset \begin{cases} c\bar{o}\{x^*(1-\partial f(\bar{x})) : x^* \in B\}, \text{ if } \bar{x} - f(\bar{x}) = 0, \\ \\ x^*(I - \partial f(\bar{x})) : x^* \in R^n, \|x^*\| = 1, \langle \bar{x} - f(\bar{x}), x^* \rangle = \|\bar{x} - f(\bar{x})\|, \\ \text{ if } \bar{x} - f(\bar{x}) \neq 0. \end{cases}$$

It is clear that $\partial f(\bar{x}) \subset LB_{n\times n}$, where we denoted by $B_{n\times n}$ a closed unique ball in $R^{n\times n}$. Besides, if $\bar{x} - f(\bar{x}) \neq 0$, then for $G \in \partial f(\bar{x})$ we get $\langle x^*(1-G), f(\bar{x}) - \bar{x} \rangle =$ $= \langle x^*, f(\bar{x}) - \bar{x} \rangle - \langle x^*G, f(\bar{x}) - \bar{x} \rangle \leq -\|\bar{x} - f(\bar{x})\| + \|G\| \|\bar{x} - f(\bar{x})\| < 0$, i.e. $-x^*(I-G) \notin N_M(\bar{x})$, for $G \in \partial f(\bar{x})$. Therefore, if $\bar{x} - f(\bar{x}) \neq 0$, then $0 \notin \partial g(\bar{x}) + N_M(\bar{x})$. This means that $f(\bar{x}) = \bar{x}$. The theorem is proved.

emark 1. Usaing the McShane lemma on continuation of Lipschitz functions in theorem 1 it sufficies to assume $f : M \to R^n$ and f is a Lipschitz function with the constant L, where $L \in (0,1)$.

Theorem 2. *Let $M \subset R^n$, $f : R^n \to R^n$ be a Lipschitz function with the constant $L, \bar{x} \in M$ be a minimum of the function g on the set M and if $x \in M$ and $f(x) \neq x$, then for $G \in \partial f(x)$ there exists $\exists z \in T_M(x)$, that satisfies the inequality $\langle x^*(I-G), z \rangle < 0$, where $x^* \in R^n$, $\|x^*\| = 1$, $\langle x - f(x), x^* \rangle = \|x - f(x)\|$*

emark 2. In lemma 2 the compactness M may be substituted by the condition: M is closed, $\|f\|$ is lower semi-compact, or X is a reflexive Banach space, M is a closed convex set, $\|f\|$ is a convex function and $\|f(x)\| \to \infty$ as $\|x\| \to \infty,\ x \in M$.

Let Y be the ordered Banach space with monotone norm, $f : X \to Y$ be a continuous function, and $e\ \ f = \{(x,y) \in X \times Y : f(x) \le y\}$. Then, from theorem 5.3.17 [4] we have

$$\partial \|f(x)\| \subset \underset{z^* \in \partial\|z\|}{\cup} \{x^* \in X^* : (x^*, z^*) \in N(e\ \ f;\ (x, f(x)))\},$$

where $z = f(x)$, $N(e\ \ f;\ (x, f(x))) = \{(x^*,\ z^*) \in X^* \times Y^* : (x^*,\ z^*)\ (v) \le 0,$ $v \in T\ \ (e\ \ f;\ (x,\ f(x)))\}$.

Lemma 3. *Let Y be the ordered Banach space with monotone norm, M be a compact subset in X, $f : X \to Y$ be a Lipschitz function and from $y^* \in \partial \|f(x)\|$, where $x \in M$, $f(x) \ne 0$, it follows that $-y^* \notin N_M(x)$. Then there exists such a point $\bar{x} \in M$, that $f(\bar{x}) = 0$.*

Lemma 3 is proved similar to lemma 2.

Note that similarly we can get the analogy of theorem 2 in the case, when X is an ordered Banach space with monotone norm and $f : M \to X$ is a continuous function.

Let's consider a subdifferential of abstractive function. For simplicity let X and Y be Banach spaces. We denote a set of linear continuous operators from X to Y by $L(X,\ Y)$.

A scalar subdifferential of functions $f : M \to Y$ at the point x is said to be a closed convex set M from $L(X,\ Y)$ that satisfies the equality: $\partial \langle y^*,\ f(x)\rangle = y^* \circ M$ for any $y^* \in Y^*$ and we denote it by $\partial_c f(x)$.

The sense of the equality $\partial \langle y^*,\ f(x)\rangle = y^* \circ M$ is in that every element $x^* \in \partial \langle y^*,\ f(x)\rangle$ may be represented in the form $\langle x^*,\ v\rangle = \langle y^*,\ Av\rangle$ for any $v \in X$, where $A \in M$.

Note that when X is a Banach space using the notion of scalar subdifferential we can get the analogy of theorem 2.

Lemma 4. *Let $\varphi : X \to R$ be a continuous function in the vicinity of x_0, $q(x) = |\varphi(x)|$ and $q(x_0) > 0$. Then*

$$\partial q(x_0) = \begin{cases} \partial\varphi(x_0) : \varphi(x_0) > 0, \\ -\partial\varphi(x_0) : \varphi(x_0) < 0. \end{cases}$$

Proof. Since φ is continuous at the point x_0, then from the definition of generalized derivative with respect to direction we have:

$$q^\circ(x_0; v) = \lim_{\varepsilon\downarrow 0} \limsup_{\substack{y \downarrow_\varphi x_0 \\ t\downarrow 0}} \inf_{\omega \in v + \varepsilon B} \frac{q(y + t\omega) - q(y)}{t} = \begin{cases} \varphi^\circ(x_0; v) : \varphi(x_0) > 0, \\ \varphi^\circ(x_0; -v) : \varphi(x_0) < 0, \end{cases}$$

where $y \downarrow \varphi^{x_0}$ means that y and $\varphi(y)$ converge to x_0 and $\varphi(x_0)$ respectively. Therefore, if $\varphi(x_0) > 0$, then

$$\partial q(x_0) = \{x^* \in X^* : q^\circ(x_0; v) \ge \langle x^*,\ v\rangle,\ v \in X\} =$$

$$= \{x^* \in X^\circ : \varphi^\circ(x_0; v) \ge \langle x^*,\ v\rangle,\ v \in X\} = \partial\varphi(x_0).$$

It is similarly verified that, if $\varphi(x_0) < 0$, then $\partial q(x_0) = -\partial\varphi(x_0)$. The lemma is proved.

Note that, if $\varphi : X \to R$ is lower semi-continuous at the point x_0 and $\varphi(x_0) > 0$, then $\partial q(x_0) = \partial\varphi(x_0)$.

Let $f = (f_1, ..., f_n) : R^n \to R^n$. Assume $\bar{g}(x) = \sum\limits_{i=1}^{n} |f_i(x) - x_i|$, $\psi(x) = \sum\limits_{i=1}^{n} f_i(x)$.

Theorem 4. *Let $M \subset R^n$ be a compact set, $f_i : R^n \to R$ be a lower semi-continuous (upper semi-continuous) function, $x_i \leq f_i(x)$ $(x_i \geq f_i(x))$, $i = \overline{1,n}$, $f(x) \in x + T_M(x)$ for any $x \in M$, $\bar{x}$be a minimum point of the function $\bar{g}(x)$in the set M, or $dom\ \psi^\circ(\bar{x};\cdot) \cap int\ T_M(\bar{x}) \neq \emptyset$, or $dom\ \psi^\circ(\bar{x};\cdot) - T_M(\bar{x};\cdot) = X$; for $x \in M$ and for any $z^* \in \partial\psi(x)$, where $f(x) \neq x$, it is fulfilled the inequality*

$$\langle z^*, f(x) - x\rangle < \sum_{i=1}^{n} (f_i(x) - x_i) \qquad \left(\langle z^*, f(x) - x\rangle > \sum_{i=1}^{n} (f_i(x) - x_i)\right).$$

Then $f(\bar{x}) = \bar{x}$.

Proof . By Weierstress theorem the function $\bar{g}$ attains minimum on the set M at some point $\bar{x}$. Therefore by theorem 2.9.8 [1] and by supposition 7.6.12 [3] we have

$$0 \in \partial\bar{g}(\bar{x}) + N_M(\bar{x}).$$

Since $\bar{g}(x) = \sum\limits_{i=1}^{n} (f_i(x) - x_i) = \psi(x) - \sum\limits_{i=1}^{n} x_i$, then $\partial\bar{g}(x) = \partial\psi(x) - l$, where $l = (1, 1, ..., 1)$. Then, for $f(x) \neq x$, by the condition we have

$$\langle z^* - l, f(x) - x\rangle = \langle z^*, f(x) - x\rangle - \sum_{i=1}^{n} (f_i(x) - x_i) < 0,$$

for any $z^* \in \partial\psi(x)$, i.e. if $x^* \in \partial\bar{g}(x)$, then $-x^* \notin N_M(x)$. Therefore $0 \in \partial\bar{g}(\bar{x}) + N_M(\bar{x})$ if and only if $f_i(\bar{x}) = \bar{x}_i$.

The second case is similarly proved. The theorem is proved.

It is clear that by changing the condition $f(x) \in x + T_M(x)$ by the condition $x \in f(x) + T_M(x)$ for any $x \in M$, we can get the analogy of theorem 4.

Lemma 5. *If $\min\limits_{x,y\in M} \|f(x) - y\| = \|f(\bar{x}) - \bar{x}\|$ where $\bar{x} \in M \subset X$ and $f(\bar{x}) \in \bar{x} + T_M(\bar{x})$, then $\bar{x}$ is a fixed point of the function f on the set M.*

Proof. By the condition $\min\{\|f(\bar{x}) - y\| : y \in M\} = \|f(\bar{x}) - \bar{x}\|$. Therefore, by theorem 2.9.8 [1] we have $0 \in \partial_y \|f(\bar{x}) - y\|_{y=\bar{x}} + N_M(\bar{x})$. Let $f(\bar{x}) \neq \bar{x}$. It is clear that $\partial_y \|f(\bar{x}) - y\|_{y=\bar{x}} = \{-x^* : x^* \in X^*, \|x^*\| = 1, \langle x^*, f(\bar{x}) - \bar{x}\rangle = \|f(\bar{x}) - \bar{x}\|\}$. Then there exists such $-\bar{x}^* \in \partial_y \|f(\bar{x}) - y\|_{y=\bar{x}}$, that $\bar{x}^* \in N_M(\bar{x})$. Therefore $\|f(\bar{x}) - \bar{x}\| = \langle \bar{x}^*, f(\bar{x}) - \bar{x}\rangle \leq 0$, i.e. $f(\bar{x}) = \bar{x}$. We get contradiction. The lemma is proved.

Let X be a Banach space, $F : X \to 2^X$, $M \subset domF = \{x \in X : F(x) \neq \varnothing\}$, $W(x) = \inf\{\|x - y\| : y \in F(x)\}$, $grF = \{(x, y) \in X \times X : y \in F(x)\}$, $grDF(x_0, y_0) = T_{grF}(x_0, y_0)$, $DF(x_0, y_0)^*(q) = \{\ \ : (q, -\ \ ) \in N_{grF}(x_0, y_0)\}$.

Lemma 6. *Let M be a closed and convex set in X, grF be closed and convex, $\bar{x} \in M$ be a minimum of the function $W(x)$ on the set M, such $\bar{y} \in F(\bar{x})$* that

$W(\bar{x}) = \|\bar{x} - \bar{y}\|$, $x_0 \in M$ and $y_0 \in F(x_0)$ be such that $W(x_0) > W(\bar{x})$ and $W(x_0) = \|x_0 - y_0\|$. Then from $\ \in DF(x_0, y_0)^*(q) + r$, where $(r, q) \in \partial \|x_0 - y_0\|$ it follows that $- \ \notin N_M(x_0)$.

Proof. By theorem 4.5.2 [3] $\ \in \partial W(x_0)$ if and only if there exists such $(r, q) \in \partial \|x_0 - y_0\|$ that $\ \in D(x_0, y_0)^*(q) + r$. By the condition $W(\bar{x}) < W(x_0)$, i.e. $x_0 \neq y_0$. Therefore

$$\partial \|x_0 - y_0\| = \{(x^*, -x^*) : x^* \in X^*, \|x^*\| = 1, \ \langle x^*, x_0 - y_0 \rangle = \|x_0 - y_0\|\}.$$

Hence, we have $\ \in DF(x_0, y_0)^*(-x^*) + x^*$ or $\ - x^* \in DF(x_0, y_0)^*(-x^*)$ for some $(x^*, -x^*) \in \partial \|x_0 - y_0\|$. By definition of adjoined mapping we have

$$\langle \ - x^*, x_0 - x \rangle \geq \langle -x^*, y_0 - y \rangle, \ (x, y) \in grF.$$

Hence it follows that

$$-\langle \ , x - x_0 \rangle \geq \langle x^*, x_0 - y_0 \rangle - \langle x^*, x - y \rangle, \ (x, y) \in grF. \qquad (1)$$

Since $W(\bar{x}) = \max\{\langle z^*, \bar{x} - \bar{y} \rangle : \|z^*\| = 1, z^* \in X^*\}$, then $W(\bar{x}) \geq \langle x^*, \bar{x} - \bar{y} \rangle$. Assuming $x = \bar{x}$, $y = \bar{y}$ from (1) we get, $-\langle \ , \bar{x} - x_0 \rangle > 0$, i.e. $- \ \notin N_M(x_0)$. The lemma is proved.

Lemma 7. *Let M be a closed convex set in X, $int\ M \neq \varnothing$, grF be closed and convex, $y_0 \in F(x_0)$ be such that $W(x_0) = \|x_0 - y_0\|$, where $x_0 \neq y_0$, and $x^* \in DF(x_0, y_0)^*(x^*)$ for $x^* \in X^*, \|x^*\| = 1$, $\langle x^*, x_0 - y_0 \rangle = \|x_0 - y_0\|$ and $0 \neq \partial W(x_0)$. Then $0 \notin \partial W(x_0) + N_M(x_0)$.*

Proof. By theorem 4.5.2 [3] $\ \in \partial W(x_0)$ if and only if there exists such a $(r, q) \in \partial \|x_0 - y_0\|$, that $\ \in DF(x_0, y_0)^*(q) + r$. Since $x_0 \neq y_0$, then $\partial \|x_0 - y_0\| = \{(x^*, -x^*) : x^* \in X^*, \ \|x^*\| = 1, \ \langle x^*, x_0 - y_0 \rangle = \|x_0 - y_0\|\}$. Let $(x^*, -x^*) \in \partial \|x_0 - y_0\|$ be such that $\ \in DF(x_0, y_0)^*(-x^*) + x^*$ or $\ - x^* \in DF(x, y_0)^*(-x^*)$. By definition of adjoined mapping we have

$$\langle \ - x^*, x_0 - x \rangle \geq \langle -x^*, y_0 - y \rangle, \ (x, y) \in grF.$$

Then it is clear that

$$\langle \ , x_0 - x \rangle \geq \langle -x^*, y_0 - y \rangle + \ \langle x^*, x_0 - x \rangle, \ \ (x, y) \in grF. \qquad (2)$$

Since $x^* \in DF(x_0, y_0)^*(x^*)$, then

$$\langle x^*, x_0 - x \rangle \geq \langle x^*, y_0 - y \rangle, \ \ (x, y) \in grF. \qquad (3)$$

It follows from (2) and (3) that $\langle - \ , x - x_0 \rangle \geq 0$ for $(x, y) \in grF$, then we have that $(- \ , z) \geq 0$ for $z \in T_M(x_0)$. Therefore $\langle - \ , z \rangle > 0$ for $z \in int\ T_M(x_0)$. Hence we get $0 \notin \partial W(x_0) + N_M(x_0)$. The lemma is proved.

Theorem 5. *Let M be a closed convex set in X, grF be closed and convex, $M \neq \varnothing$, or $0 \in int(dom\ F - M)$, $\bar{x} \in M$ be a minimum of the function $W(x)$ on the set M and $\bar{y} \in F(\bar{x})$ be such that $W(\bar{x}) = \|\bar{x} - \bar{y}\|$, let for any $x_0 \in M$, and for $y_0 \in F(x_0)$, where $W(x_0) = \|x_0 - y_0\| > 0$, and $x^* \in N^{(y_0)}_{F(x_0)}$, where $\|x^*\| = 1$, $\langle x^*, x_0 - y_0 \rangle = \|x_0 - y_0\|$ there exist such points $\tilde{x} \in M$ and $\tilde{y} \in F(\tilde{x})$ that $\|x_0 - y_0\| > \langle x^*, \tilde{x} - \tilde{y} \rangle$. Then $\bar{x} = \bar{y}$, i.e. $\bar{x} \in F(\bar{x})$.*

Proof. Since $\bar{x} \in M$ minimizes the function W on the set M, then $0 \in \partial W(\bar{x}) + N_M(\bar{x})$. By theorem 4.5.2 [3] $^- \in \partial W(\bar{x})$ if and only if there exists such $(\bar{r}, \bar{q}) \in \partial\|\bar{x} - \bar{y}\|$, that $^- \in DF(\bar{x}, \bar{y})^*(\bar{q}) + \bar{r}$. If $\bar{x} \neq \bar{y}$, then

$$\partial\|\bar{x} - \bar{y}\| = \{(x^*, -x^*) : x^* \in X^*,\ \|x^*\| = 1,\ \ \langle x^*, \bar{x} - \bar{y}\rangle = \|\bar{x} - \bar{y}\|\}.$$

Let $(\bar{x}^*, -\bar{x})^* \in \partial\|\bar{x} - \bar{y}\|$ be such that $^- - \bar{x}^* \in DF(\bar{x}, \bar{y})^*(-\bar{x}^*)$. Hence, we have

$$\langle ^- - \bar{x}^*, \bar{x} - x\rangle \geq \langle -\bar{x}^*, \bar{y} - y\rangle,\ x \in M\ \ \forall y \in F(x).$$

It is clear that $\bar{x}^* \in N^{(\bar{y})}_{F(\bar{x})}$ and

$$\langle -^-, x - \bar{x}\rangle \geq \langle \bar{x}^*, \bar{x} - \bar{y}\rangle - \ \langle \bar{x}^*, x - y\rangle,\ x \in M,\ \ y \in F(x).$$

By the condition there exists such $\tilde{x} \in M$ and $\tilde{y} \in F(\tilde{x})$ that $\langle \bar{x}^*, \bar{x} - \bar{y}\rangle - \langle \bar{x}^*, \tilde{x} - \tilde{y}\rangle > 0$. Therefore, $\langle -^-, \tilde{x} - \bar{x}\rangle > 0$, i.e. $-^- \notin N_M(\bar{x})$. Then, it is clear that $0 \notin \partial W(\bar{x}) + N_M(\bar{x})$, i.e. we get a contradiction. The theorem is proved.

Corollary 2. *If M is a closed convex set in X, int M is non-empty, grF is closed and convex, the point $\bar{x} \in M$ is a minimum of the function $W(x)$ in the set M and $\bar{y} \in F(\bar{x})$ are such that $W(\bar{x}) = \|\bar{x} - \bar{y}\|$ and for any $x^* \in X^*$, $\|x^*\| = 1$ there exist such points $\tilde{x} \in M$ and $\tilde{y} \in F(\tilde{x})$ that $\langle x^*, \tilde{x} - \tilde{y}\rangle \leq 0$, then $\bar{x} = \bar{y}$.*

emark 3. If X is a Hilbert space, then the condition $\|\bar{x} - \bar{y}\| > \langle \bar{x}^*, \tilde{x} - \tilde{y}\rangle$, where $\bar{x}^* \in X^*$, $\|\bar{x}^*\| = 1$, $\langle \bar{x}^*, \bar{x} - \bar{y}\rangle = \|\bar{x} - \bar{y}\|$ is equivalent to the condition: $\|\bar{x} - \bar{y}\|^2 > \langle \bar{x} - \bar{y},\ \tilde{x} - \tilde{y}\rangle$.

emark 4. Let $\bar{y} \in F(\bar{x})$ be such that $W(\bar{x}) = \|\bar{x} - \bar{y}\| > 0$ and for any $x \in M$ the set $F(x)$ be convex, bounded and closed. Show that if the inequality $\rho_x(F(\bar{x}), F(\bar{y})) < \|\bar{x} - \bar{y}\|$ is fulfilled, then there exists such $\tilde{y} \in F(\bar{y})$ that $\langle \bar{x}^*, \bar{x} - \tilde{y}\rangle < \|\bar{x} - \bar{y}\|$ for $\bar{x}^* \in N^{(\bar{y})}_{F(\bar{x})}$, $\|\bar{x}^*\| = 1$, $\langle \bar{x}^*, \bar{x} - \bar{y}\rangle = \|\bar{x} - \bar{y}\|$.

From $\bar{x}^* \in N^{(\bar{y})}_{F(\bar{x})}$ it follows that $\langle \bar{x}^*, \bar{y}\rangle = \max\{\langle \bar{x}^*, y\rangle : y \in F(\bar{x})\}$.

Let $\tilde{y} \in F(\bar{y})$ be such that $\langle \bar{x}^*, \tilde{y}\rangle = \max\{\langle \bar{x}^*, z\rangle : z \in F(\bar{y})\}$. Using the formula

$$\rho_x(A, B) = \sup\{|S_A(x^*) - S_B(x^*)| : x \in X^*,\ \|x^*\| \leq 1\},$$

where A and B are closed bounded sets in X, we get

$$|\langle \bar{x}^*, \bar{y} - \tilde{y}\rangle| = \left|\max_{y \in F(\bar{x})} \langle \bar{x}^*, y\rangle - \max_{y \in F(\bar{y})} \langle \bar{x}^*, z\rangle\right| \leq \rho_x(F(\bar{x}), F(\bar{y})) < \|\bar{x} - \bar{y}\|.$$

Lemma 8. *If $\min\limits_{x,z \in M} \inf\limits_{y \in F(x)} \|z - y\| = \min\limits_{y \in F(\bar{x})} \|\bar{x} - y\| = \|\bar{x} - \bar{y}\|$, where $\bar{y} \in F(\bar{x})$, $\bar{x} \in M$, $F(\bar{x})$ is a closed convex set and $F(\bar{x}) \cap (\bar{x} + T_M(\bar{x})) \neq \varnothing$, then $\bar{y} = \bar{x}$, i.e. $\bar{x} \in F(\bar{x})$.*

Proof. Assume $\Phi(z, x) = \inf\{\|z - y\| : y \in F(x)\}$. Since $\min\limits_{z \in M} \Phi(z, \bar{x}) = \Phi(\bar{x}, \bar{x}) = \|\bar{x} - \bar{y}\|$, then $0 \in \partial_z \Phi(z, \bar{x})_{z = \bar{x}} + N_M(\bar{x})$. Using the supposition 4.5.1 [3] we get that, if $\bar{y} \neq \bar{x}$, then $x^* \in \partial_z \Phi(z, \bar{x})_{z = \bar{x}}$ if and only if, then $x^* \in N^{(\bar{y})}_{F(\bar{x})}$, $\|x^*\| = 1$, $\langle x^*, \bar{x} - \bar{y}\rangle = \|\bar{x} - \bar{y}\|$. Let $\bar{x}^* \in \partial_z \Phi(z, \bar{x})_{z = \bar{x}}$ be such that $-\bar{x}^* \in N_M(\bar{x})$. Then by the condition, we get

$$\max\{\langle y - \bar{x},\ \bar{x}^*\rangle : y \in F(\bar{x})\} = -\|\bar{x} - \bar{y}\| \geq 0.$$

Hence it follows $\bar{y} = \bar{x}$. The lemma is proved.

Let X and Y be Banach spaces, $F_1 : X \to 2^Y$, $W_0(x) = \inf\{\|y\| : y \in F_1(x)\}$, $\bar{W}(x, y^*) = \inf\{\langle y, y^* \rangle : y \in F_1(x)\}$. If $F_1(x)$ is convex and closed, then (see [6]) $W_0(x) = \sup\{\bar{W}(x, y^*) : \|y^*\| \le 1\}$. Therefore, if grF_1 is convex and closed, then $x \to W_0(x)$ is a convex function. It is clear $dom\ F_1^{-1} = F_1(X)$.

Lemma 9. *Let grF_1 be convex and closed, $\bar{y} \in F_1(\bar{x})$ where $\bar{y} \in int\ domF_1^{-1}$, such that $W_0(\bar{x}) = \|\bar{y}\|$ and $0 \in \partial W_0(\bar{x})$. Then $\bar{y} = 0$, i.e. $0 \in F_1(\bar{x})$.*

Proof. By theorem 4.5.2 [3] $^{-} \in \partial W_0(\bar{x})$ if and only if there exists such $\bar{q} \in \partial\|\bar{y}\|$, that $^{-} \in DF_1(\bar{x}, \bar{y})^*(\bar{q})$. Since $\bar{y} \in int\ dom\ F_1^{-1}$, then $domD\left(F_1^{-1}\right)(\bar{y}, \bar{x}) = Y$. Then by lemma 2.1.2 [7], we get that $D\left(F_1^{-1}\right)(\bar{y}, \bar{x})^* = D(F_1)((\bar{x}, \bar{y})^*)^{-1}$ is bounded.

By lemma 2.1.1 [7] adjoined mapping $D\left(F_1^{-1}\right)(\bar{y}, \bar{x})^*$ is bounded, if and only if, then $D\left(F_1^{-1}\right)(\bar{y}, \bar{x})^*(0) = \{0\}$. Since $^{-} = 0$, we get that $\bar{q} = 0$, i.e. $0 \in \partial\|\bar{y}\|$. Hence, it follows that $\bar{y} = 0$. The lemma is proved.

Theorem 6. *Let $M \subset domF_1$ be a closed convex set, grF_1 be closed and convex, $\bar{x} \in M$ be a minimum of the function $W_0(x)$ on the set M and $\bar{y} \in F_1(\bar{x})$ be such that $W_0(\bar{x}) = \|\bar{y}\|$. Besides, let either $int\ M \ne \varnothing$ or $W_0(x)$ be continuous at some point $x_1 \in M$. Then, if for any $x_0 \in M$ and $y_0 \in F_1(x_0)$, where $W_0(x_0) = \|y_0\| > 0$, and for $-y^* \in N_{F(x_0)}^{(y_0)}$, where $\|y^*\| = 1$, $\langle y^*, y_0 \rangle = \|y_0\|$, there exist such points $\tilde{x} \in M$ and $\tilde{y} \in F_1(\tilde{x})$, that $\|y_0\| > \langle y^*, \tilde{y} \rangle$ then $\bar{y} = 0$, i.e. $0 \in F_1(\bar{x})$.*

Proof. Since $\bar{x} \in M$ minimizes the function W_0 on the set M, then by theorem 4.4 [2] $0 \in \partial W_0(\bar{x}) + N_M(\bar{x})$. By theorem 5.4.2 [3] $^{-} \in \partial W_0(\bar{x})$ if and only if there exists such $\bar{q} \in \partial\|\bar{y}\|$, that $^{-} \in \partial F_1(\bar{x}, \bar{y})^*(\bar{q})$. If $\bar{y} \ne 0$, then

$$\partial\|\bar{y}\| = \{\bar{y}^* \in Y^* : \|\bar{y}^*\| = 1,\ \langle \bar{y}^*, \bar{y} \rangle = \|\bar{y}\|\}.$$

Let $\bar{y}^* \in \partial\|\bar{y}\|$ be such that $^{-} \in DF_1(\bar{x}, \bar{y})^*(\bar{y}^*)$. Then, it is clear that

$$\langle ^{-}, \bar{x} - x \rangle \ge \langle \bar{y}^*, \bar{y} - y \rangle,\ x \in M,\quad y \in F_1(x).$$

By the condition there exists $\tilde{x} \in M$ and $\tilde{y} \in F_1(\tilde{x})$, that $\|\bar{y}\| = \langle \bar{y}^*, \bar{y} \rangle > \langle \bar{y}^*, \tilde{y} \rangle$. Then it is clear that $\langle ^{-}, \bar{x} - \tilde{x} \rangle > 0$, i.e. $-^{-} \notin N_M(\bar{x})$. Hence, we have $0 \notin \partial W_0(\bar{x}) + N_M(\bar{x})$. The obtained contradiction means that $\bar{y} = 0 \in F_1(\bar{x})$. The theorem is proved.

Let $a : X \to 2^Y$, $W_1(x) = \inf\{\|x - y\|^2 : y \in a(x)\}$, $Da(x_0, y_0)x = \{y \in X : (x, y) \in T_{gra}(x_0, y_0)\}$, $M \subset dom\ a$. It there exists such a vicinity U of the point x_0 and a compact $V \subset X$, that $a(U) \subset V$ and $a(x)$ is non-empty and compact for all $x \in U$, then a is said to be uniformly compact at the point x_0.

Lemma 10. *Let X be a Hilbert space, many-valued mapping a be closed, $x_0 \in M$ and $y_0 \in a(x_0)$ be such that $W_1(x_0) = \|x_0 - y_0\|^2$, the set $a(x_0)$ be convex, many valued mapping a be uniformly compact at the point x_0, $intT_M(x_0) \ne \varnothing$, $T_M(x_0) \subset dom\ Da(x_0, y_0)$ and $(x_0 - y_0) \in Da(x_0, y_0)^*(x_0 - y_0)$. Then, if $x_0 \ne y_0$ and $0 \notin \partial W_1(x_0)$, then $0 \notin \partial W_1(x_0) + N_M(x_0)$.*

Proof. By theorem 2.11 [6] $^{-} \in \partial W_1(x_0)$, then $(^{-}, 0) \in (2(x_0 - y_0), -2(x_0 - y_0)) + N_{gr}(x_0, y_0)$ or $^{-} - 2(x_0 - y_0)^* \in Da(x_0, y_0)^*(2(y_0 - x_0))$. Therefore, $\langle ^{-} - 2(x_0 - y_0), x \rangle + 2\langle x_0 - y_0, y \rangle \le 0$ for $(x, y) \in T_{gra}(x_0, y_0)$. Then $\langle -^{-}, x \rangle \ge 2\langle y_0 - x_0, x \rangle + 2\langle x_0 - y_0, y \rangle$ for $(x, y) \in T_{gra}(x_0, y_0)$. By the condition $(x_0 - y_0) \in$

$Da(x_0, y_0)^*(x_0 - y_0)$, i.e. $-\langle x_0 - y_0, x\rangle + \langle x_0 - y_0, y\rangle \geq 0$ for $(x, y) \in T_{gra}(x_0, y_0)$. Since $T_M(x_0) \subset dom\ Da(x_0, y_0)$, then $\langle -\bar{\ }, x\rangle \geq 0$ for $x \in T_M(x_0)$. It is clear that $\bar{\ } \neq 0$, therefore for $z \in int\ T_M(x_0)$ the inequality $\langle -\bar{\ }, z\rangle \geq 0$ is fulfilled, i.e. $-\bar{\ } \notin N_M(x_0)$. Hence, we have $0 \notin \partial W_1(x_0) + N_M(x_0)$. The lemma is proved.

Theorem 7. *Let X be a Hilbert space, a multi-value mapping a be closed, the set $a(x)$ be non-empty and convex for $x \in M$, $\bar{x} \in M$ be a minimum of the function $W_1(x)$ on the set M and $\bar{y} \in a(\bar{x})$ be such that $W_1(\bar{x}) = \|\bar{x} - \bar{y}\|^2$ there exist a hypertanget to M at the point $\bar{x}$, $dom\ Da(\bar{x}, \bar{y}) \cap int\ T_M(\bar{x}) \neq \varnothing$, the mapping a uniformly compact at the point $\bar{x}$, for any $x_0 \in M$ and for $y_0 \in a(x_0)$, where $W_1(x_0) = \|x_0 - y_0\|^2 > 0$ there exist such points $\tilde{x} \in T_M(x_0)$ and $\tilde{y} \in Da(x_0, y_0)\tilde{x}$ that $\langle x_0 - y_0, \tilde{x} - \tilde{y}\rangle < 0$. Then $\bar{x} = \bar{y}$, i.e. $\bar{x} \in a(\bar{x})$.*

Proof. By theorem 11.2 [6] $\bar{\ } \in \partial W_1(\bar{x})$, then $(\bar{\ }, 0) \in 2(\bar{x} - \bar{y}, \bar{y} - \bar{x}) + N_{gra}(\bar{x}, \bar{y})$. Since

$$W_1^0(\bar{x}; \upsilon) = \sup\{\langle\ , \upsilon\rangle :\ \in \partial W_1(\bar{x})\} \leq$$

$$\leq \sup\{\langle\ , \upsilon\rangle : (\ , 0) \in 2(\bar{x} - \bar{y}, \bar{y} - \bar{x}) + N_{gra}(\bar{x}, \bar{y})\} =$$

$$= \sup\{2\langle \bar{x} - \bar{y}, \upsilon\rangle + \langle x^*, \upsilon\rangle : (x^*, 2(\bar{x} - \bar{y})) \in N_{gr}(\bar{x}, \bar{y})\},$$

then $dom\ W_1^0(\bar{x}; \cdot) \supset dom\ Da(\bar{x}, \bar{y})$. By the condition $\bar{x} \in M$ minimizes the function W_1 on the set M, then by theorem 2.9.8 [1] we get $0 \in \partial W_1(\bar{x}) + N_M(\bar{x})$. Let $\bar{x} = \bar{y}$. Since $(\bar{\ } - 2(\bar{x} - \bar{y}),\ 2(\bar{x} - \bar{y})) \in N_{gra}(\bar{x}, \bar{y})$, then

$$\langle \bar{\ } - 2(\bar{x} - \bar{y}), x\rangle + 2\langle \bar{x} - \bar{y}, y\rangle \leq 0,\ (x, y) \in T_{gra}(\bar{x}, \bar{y}).$$

Then $\langle -\bar{\ }, x\rangle \geq 2\langle \bar{y} - \bar{x}, x\rangle + 2\langle \bar{x} - \bar{y}, y\rangle$ for $(x, y) \in T_{gra}(\bar{x}, \bar{y})$. By the condition there exists such $(\tilde{x}, \tilde{y}) \in T_{gra}(\bar{x}, \bar{y})$ that $\langle -\bar{\ }, \tilde{x}\rangle \geq 2\langle \bar{y} - \bar{x}, \tilde{x}\rangle + 2\langle \bar{x} - \bar{y}, \tilde{y}\rangle = 2\langle \bar{y} - \bar{x}, \tilde{x} - \tilde{y}\rangle > 0$, i.e. $\langle -\bar{\ }, \tilde{x}\rangle > 0$ and $\tilde{x} \in T_M(\bar{x})$. Therefore $-\ \notin N_M(\bar{x})$. Then it is clear that $0 \notin \partial W_1(\bar{x}) + N_M(\bar{x})$, i.e. we get a contradiction. The theorem is proved.

emark 5. If X is a reflexive Banach space and the square of the norm is everywhere strictly differentiable, then theorem 7 is also true. Besides, we can substitute the convexity of the set $a(x)$ by the condition: the set $\{y \in a(x) : W_1(x) = \|x - y\|^2\}$ consists of a unique point.

By $K_V(X)$ we denote a totality of all non-empty convex compact subsets, and let $a : X \to K_V(X)$. Assume $S_a(x, x^*) = \sup\{\langle x^*, y\rangle : y \in a(x)\}$, where $x^* \in X^*$.

The mapping a is said to be weakly uniformly differentiable (w.u.d.) at the point x_0 the direction of $\bar{x}$ if S_a is lower w.u.d. at the points (x_0, x^*), $x^* \in X^*$, in the direction of $\bar{x}$, i.e. there exists $S_a'(x_0, x^*; \bar{x})$ and

$$\overline{\lim_{t\downarrow 0, z^* \to x^*}} \frac{1}{t}(S_a(x_0 + t\bar{x}, z^*) - S_a(x_0, z^*)) \geq S_a'(x_0, x^*; \bar{x}).$$

Let $z_0 = (x_0, y_0) \in gra$, $\hat{T}_{gra}^H(z_0) = \left\{\bar{z} \in X \times X : \overline{\lim_{t\downarrow 0}} \frac{d_a(z_0 + t\bar{z})}{t} = 0\right\}$, where $d_a(z) = \inf\{\|y - \upsilon\| : \upsilon \in a(x)\}$, $z = (x, y)$ and $\hat{D}_H a(z_0; \bar{x}) = \{\bar{y} \in X : (\bar{x}, \bar{y}) \in \hat{T}_{gra}^H(z)\}$. It is clear that $\hat{D}_H a(z_0; \bar{x}) = \underline{\lim_{t\downarrow 0}} \frac{1}{t}(a(x_0 + t\bar{x}) - y_0)$.

We'll say that many valued mapping a admits the first order approximation on the point $z_0 = (x_0, y_0) \in gra$ in the direction of $\bar{x} \in X$, if for any sequence $\{y_k\}$ is such that as $y_k \in a(x_0 + \varepsilon_k \bar{x})$, $k = 1, 2, ...,$ $\varepsilon_k \downarrow 0$, $y_k \to y_0 \in a(x_0)$ as $k \to \infty$ it is valid $y_k = y_0 + \varepsilon_k z_k + 0(\varepsilon_k)$, where $z_k \in \hat{D}_H a(z_0; \bar{x})$, $\varepsilon_k z_k \to 0$ as $k \to \infty$.

Assume $\gamma(x_0, M) = \{\bar{x} \in X : \exists \varepsilon_0 > 0,\ x_0 + \varepsilon \bar{x} \in M,\ \varepsilon \in [0, \varepsilon_0]\}$.

Theorem 8. *Let a compact set $M \subset X$ be such that for any $x_0 \in M$ the set $\gamma(x_0, M)$ is non-empty, $a : X \to K_V(X)$, for $x_0 \in M$ and $y_0 \in a(x_0)$, where $W_1(x_0) = \|x_0 - y_0\|^2 > 0$, there exists such $\tilde{u} \in \gamma(x_0, M)$ that $\inf\{\langle x_0 - y_0, \tilde{u} - \tilde{v}\rangle : \tilde{v} \in \hat{D}_H a(z_0; \tilde{u})\} < 0$ and one of the conditions be fulfilled:*

1) X is finite-dimensional, mapping a is continuous by Housdorff and w.u.d. for all points $x_0 \in M$ in all directions of u;

2) X is a Hilbert space, the mapping a is upper semi-continuous and at each point (x_0, y_0) (where $x_0 \in M$, $y_0 \in a(x_0)$ and $W_1(x_0) = \|x_0 - y_0\|^2$) it admits the first order approximation in all directions of u.

Then there exist such a point $\bar{x} \in M$ that $\bar{x} \in a(\bar{x})$.

Proof. Having assumed $\Phi(x) = -\varphi(x) = \sup\{-\|x - y\|^2 : y \in a(x)\}$ under conditions 1) of theorem 5.3, under condition 2) from corollary 1 of theorem 7.1 [6] we get

$$\Phi'(x_0; u) = \sup_{v \in \hat{D}_H a(z_0; u)} \langle (-2(x_0 - y_0), 2(x_0 - y_0)), (u, v)\rangle .$$

Hence, we have

$$\varphi'(x_0; u) = \underset{v \in \hat{D}_H a(z_0; u)}{2\inf} \langle (x_0 - y_0, y_0 - x_0),\ (u, v)\rangle = \underset{v \in \hat{D}_H a(z_0; u)}{2\inf} \langle x_0 - y_0,\ u - v\rangle .$$

If the point $\bar{x} \in M$ minimizes the function $\varphi(x)$ on the set M, then $\varphi'(\bar{x}; u) \geq 0$ for $u \in \gamma(\bar{x}; M)$. Since a is upper semi-continuous, then the function φ is lower semi-continuous (see [8]). Therefore, there exist a point $\bar{x} \in M$ which minimizes the function φ on the set M. Let $\bar{y} \in a(\bar{x})$ be such that $W_1(\bar{x}) = \|\bar{x} - \bar{y}\|^2$. If $\bar{x} \neq \bar{y}$, then by the condition there exist such $\bar{u} \in \gamma(\bar{x}; M)$ that $\inf\{\langle \bar{x} - \bar{y}, \bar{u} - \bar{v}\rangle : \bar{v} \in \hat{D}_H a(\bar{z}; \bar{u})\} < 0$, where $z = (\bar{x}, \bar{y})$, i.e. there exit such $\bar{u} \in \gamma(\bar{x}; M)$, that $\varphi'(\bar{x}, \bar{y}) < 0$. We get a contradiction. We have $\bar{x} = \bar{y}$. The theorem is proved.

Note that under condition 1) of theorem 8 the condition $\inf\{\langle x_0 - y_0, \tilde{u} - \tilde{v}\rangle : \tilde{v} \in \hat{D}_H a(z_0; \tilde{u})\} < 0$ is equivalent to the condition $\langle x_. - y_0, \tilde{u}\rangle + W'_a(x_0, y_0 - x_0, \tilde{u}) < 0$, where $W_a(x, x^*) = \inf\{\langle x^*, y\rangle : y \in a(x)\}$.

emark 6. The corresponding results are true for the zeros of many-valued mapping and the obtained results may be generalized for separable local convex spaces. Let $a : M \to 2^Y$, where $M \subset X$, $a(x)$ is non-empty and convex, X and Y be separable local convex spaces. Besides, let V be a convex balanced vicinity of zero in Y^*, and ∂V be a set boundary points of the set V. Denote $K_a(x, y^*) = \inf\{|\langle y^*, y\rangle| : y \in a(x)\}$ and $\Phi(x) = \sup\{K_a(x, y^*) : y^* \in \partial V\}$. It is clear that $\Phi(x) = \sup_{y^* \in V} \inf_{y \in a(x)} \langle y^*, y\rangle$ and zeros of mapping is a minimum of the function Φ and we can similarly show that under same conditions the point of minimum of the function Φ on the set M is the zero of the mapping a.

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