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A PROBLEM ON PERIODIC TRAJECTORIES OF CIRCULAR DYNAMIC BILLIARD AND ITS APPLICATIONS

Abstract

It is considered a problem on the construction of periodic trajectories of billiard in a circle in the presence of small impressed forces. A two-step iteration procedure in non-smooth dynamic system developed by the author is used for its solution. A bundle of two satellites in cyclic orbit is considered as an example.

1. Problem statement.

Mathematical billiard is material point moving by inertia interior to some convex plane domain D with smooth boundary whose reflection occurs by the law of perfectly elastic collision: the absolute value of velocity remains, the reflection angle equals the angle of incidence. The development of this model is a dynamic billiard: some impressed forces affect on the point. The papers [3, 4] are devoted to the investigation of special cases of a dynamic billiard.

We shall consider the impressed forces to be small and time independent. In this case the equations of motion interior to a unique radius circle have the form

$$\ddot{x} = \mu X\left(\mu, x, y, \dot{x}, \dot{y}\right), \quad \ddot{y} = \mu Y\left(\mu, x, \dot{x}, \dot{y}\right), \quad x^2 + y^2 < 1 \tag{1}$$

where the origin OXY coincides with the center of the circle, $\mu << 1$.

If $\mu = 0$, then in intervals between the collisions upon the boundary the trajectories are linear, and all units are equidistant from center [1, 2]. To describe the trajectory we use Birkhoff coordinates s, α representing the parameter of the curve (in the considered case as s we can take an angle between radius drawn to the current point of the curve, and axis OX) and angle between the unit of polygon and tangent to the curve at the reflection point. Each trajectory of a mathematical billiard is uniquely characterized by the sequence $\{s_k, \alpha_k\}$ $(k = 1, 2, ...)$ and for a circular domain we have

$$s_{k+1} = s_k + 2\alpha_k, \quad \alpha_k \equiv \alpha^* = const \qquad (k = 1, 2, ...) \tag{2}$$

It is easily seen that formulae (2) define periodic or quasiperiodic trajectory depending on the number α^*: if $\alpha^* = \pi m/n$, where m and n are mutually prime integers, then the trajectory consists of n units. But if the numbers α^* and π are rationally incommensurable, then the trajectory is quasiperiodic (nonclosed).

The present paper is devoted to the construction of periodic trajectories of system (1), transferring as $\mu \to 0$ to periodic trajectory of system (2).

2. A method for constructing periodic trajectories.

To construct the solutions of system (1) it is necessary to know the initial values of coordinates x and y and their time derivatives. Add to Birkhoff coordinates s and α the velocity value υ_k at the moment of the k-th reflection, then

$$x_k = \cos s_k, \quad y_k = \sin s_k, \quad \dot{x}_k = -\upsilon_k \sin(s_k + \alpha_k), \quad \dot{y}_k = \upsilon_k \cos(s_k + \alpha_k) \qquad (3)$$

Let's solve the Cauchy problem (1), (3) on such a time interval where it is valid the inequality

$$x^2 + y^2 < 1 \qquad (4)$$

The moment that the billiard reaches the frontier corresponds to the converting of inequality (4) into equality. For this moment we calculate the value s_{k+1}, angle of incidence β_{k+1} and velocity υ_{k+1} by the formulae similar to (3). Then we use the reflection law

$$\alpha_{k+1} = \pi - \beta_{k+1} \qquad (5)$$

As a result, by formula (3) we get initial conditions for constructing the next union of a billiard trajectory.

Such an approach admits to construct approximately a trajectory of dynamic billiard on a finite time interval. However, it is not sufficient for qualitative analysis of the system in unlimited time interval, including the construction of periodic trajectories and study of their stability. To solve these problems we use the method developed in [5] and [6] for systems with small parameter and nonanalytic right hand-side.

Let $\alpha^* = \pi m/n$, then formulae (2) define a two-parametric family of periodic trajectories of a mathematical billiard (i.e. in formulae (1) $\mu = 0$). We can take the initial apex coordinate s_1 and velocity of particle υ as parameters. Because of periodicity the $n+1$-th unit of the broken line coincides with its first unit, i.e.

$$s_{n+1} = s_1 \pmod{2\pi}, \quad \alpha_{n+1} = \alpha_1, \quad \upsilon_{n+1} = \upsilon_1 \qquad (6)$$

Under the made assumptions, periodicity conditions (6) are fulfilled for $\mu = 0$, it is necessary to achieve their fulfilment also for $\mu \neq 0$. Let the connection between the characteristics of two successive units be described by the fucntion Γ, i.e.

$$(s_{k+1}, \alpha_{k+1}, \upsilon_{k+1}) = \Gamma(s_k, \alpha_k, \upsilon_k, \mu) \qquad (7)$$

We represent periodicity condition (6) in the form

$$\Gamma^n(s_1, \alpha_1, \upsilon_1, \mu) = (s_1 + 2\pi m, \alpha_1, \upsilon_1) \qquad (8)$$

Because of (2) for $\mu = 0$ the mapping Γ is linear:

$$\Gamma^n(s_1, \alpha_1, \upsilon_1, 0) = (s_1 + 2n\alpha_1, \alpha_1, \upsilon_1) \qquad (9)$$

If $\alpha_1 = \alpha^*$ then identities (8) are fulfilled for any s_1, υ_1.

Expand the function (7) in powers of μ.

$$\Gamma(s,\alpha,\upsilon,\mu)=\Gamma_0(s,\alpha,\upsilon)+\mu\Gamma_1(s,\alpha,\upsilon)+....\Gamma_0(s,\alpha,\upsilon)=(s+2\alpha,\alpha,\upsilon) \quad (10)$$

Then

$$\Gamma^n(s_1,\alpha_1,\upsilon_1,\mu)=(s_1+2n\alpha_1,\alpha_1,\upsilon_1)+$$

$$+\mu\Gamma_1^n+...\Gamma_1^n=\frac{\partial}{\partial\mu}\prod_{k=1}^{n}\Gamma(s_k,\alpha_k,\upsilon_k,0) \quad (11)$$

Condition (8) takes the form

$$(2n\alpha_1-2\pi m,0,0)+\mu\Gamma_1^n(s_1,\alpha_1,\upsilon_1)+O\left(\mu^2\right)=0 \quad (12)$$

The given vector relation is equivalent to the system of three scalar equations with respect to the unknown s_1,α_1,υ_1. For $\mu=0$ this system is degenerate, therefore, its solution for small $\mu\neq 0$ exists only at some additional conditions.

Find the roots of amplitude equations and then depending on their multiplicity we use the two-step iteration procedure constructed in [5, 6].

Note that an important special case of dynamic billiard, when impressed forces posses a generalized potential, i.e. system (1) admits the first integral. In this case we can lower the order of system (12) up to the second order, since from equalities

$$\alpha_{n+1}=\alpha_1, \qquad s_{n+1}=s_1 \quad (13)$$

periodicity of trajectory follows automatically.

3. Two-unit trajectories of body bundle in orbit.

A bundle of satellites moving along cyclic orbit is considered as an example.

Until stretched the cable connecting the satellites doesn't obstruct their separate motion. At stretching moment of cable there happens a collision. Neglecting dissipation we shall consider this collision to be elastic. The length of the cable will be considered negligibly small in comparison with the radius of orbit.

The motion equations of the system with respect to the center of mass O in orbital system of coordinates can be written in Lagrange form with Lagrange function

$$L=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)+\Omega(x\dot{y}-y\dot{x})+\frac{3}{2}\Omega^2x^2, \quad x^2+y^2\leq 1 \quad (14)$$

where Ω is angular velocity of orbital motion, the axis OX is directed along radius drawn from the attraction center, the axis OY is directed along the tangent to orbit. By deriving formula (14) the normalization of a unit of mass and length was performed.

If in (11) we put $\Omega=0$, we get the Lagrangian of a mathematical billiard in cyclic domain. Considering Ω a small parameter we'll construct periodic trajectories of a dynamic billiard (14). From practical point of view two-unit trajectories of a mathematical billiard are most important: the motion of satellites along such a trajectory leads to their collision. This form of accident should be stipulated by designing the cable system, selecting its turning velocity from the safety reason.

The Lagrange equations for fucntion (14) are of the form

$$\frac{d}{dt}(\dot{x}-\Omega y)-3\Omega^2 x=0, \quad \frac{d}{dt}(\dot{y}+\Omega x)=0 \tag{15}$$

The second equation (15) is integrated at once

$$\dot{y}+\Omega x=C=const \tag{16}$$

Putting equality (16) into the first from equation (15), we get

$$\ddot{x}-2\Omega^2 x=C\Omega \tag{17}$$

Using formulae (16) and (17) it is easy to construct a general solution of linear system (15)

$$\begin{gathered} x=-\tfrac{C}{2\Omega}+A\cosh\sqrt{2}\Omega t+B\sinh\sqrt{2}\Omega t, \\ y=D+\tfrac{3}{2}Ct-\tfrac{A}{\sqrt{2}}\sinh\sqrt{2}\Omega t-\tfrac{B}{\sqrt{2}}\cosh\sqrt{2}\Omega t \end{gathered} \tag{18}$$

where A,B,C,D are arbitrary constants. Expressing the latter from initial conditions represent (8) with regard to (3) where $k=1$, $\alpha_1=\pi/2+\xi_1$ in the form

$$\begin{gathered} x(t)=\cos s_1\left(1+\frac{3}{2}\Omega^2\tau^2\right)- \\ -v_1\tau\cos(s_1+\xi_1)\left(1+\frac{1}{3}\Omega^2\tau^2\right)-\frac{1}{2}\Omega v_1\tau^2\sin(s_1+\xi_1) \\ y(t)=\sin s_1-v_1\tau\sin(s_1+\xi_1)\left(1-\frac{1}{6}\Omega^2\tau^2\right)+\frac{1}{2}\Omega v_1\tau^2\cos(s_1+\xi_1)\ . \end{gathered} \tag{19}$$

where $\tau=t-t_1$ and the terms of the third and highest orders of smallness are rejected.

Time in flight determine from the condition $x^2(t)+y^2(t)=1$. As a result of simple but laborious calculations we arrive at:

$$\begin{gathered} \tau=\frac{2\cos\xi_1}{v_1}\left(1+\frac{\Omega}{v_1}\sin\xi_1\right)+\frac{\Omega^2}{v_1^3}\left(2\sin^2\xi_1\cos\xi_1-\Phi\right) \\ \Phi=6\cos\xi_1\cos^2 s_1+\frac{8}{3}\cos^3\xi_1\left(2\cos^2(s_1+\xi_1)-\sin^2(s_1+\xi_1)\right)+ \\ +2\cos^3\xi_1-\frac{44}{3}\cos^2\xi_1\cos s_1\cos(s_1+\xi_1)+\frac{4}{3}\cos^2\xi_1\sin s_1\sin(s_1+\xi_1) \end{gathered} \tag{20}$$

Putting the obtained expression into formula (19), we find

$$x_2 = -\cos(s_1 + 2\xi_1) - 2\mu\sin(s_1 + 2\xi_1)\cos\xi_1 + 6\mu^2\cos s_1\cos^2\xi_1 -$$

$$-\mu^2\left(\cos(s_1+\xi_1)\left(2\sin^2\xi_1\cos(\xi_1-\Phi)\right) + \tfrac{8}{3}\cos^3\xi_1\cos(s_1+\xi_1)\right) -$$

$$-4\mu^2\sin\xi_1\cos^2\xi_1\sin(s_1+\xi_1)$$

$$\dot{x}(\tau) = -v_1\cos(s_1+\xi_1) - 2\Omega\sin(s_1+\xi_1) +$$

$$+\mu^2 v_1\left(6\cos\xi_1\cos s_1 - 4\cos^2\xi\cos(s_1+\xi_1) -\right.$$

$$\left.-2\sin\xi_1\cos\xi_1\sin(s_1+\xi_1)\right) \qquad (21)$$

$$y_2 = -\sin(s_1+\xi_1) + 2\mu\cos\xi_1\cos(s_1+2\xi_1) +$$

$$+\mu^2\left(\Phi - 2\sin^2\xi_1\cos\xi_1\right)\sin(s_1+\xi_1) +$$

$$+\tfrac{4}{3}\mu^2\cos^3\xi_1\sin(s_1+\xi_1) + 4\mu^2\sin\xi_1\cos^2\xi_1\cos(s_1+\xi_1)$$

$$\dot{y}(\tau) = -v_1\sin(s_1+\xi_1) + 2\Omega\cos\xi_1\cos(s_1+\xi_1) +$$

$$+2\mu^2 v_1\cos\xi_1\sin(s_1+2\xi_1)$$

where $\mu = \Omega/v_1$.

To find the characteristics of the second unit of the broken line we use the formula

$$v_2 = \sqrt{(\dot{x}(\tau))^2 + (\dot{y}(\tau))^2}, \quad x_2 = \cos s_2, \quad \dot{x}(\tau) = -\sin(s_2 - \alpha_2) \qquad (22)$$

Putting expression (21) into (22) we get

$$v_2 = v_1 - \frac{3}{2}\Omega\mu^2\sin 2\xi_1\sin 2(s_1+\xi_1) + O\left(\Omega^3\right) \qquad (23)$$

$$s_2 = s_1 + \pi + 2\xi_1 - 2\mu\cos\varepsilon_1 + \mu^2\Psi + O\left(\Omega^3\right)$$

$$\xi_2 = \alpha_2 - \frac{\pi}{2} = \xi_1 + \mu^2\Xi + O\left(\Omega^3\right)$$

$$\Xi = \Psi - \frac{3}{2}\sin 2\xi_1\sin 2(s_1+\xi_1)\,tg(s_1+\xi_1) -$$

$$-2\cos^2\xi_1 tg(s_1+\xi_1)\,2 + 2\frac{\cos\xi_1\cos(s_1+2\xi_1)}{\cos(s_1+\xi_1)}$$

$$\Psi = 2tg(s_1+2\xi_1)\cos^2\xi_1 - 4\frac{\sin\xi_1\cos^2\xi_1\cos(s_1+\xi_1)}{\cos(s_1+2\xi_1)} -$$

$$-\frac{\sin(s_1+\xi_1)}{\cos(s_1+2\xi_1)}\left(\frac{4}{3}\cos^3\xi_1 + 2\sin^2\xi_1\cos\xi_1 + \Phi\right)$$

By composing bifurcation equation (12), for two-unit billiard trajectory we put $\xi_1 = 0$, then because of (23)

$$\begin{gathered} \upsilon_2 = \upsilon_1 + O\left(\mu^3\right), \quad s_2 = s_1 + \pi + 2\xi_1 + O\left(\mu\right), \quad \xi_2 = \mu^2 \Xi\left(s_1, 0\right) + O\left(\mu^3\right) \\ s_3 = s_1 + 4\xi_1 + O\left(\mu\right), \; \xi_3 = \xi_1 + \mu^2\left(\Xi\left(s_1, 0\right) + \Xi\left(s_1 + \pi, 0\right)\right) + O\left(\mu^3\right) \end{gathered} \tag{24}$$

Assuming in (24) $\xi_3 = \xi_1$ after reduction by Ω^2 we get an amplitude equation in the form

$$\sin s_1 \cos s_1 - tg\ s_1 + 1 = 0 \tag{25}$$

Equation (25) has the roots $s_1 = s^* + \pi l,\ l \in Z$, where $s^* \approx 55^o 42'$. The same trajectory of mathematical billiard generating periodical solutions of orbital bundle corresponds to these roots. This trajectory lies on a line inclined to orbit line at an angle of $\pi/2 - s^*$.

Note that a mathematical billiard trajectory generates a whole family of periodic solutions of the considered system whose parameter may be a value of energy integral.

Take into account the linearity of equations (15): change of independent variable by formula $t = \gamma t'$ preserves the form of these equations but with new value of the parameter $\Omega' = \Omega/\gamma$. Therefore, we restrain ourselves to the construction of such a periodic solution for which $\upsilon_1 = 1$, here $\Omega = \mu$.

We'll look for the solution of system (8) at $\upsilon_1 = 1$. As initial values of variables we take $s_1^0 = s^*$, $\alpha_1^0 = \pi/2$. For these values we calculate $\Gamma^2\left(s_1^0, \alpha_1^0, 1, \mu\right)$. To this end we have to compose expressions (18) and then solve the equation $x^2(t) + y^2(t) = 1$ to determine the first collision moment, and provide boundary conditions by formulae (3) and (5). Then all these operations are repeated for the second time. As a result we get the relation

$$s_3^0 = \Gamma_1^2\left(s_1^0, \alpha_1^0, 1, \mu\right) \tag{26}$$

Comparing (26) with (24), we get the first step of iteration in he form

$$\xi_1^1 = \xi_1^0 - \frac{1}{4}\left(s_3^0 - s_1^0\right) \tag{27}$$

To perform the second step of iterative procedure it is necessary [5] to calculate $\Gamma^2\left(s_1^0, \alpha_1^1, 1, \mu\right)$, and then to consider the equation

$$\xi_3^0 = \Gamma_2^2\left(s_1^0, \alpha_1^1, 1, \mu\right) \tag{28}$$

Allowing for (24), we have

$$s_1^1 = s_1^0 - \frac{1}{\Delta}\left(\xi_3^0 - \xi_1^0\right), \quad \Delta = \mu^2 \frac{\partial \Psi}{\partial s} = 4\mu^2\left(\cos 2s^* - \sec^2 s^*\right) \approx -14\mu^2 \tag{29}$$

Then the described procedure is repeated successively to determine by formula (27) and (29) the values $\xi_1^2, s_1^2, \xi_1^3, s_1^3$ and etc. the advantage of this method is in possibility of error estimation at each step of iteration.

The initial estimation may be determined from the comparison of diametral trajectory with parameter s^* and curve (19) with parameters

$$s_1 = s^*, \quad \xi_1 = 0, \quad \upsilon_1 = 1, \; \Omega = \mu$$

Moment $\tau = 2$ corresponds to the tension of cable, here

$$\begin{aligned} x\,(2) &= \left(-1 + \frac{10}{3}\mu^2\right)\cos s_1 - 2\mu \sin s_1 \\ y\,(2) &= \left(-1 + \frac{4}{3}\mu^2\right)\sin s_1 + 2\mu \cos s_1 \end{aligned} \tag{30}$$

Since for diametral trajectory the values of variables are expressed by the same formulae (30), where $\mu = 0$, we can estimate the initial error as 2μ.

If we put $\tau = 1$ in formulae (19), we get the estimate of minimal distance between the satellites in the form

$$r\,(1) = \sqrt{x^2\,(1) + y^2\,(1)} = \frac{1}{2}\mu \tag{31}$$

Proceeding from (31), w conclude that for providing safe distance between the satellites the d velocity of the cable turn must satisfy the inequality

$$\upsilon > \frac{1}{2}\frac{\Omega l^2}{d}$$

where l is the length of the cable.

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