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STRONG SOLVABILITY OF THE FIRST BOUNDARY VALUE PROBLEM FOR DEGENERATE ELLIPTIC-PARABOLIC EQUATIONS OF SECOND ORDER

Abstract

In the work the first boundary value problem is considered for degenerate elliptic-parabolic equations of second order with, generally speaking, discontinuous coefficients. It's supposed that a matrix of senior coefficients satisfies parabolic Cordes condition with respect to space variables. A unique strong (almost everywhere) solvability is established for above mentioned problem in the corresponding weighted Sobolev space.

Introduction.

Let $\mathbf{E}_n$ be an n-dimensional Euclidean space of points $x = (x_1, ..., x_n)$, Ω be a bounded domain in $\mathbf{E}_n$ with a boundary $\partial\Omega$, $\partial\Omega \in C^2$, Q_T be a cylinder $\Omega \times (0, T)$, where $T \in (0, \infty)$.

Let's consider in Q_T the first boundary value problem

$$\mathcal{L}u = \sum_{i.j=1}^{n} a_{ij}(x,t)\, u_{ij} + \psi(x,t)\, u_{tt} - u_t = f(x,t), \tag{1}$$

$$u|_{\Gamma(Q_T)} = 0, \tag{2}$$

where for $i, j = \overline{1, n}$ $u_{ij} = \dfrac{\partial^2 u(x,t)}{\partial x_i \partial x_j}$, $u_i = \dfrac{\partial u}{\partial x_i}$, $u_{it} = \dfrac{\partial^2 u}{\partial x_i \partial t}$, $\Gamma(Q_T) = (\partial\Omega \times [0,T]) \cup (\Omega \times \{(x,t) : t = 0\})$ is a parabolic boundary of the domain Q_T and $\psi(x,t) = \lambda(\rho)\, \omega(t)\, \varphi(T-t)$, $\rho = \rho(x) = dist(x, \partial\Omega)$.

Assume that the coefficients of the operator $\mathcal{L}$ satisfy the following conditions: $\|a_{ij}(x,t)\|$ is a real symmetrical matrix with elements measurable in Q_T and for any $(x,t) \in Q_T$ and $\xi \in \mathbf{E}_n$ the following inequalities are true

$$\gamma|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\, \xi_i \xi_j s \le \gamma^{-1}|\xi|^2 \tag{3}$$

where $\gamma \in (0, 1]$ is a constant

$$\sigma = \sup_{Q_T} \left(\sum_{i,j=1}^{n} a_{ij}^2(x,t) \Big/ \sum_{i=1}^{n} a_{ii}(x,t) \right)^2 < \frac{1}{n - \lambda^2}, \tag{4}$$

where

$$\lambda = \frac{\inf\limits_{Q_T} \sum\limits_{i=1}^{n} a_{ii}(x,t)}{\sup\limits_{Q_T} \sum\limits_{i=1}^{n} a_{ii}(x,t)},$$

$$\lambda(\rho)\geq 0,\quad \lambda(\rho)\in C^1[0,diam\ \Omega],\quad \left|\lambda'(\rho)\right|\leq p\sqrt{\lambda(\rho)},\tag{5}$$

$$\omega(t)\geq 0,\quad \omega(t)\in C^1[0,T],\tag{6}$$

$$\varphi(z)\geq 0,\quad \varphi'(z)\geq 0,\quad \varphi(z)\in C^1[0,T],\ \varphi(0)=\varphi'(0)=0,$$

$$\varphi(z)\geq \beta z\varphi'(z),\tag{7}$$

where p and β are positive constants.

The condition (4) is called Cordes condition and is taken within non-degenerate linear transformation in the following sense: the domain Q_T can be covered by finite number of domains $Q^1,...,Q^M$ so, that in each Q^i such a non-degenerate linear transformation of coordinates exists, that the matrix of senior coefficients of an image of the operator $\mathcal{L}$ satisfies the condition (4) in image of Q^i, $i=\overline{1,M}$.

The purpose of this work is to prove a unique strong (almost everywhere) solvability of the first boundary value problem (1)-(2) in the corresponding weighted Sobolev space for any $f(x,t)\in L_2(Q_T)$. Let's note that for similar equations with one space variable the first fundamental result in this direction was obtained by Keldysh [1]. We will also mention the works [2]-[4] where strong solvability of the boundary value problem (1)-(2) is established for equations with smooth coefficients. For the case when $\psi(x,t)=\varphi(T-t)$ the corresponding result was obtained in the work [5] for equations whose main part satisfies the parabolic Cordes condition. As to the second order elliptic and parabolic equations of non-divergence structure, satisfying the condition of Cordes type, we will mention the works [6]-[13] in this connection. We'd also note that the questions of weak solvability of the first boundary value problem for degenerate second order elliptic-parabolic equations of divergence structure were studied in the works [14]-[15]. As a base of our considerations in the given work we take the coercive estimate for operators of $\mathcal{L}$-type established in he work by the author [16].

1^0. Estimate for a model operator.

At first we introduce some denotations and definitions. Let $W_2^{1,0}(Q_T)$, $W_2^{2,0}(Q_T)$, $W_2^{2,1}(Q_T)$ and $W_{2,\psi}^{2,2}(Q_T)$ be Banach spaces of functions $u(x,t)$ given on Q_T with finite norms

$$\|u\|_{W_2^{1,0}(Q_T)}=\left(\int\limits_{Q_T}\left(u^2+\sum_{i=1}^n u_i^2\right)dxdt\right)^{1/2},$$

$$\|u\|_{W_2^{2,0}(Q_T)}=\left(\int\limits_{Q_T}\left(u^2+\sum_{i=1}^n u_i^2+\sum_{i,j=1}^n u_{ij}^2\right)dxdt\right)^{1/2},$$

$$\|u\|_{W_2^{2,1}(Q_T)}=\|u\|_{W_2^{2,0}(Q_T)}+\|u_t\|_{L_2(Q_T)}$$

$$\|u\|_{W_{2,\psi}^{2,2}(Q_T)}=\left(\int\limits_{Q}\left(u^2+\sum_{i=1}^n u_i^2+\sum_{i,j=1}^n u_{ij}^2+u_t^2+\psi^2(x,t)\,u_{tt}^2+\psi(x,t)\sum_{i=1}^n u_{it}^2\right)dxdt\right)^{1/2}\tag{8}$$

respectively.

Let $\overset{\circ}{W}{}^{2,2}_{2,\psi}(Q_T)$ be a subspace of $W^{2,2}_{2,\psi}(Q_T)$ that has a set of all functions from $C^\infty\left(\overline{Q_T}\right)$ vanishing on $\Gamma(Q_T)$ as a dense set. For $R>0$, $x^0\in \mathbf{E}_n$ we denote a ball $\left\{x:\left|x-x^0\right|<R\right\}$ by $B_R\left(x^0\right)$ and a cylinder $B_R\left(x^0\right)\cap(0,T)$ by $Q_T^R\left(x^0\right)$. Let $\overline{B_R}\left(x^0\right)\subset\Omega$. We say that $u(x,t)\in A\left(Q_T^R\left(x^0\right)\right)$ if $u(x,t)\in C^\infty\left(\overline{Q}_T^R\left(x^0\right)\right)$, $u|_{t=0}=0$ and supp $u\subset\overline{Q}_T^\rho\left(x^0\right)$ for some $\rho\in(0,R)$.

Everywhere further the notation $C(\cdot)$ means that a positive constant C depends only on the contents of brackets.

Our goal is to establish a unique strong solvability of the boundary value problem (1)-(2) by means of coercive estimate obtained in the work [16] and method of continuation by parameter. For this purpose we have to prove independently the solvability of the problem mentioned for some model equation from the class under consideration. As a model operator we take the following one

$$\mathcal{L}_0=\Delta+\varphi(T-t)\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial t},$$

where $\Delta=\sum\limits_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is a Laplace operator and fucntion $\varphi(z)$ satisfies the conditions (7).

Everywhere further we limit ourselves to consideration of the most interesting case, when $\varphi(z)>0$ at $z>0$. If $\varphi(z)\equiv0$, then the equation (1) is parabolic and the corresponding result on solvability of the first boundary value problem was obtained in [8]. But if $\varphi(z)=0$ at $z\in\left[0,z^0\right]$, then the solution of the problem (1)-(2) can be obtained by assembling of the solution $u(x,t)$ of the problem in a cylinder Q_{z^0} and the solution $v(x,t)$ of the first boundary value problem for parabolic equation in a cylinder $\Omega\times\left(z^0,T\right)$ with boundary conditions $v\left(x,z^0\right)=u\left(x,z^0\right)$, $v|_{\partial\Omega\times[z^0,T]}=0$.

Let's fix an arbitrary $\varepsilon\in(0,T)$ and introduce a fucntion $\varphi_\varepsilon(z)$ in the following way: $\varphi_\varepsilon(z)=\varphi(\varepsilon)-\frac{\varphi'(\varepsilon)\varepsilon}{m}+\frac{\varphi'(\varepsilon)}{m\varepsilon^{m-1}}z^m$ at $z\in[0,\varepsilon)$, $\varphi_\varepsilon(z)=\varphi(z)$ at $z\in[\varepsilon,T]$, where $m=\frac{2}{\beta}$. It's easy to see that $\varphi_\varepsilon(z)\in C^1[0,T]$. Let's show that for $z\in[0,T]$

$$\varphi_\varepsilon(z)\geq\frac{1}{2}\varphi(z) \tag{9}$$

It's enough to prove (9) for $z\in[0,\varepsilon)$. It's clear that due to monotonicity of $\varphi(z)$ the inequality (9) will be fulfilled if

$$\varphi(\varepsilon)-\frac{\varphi'(\varepsilon)\varepsilon}{m}\geq\frac{1}{2}\varphi(\varepsilon),$$

or $\varphi(\varepsilon)\geq\frac{2}{m}\varphi'(\varepsilon)\varepsilon$.

But the last estimate is true because of (7). Hence the inequality (9) has been proved.

Without losing of generality we consider $m>1$. Then

$$q_\varepsilon(T)=\sup_{[0,T]}\varphi_\varepsilon'(z)\leq q(T)=\sup_{[0,T]}\varphi'(z). \tag{10}$$

Indeed, $q_\varepsilon(T)\leq\max\left\{\sup\limits_{[0,\varepsilon]}\varphi'(\varepsilon)\left(\frac{z}{\varepsilon}\right)^{m-1},q(T)\right\}=\max\left\{\varphi'(\varepsilon),q(T)\right\}=q(T)$.

We say, that $u(x,t)\in B\left(Q_T^R\left(x^0\right)\right)$, if $u(x,t)\in A\left(Q_T^R\left(x^0\right)\right)$ and, besides, $u|_{t=T}=u_t|_{t=T}=0$.

Let's define the operator

$$\mathcal{L}_\varepsilon=\Delta+\varphi_\varepsilon(T-t)\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial t}.$$

Lemma 1. *If $\varphi(z)$ satisfies the conditions (7), then there exists such $T_1(\varphi,n)$, that at $T\leq T_1$ for any fucntion $u(x,t)\in B\left(Q_T^R\left(x^0\right)\right)$ the estimate is true:*

$$\int\limits_{Q_T^R(x^0)}\left(\sum_{i,j=1}^n u_{ij}^2+u_t^2+\varphi_\varepsilon^2(T-t)\,u_{tt}^2+\varphi_\varepsilon(T-t)\sum_{i=1}^n u_{it}^2\right)dxdt\leq$$

$$\leq\left(1+2(n+1)\,q(T)\right)\int\limits_{Q_T^R(x^0)}\left(\mathcal{L}_\varepsilon u\right)^2dxdt \tag{11}$$

Proof. For simplicity we'll write Q instead of $Q_T^R\left(x^0\right)$

$$\int\limits_Q\left(\mathcal{L}_\varepsilon u\right)^2dxdt\geq\int\limits_Q\sum_{i,j=1}^n u_{ij}^2dxdt+\int\limits_Q\varphi_\varepsilon^2(T-t)\,u_{tt}^2dxdt+\int\limits_Q u_t^2dxdt+$$

$$+2\int\limits_Q\varphi_\varepsilon(T-t)\,u_{tt}\Delta udxdt-2\int\limits_Q\varphi_\varepsilon(T-t)\,u_{tt}u_tdxdt \tag{12}$$

But on the other hand

$$2\int\limits_Q\varphi_\varepsilon(T-t)\,u_{tt}\Delta udxdt-2\int\limits_Q\sum_{i=1}^n\left(\varphi_\varepsilon(T-t)\,u_{ii}\right)_t u_tdxdt$$

$$\left(as\ \ u_{ii}|_{t=0}=u_{ii}|_{t=T}=0\right)=2\int\limits_Q\varphi'_\varepsilon(T-t)\sum_{i=1}^n u_{ii}u_tdxdt-$$

$$-2\int\limits_Q\varphi_\varepsilon(T-t)\sum_{i=1}^n u_{iit}u_tdxdt\geq-q_\varepsilon(T)\int\limits_Q\sum_{i,j=1}^n u_{ij}^2dxdt-$$

$$-nq_\varepsilon(T)\int\limits_Q u_t^2dxdt+2\int\limits_Q\varphi_\varepsilon(T-t)\sum_{i=1}^n u_{it}^2dxdt \tag{13}$$

and similarly

$$-2\int\limits_Q\varphi_\varepsilon(T-t)\,u_{tt}u_tdxdt=-\int\limits_Q\varphi'_\varepsilon(T-t)\,u_t^2dxdt+$$

$$+\varphi_\varepsilon(T)\int\limits_B u_t^2(x,0)\,dx\ \ \left(as\ \ u_t|_{t=T}=0\right)\geq-q_\varepsilon(T)\int\limits_Q u_t^2dxdt\ . \tag{14}$$

Owing to (7) $q(T)\to 0$ at $T\to 0$. Choosing T_1 so small that $(n+1)\,q(T_1)\leq\dfrac{1}{2}$, we have *at* $T\leq T_1$

$$\frac{1}{1-(n+1)\,q(T)}\leq 1+2(n+1)\,q(T)$$

Using this and (10), and proving like in lemma 1, from the work [16], we get the estimate (11) on the base of (12)-(14).

Lemma has been proved.

Lemma 2. *Let $\varphi(z)$ satisfy the conditions (7), $\mathcal{L}$* at $\varepsilon>0$ *have the same meaning as in lemma 1. Then at $T\leq T_2(\varphi,n,\Omega)$ for any function $u(x,t)\in\dot{W}_{2,\varphi_\varepsilon}^{2,2}(Q_T)$ the estimate is true:*

$$\|u\|_{W_{2,\varphi_\varepsilon}^{2,2}(Q_T)}\leq C_1(\varphi,n,\Omega)\,\|\mathcal{L}_\varepsilon u-\mu u\|_{L_2(Q_T)}\,, \tag{15}$$

where $\mu = \frac{1}{T}$; $W^{2,2}_{2,\varphi_\varepsilon}(Q_T)$ *is a Banach space of functions* $u(x,t)$ *given on* Q_T *with the finite norm defined by the equality (8), where function* ψ *is replaced by* φ_ε; $\dot{W}^{2,2}_{2,\varphi_\varepsilon}(Q_T)$ *is a completion of a set of all functions from* $C^\infty(\overline{Q}_T)$ *vanishing on* ∂Q_T *with respect to a norm of the space* $W^{2,2}_{2,\varphi_\varepsilon}(Q_T)$.

Proof . It's enough to prove the lemma for functions $u(x,t) \in C^\infty(\overline{Q}_T)$, $u|_{\partial Q_T=0}$.

Let's note that according to the above mentioned $q(T_1) \le 1$. Then reasoning as in the proof of coercive estimate [16], we derive from (11) the existence of such $T_3(\varphi, n, \Omega) \le T_1$, that if $T \le T_3$, then for any function $v(x,t) \in C^\infty(\overline{Q}_T)$, $v|_{\Gamma(Q_T)} = 0$, $v|_{t=T} = v_t|_{t=T} = 0$ the estimate is true:

$$\|v\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)} \le c_2(\varphi, n, \Omega)\left(\|\mathcal{L}_\varepsilon v\|_{L_2(Q_T)} + \|v\|_{L_2(Q_T)}\right) \tag{16}$$

Let $T \le \frac{T_3}{2}$. We take $R = \frac{T}{4}$, and let $u(x,t) \in C^\infty(\overline{Q}_T)$, $u|_{\partial Q_T} = 0$. Let's consider such a function $\zeta(t) \in C^\infty[0,T]$, that $\zeta(t) = 1$ at $t \in [0, T-R]$, $\zeta(t) = 0$ at $t \in \left[T - \frac{R}{2}, T\right]$, $0 \le \zeta(t) \le 1$ and

$$|\zeta'(t)| \le \frac{C_3}{R}, \quad |\zeta''(t)| \le \frac{C_3}{R^2} \tag{17}$$

Putting in (16) $v(x,t) = u(x,t)\zeta(t)$ and taking into account (17), we get

$$\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_{T-R})} \le C_2\left(\|\mathcal{L}_\varepsilon(u\zeta)\|_{L_2(Q_T)} + \|u\|_{L_2(Q_T)}\right) \le$$

$$\le C_2\left(\|\mathcal{L}_\varepsilon(u)\|_{L_2(Q_T)} + \left(\frac{C_3}{R} + 1\right)\|u\|_{L_2(Q_T)}\right) +$$

$$+\frac{2C_3}{R}\|\varphi_\varepsilon u_t\|_{L_2(Q_T)} + \frac{C_3}{R^2}\|\varphi_\varepsilon u\|_{L_2(Q_T)} \tag{18}$$

From the conditions (7) it follows, that $\sup\limits_{[0,T]}\varphi(z) \le C_4(\varphi)\cdot T$. So, taking into consideration, that $\sup\limits_{[0,T]}\varphi_\varepsilon(z) = \sup\limits_{[0,T]}\varphi(z)$, we conclude

$$\|\varphi_\varepsilon u\|_{L_2(Q_T)} \le C_4 T\|u\|_{L_2(Q_T)} \tag{19}$$

On the other hand for any $\alpha' > 0$ the interpolation inequality takes place

$$\|\varphi_\varepsilon u_t\|_{L_2(Q_T)} \le C_4 T\alpha'\|\varphi_\varepsilon u_{tt}\|_{L_2(Q_T)} + \frac{3}{\alpha'}\|u\|_{L_2(Q_T)} \tag{20}$$

Indeed, let's fix an arbitrary α' and consider for $\nu > 0$ the integral

$$k = \int\limits_{Q_T}\left[v\varphi^2_\varepsilon(T-t)u_{tt} + \frac{1}{\nu}u\right]^2 dxdt.$$

It's clear, that $k \ge 0$. Simultaneously

$$k = v^2\int\limits_{Q_T}\varphi^4_\varepsilon(T-t)u^2_{tt}dxdt + \frac{1}{\nu^2}\int\limits_{Q_T}u^2dxdt +$$

$$+2\int\limits_{Q_T}\varphi^2_\varepsilon(T-t)u_{tt}udxdt \le c^2_4T^2\nu^2\int\limits_{Q_T}\varphi^2_\varepsilon(T-t)u^2_{tt}dxdt +$$

$$+\frac{1}{\nu^2}\int\limits_{Q_T} u^2dxdt-2\int\limits_{Q_T}\varphi_\varepsilon^2(T-t)\,u_t^2dxdt+4\int\limits_{Q_T}\varphi_\varepsilon(T-t)\,\varphi'_\varepsilon(T-t)\,u\;u_tdxdt.$$

Besides, using the fact, that $q(T)\le 1$ as well as the inequality (10), we get

$$4\int\limits_{Q_T}\varphi_\varepsilon(T-t)\,\varphi'_\varepsilon(T-t)\,u\;u_tdxdt\le\int\limits_{Q_T}\varphi_\varepsilon^2(T-t)\,u_t^2dxdt+$$

$$+4\int\limits_{Q_T}\left(\varphi'_\varepsilon(T-t)\right)^2u^2dxdt\le\int\limits_{Q_T}\varphi_\varepsilon^2(T-t)\,u_t^2dxdt+4\int\limits_{Q_T}u^2dxdt \tag{21}$$

From (20)-(21) it follows, that

$$\int\limits_{Q_T}\varphi_\varepsilon^2(T-t)\,u_t^2dxdt\le c_4^2T^2v^2\int\limits_{Q_T}\varphi_\varepsilon^2(T-t)\,u_{tt}^2dxdt+\left(\frac{1}{\nu^2}+4\right)\int\limits_{Q_T}u^2dxdt\;.$$

Now it's enough to put $\nu=\min\{\alpha',1\}$ to prove the inequality (20).
Using (19) and (20) in (18) we conclude that for any $\alpha'>0$ the inequality is true

$$\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_{T-R})}\le C_2\|\mathcal{L}_\varepsilon(u)\|_{L_2(Q_T)}+8\alpha' C_2C_3C_4\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)}+$$

$$+\frac{C_5(\varphi,n,\Omega)}{\alpha' R}\|u\|_{L_2(Q_T)} \tag{22}$$

Let's fix an arbitrary $\alpha>0$ and choose $\alpha'=\dfrac{\alpha}{8C_2C_3C_4}$. Then from (22) it follows, that

$$\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_{T-R})}\le C_2\|\mathcal{L}_\varepsilon(u)\|_{L_2(Q_T)}+\alpha\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)}+\frac{C_6(\varphi,n,\Omega)}{\alpha T^2}\|u\|_{L_2(Q_T)} \tag{23}$$

Similarly we can show that if

$$Q'=\Omega\times(T-2R,T+2R)\,,\;\;Q''=\Omega\times(T-R,T+R)\,,$$

$$S(Q')=\partial\Omega\times[T-2R,T+2R]\,,$$

then for any function $\omega(x,t)\in C^\infty\left(\bar{Q}'\right)$, $\omega|_{S(Q')}=0$ at any $\alpha>0$ the estimate is true

$$\|\omega\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q'')}\le C_2\|\mathcal{L}_\varepsilon\omega\|_{L_2(Q')}+\alpha\|\omega\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q')}+\frac{C_7(\varphi,n,\Omega)}{\alpha T}\|\omega\|_{L_2(Q_T)}\,. \tag{24}$$

Let $Q'_+=\Omega\times(T-2R,T)\,,\;Q'_-=\Omega\times(T,T+2R)\,,\;Q''_+=\Omega\times(T-R,T)$. We extend the fucntion $u(x,t)$ in an odd way and $\varphi_\varepsilon(T-t)$ in an even way through the hyperplane $t=T$ from Q'_+ to Q'_-. We denote the extended functions again by $u(x,t)$ and $\varphi_\varepsilon(T-t)$ respectively. Putting in (24) $\omega=u$ and taking into account the equality

$$\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q'')}\le\sqrt{2}\,\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q'_+)}$$

and similar equalities for norms $\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q')}$, $\|u\|_{L_2(Q')}$ and $\|\mathcal{L}_\varepsilon\omega\|_{L_2(Q')}$, we get

$$\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q''_+)}\le C_2\|\mathcal{L}_\varepsilon u\|_{L_2(Q'_+)}+\alpha\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q'_+)}+\frac{C_7}{\alpha T}\|u\|_{L_2(Q'_+)} \tag{25}$$

Uniting (23), (25) and choosing the corresponding α, we conclude

$$\|u\|^2_{W^{2,2}_{2,\varphi_\varepsilon}(Q_+)}\le C_8(\varphi,n,\Omega)\left(\|\mathcal{L}_\varepsilon u\|^2_{L_2(Q_T)}+\frac{1}{T^2}\|u\|^2_{L_2(Q_T)}\right) \tag{26}$$

On the other hand recalling that $\mu = \frac{1}{T}$, we have

$$\int\limits_{Q_T} (\mathcal{L}_\varepsilon u - \mu u)^2 \, dxdt = \|\mathcal{L}_\varepsilon u\|^2_{L_2(Q_T)} + \mu^2 \|u\|^2_{L_2(Q_T)} -$$

$$-2\mu \int\limits_{Q_T} u \mathcal{L}_\varepsilon u dxdt = \|\mathcal{L}_\varepsilon u\|^2_{L_2(Q_T)} + \mu^2 \|u\|^2_{L_2(Q_T)} + k_1 \tag{27}$$

Besides,

$$k_1 = -2\mu \int\limits_{Q_T} u \left(\Delta u + \varphi_\varepsilon (T-t) u_{tt} - u_t\right) dxdt = 2\mu \int\limits_{Q_T} \sum_{i=1}^{n} u_i^2 dxdt -$$

$$-2\mu \int\limits_{Q_T} \varphi_\varepsilon (T-t) u \; u_{tt} dxdt + \mu \int\limits_{Q_T} \left(u^2\right)_t dxdt \geq$$

$$\geq 2\mu \int\limits_{Q_T} \varphi_\varepsilon (T-t) u_t^2 dxdt - 2\mu \int\limits_{Q_T} \varphi'_\varepsilon (T-t) u \; u_t dxdt. \tag{28}$$

Let's show that for $z \in (0, T)$ the inequality is true

$$\varphi_\varepsilon (z) \geq \beta z \varphi'_\varepsilon (z) . \tag{29}$$

Owing to (7) it's enough to prove (29) only for $z \in (0, \varepsilon)$. But for such z (29) is equivalent to the inequality

$$\varphi (\varepsilon) - \frac{\varphi' (\varepsilon) \varepsilon}{m} \geq \frac{\varphi' (\varepsilon) z^m}{m \varepsilon^{m-1}}, \quad where \quad m = \frac{2}{\beta}.$$

But the last inequality is true, if the estimate takes place

$$\varphi (\varepsilon) \geq \frac{2}{m} \varphi' (\varepsilon) \varepsilon \tag{30}$$

Now it is sufficient to note that (30) is fulfilled owing to (7). Hence from (28), (29) and (10) we obtain

$$k_1 \geq -\frac{\mu}{2} \int\limits_{Q_T} \frac{\left[\varphi'_\varepsilon (T-t)\right]^2}{\varphi_\varepsilon (T-t)} u^2 dxdt \geq \frac{\mu}{2\beta} \int\limits_{Q_T} \frac{\varphi'_\varepsilon (T-t)}{T-t} u^2 dxdt \geq$$

$$\geq \frac{-\mu q (T) T}{2\beta} \int\limits_{Q_T} \frac{u^2}{(T-t)^2} dxdt. \tag{31}$$

We apply the Hardy inequality according to which

$$\int\limits_{Q_T} \frac{u^2}{(T-t)^2} dxdt \leq 4 \int\limits_{Q_T} u_t^2 dxdt \; . \tag{32}$$

Then from (27), (31) and (32) we conclude

$$\|\mathcal{L}_\varepsilon u\|^2_{L_2(Q_T)} + \mu^2 \|u\|^2_{L_2(Q_T)} \leq \|\mathcal{L}_\varepsilon u - \mu u\|^2_{L_2(Q_T)} + \frac{2q (T)}{\beta} \|u\|^2_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)} \tag{33}$$

Now we'll choose such a small $T_4 (\varphi, n, \Omega, \beta)$, that $q (T_4) \leq \dfrac{\beta}{4C_8}$ and fix $T_2 = \min \left\{ \frac{T_3}{2}, T_4 \right\}$.

Then from (26) and (33) the needed estimate (15) follows.
Lemma has been proved.

2^0. Solvability of the problem for a model equation.
Let's consider the operator

$$\mathcal{L}'_0 = \Delta + \psi(x,t)\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t},$$

where $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is Laplace operator.

Lemma 3. *If $\psi(x,t)$ satisfies the conditions (5)-(7) then at $T \le T_5(\psi)$; $\tau \in [0,1]$ for any fucntion $u(x,t) \in A\left(Q_T^R(x^0)\right)$ the estimate is true*

$$\int\limits_{Q_T^R(x^0)} \left(\sum_{i,j=1}^{n} u_{ij}^2 + u_t^2 + \psi^2(x,t)\, u_{tt}^2 + \psi(x,t)\sum_{i=1}^{n} u_{it}^2\right) dxdt \le$$

$$\le (1 + S_2 D(T)) \int\limits_{Q_T^R(x^0)} \left(\mathcal{L}'_0 u - \frac{\tau}{T}u\right)^2 dxdt\ , \tag{34}$$

where $S_2 = S_2(\psi, n)$ is some constant

$$D(T) = q_1(T) + q(T),$$

$$q_1(T) = \sup_{t\in[0,T]} \varphi(t),\quad q(T) = \sup_{t\in[0,T]} \varphi'(t).$$

Proof. It's enough to consider the case of $\tau > 0$. We denote $\frac{\tau}{T}$ by μ'. We have

$$I_1 = \int\limits_Q \left(\mathcal{L}'_0 u - \mu' u\right)^2 dxdt = \int\limits_Q \left(\mathcal{L}'_0 u\right)^2 dxdt +$$

$$+ (\mu')^2 \int\limits_Q u^2 dxdt - 2\mu' \int\limits_Q u\Delta u dxdt + 2\mu' \int\limits_Q u\ u_t dt - 2\mu' \int\limits_Q \psi(x,t)\, u_{tt} u dxdt. \tag{35}$$

In [16] the estimate has been obtained

$$\int\limits_Q \left(\sum_{i,j=1}^{n} u_{ij}^2 + u_t^2 + \psi^2(x,t)\, u_{tt}^2 + \psi(x,t)\sum_{i=1}^{n} u_{it}^2\right) dxdt \le$$

$$\le (1 + D(T)\times S) \int\limits_Q \left(\mathcal{L}'_0 u\right)^2 dxdt\ ,$$

where $S(\psi, n)$ is some constant.
We can rewrite it in the following way

$$\int\limits_Q \left(\mathcal{L}'_0 u\right)^2 dxdt \ge \frac{1}{1 + SD(T)} \int\limits_Q \left(\sum_{i,j=1}^{n} u_{ij}^2 + u_t^2 + \psi^2 u_{tt}^2 + \psi \sum_{i=1}^{n} u_{it}^2\right) dxdt.$$

But

$$\frac{1}{1 + SD(T)} = 1 - \frac{SD(T)}{1 + SD(T)} \ge 1 - SD(T),$$

and

$$\int_Q (\mathcal{L}_0' u)^2 \, dxdt \geq (1 - SD(T)) \int_Q \left(\sum_{i,j=1}^n u_{ij}^2 + u_t^2 + \psi^2 u_{tt}^2 + \psi \sum_{i=1}^n u_{it}^2 \right) dxdt.$$

We will use the obtained estimate to estimate the first addend in (35).
For the third addend in (35) we've

$$-2\mu' \int_Q u \Delta u dxdt = 2\mu' \int_Q \sum_{i=1}^n u_i^2 dxdt \geq 0,$$

and for the fourth one

$$2\mu' \int_Q u u_t dxdt = \mu' \int_B u^2 (x, T) \, dx \geq 0,$$

Let's consider the fifth addend in (35) in detail:

$$-2\mu' \int_Q \psi(x,t) \, u_{tt} u dxdt = -2\mu' \int_Q \varphi(T-t) \, \lambda(x) \, \omega(t) \, u \; u_{tt} dxdt =$$

$$= -2\mu' \int_Q \psi(x,t) \, u_t^2 dxdt - 2\mu' \int_Q \varphi'(T-t) \, \lambda(x) \, \omega(t) \, u \; u_t dxdt +$$

$$+2\mu' \int_Q \varphi(T-t) \, \lambda(x) \, \omega'(t) \, u \; u_t dxdt \geq$$

$$\geq -2\mu' \int_Q \varphi'(T-t) \, \lambda(x) \, \omega(t) \, |u| \; |u_t| dxdt -$$

$$-2\mu' \int_Q \varphi(T-t) \, \lambda(x) \, |\omega'(t)| \; |u| \; |u_t| dxdt \geq$$

$$\geq -\mu' C_9(\lambda) \, C_{10}(\omega) \, \alpha q(T) \int_Q u_t^2 dxdt - \frac{\mu'}{\alpha} C_9 C_{10} q(T) \int_Q u^2 dxdt -$$

$$-\mu' C_9 C_{11}(\omega) \, \alpha q_1(T) \int_Q u_t^2 dxdt - \frac{\mu'}{\alpha} C_9 C_{11} q_1(T) \int_Q u^2 dxdt \tag{36}$$

Let's take $C_{12} = \max\{C_{10}, C_{11}\}, \quad C_{13} = C_9 C_{12}$.
Then continuing our reasoning we obtain from (36)

$$-2\mu' \int_Q \psi(x,t) \, u_{tt} u dxdt \geq -\mu' C_{13} \alpha D(T) \int_Q u_t^2 dxdt - \frac{\mu'}{2} C_{13} D(T) \int_Q u^2 dxdt. \tag{37}$$

Let $T \leq T_5(\psi)$ be so small, that $C_{13} D(T) \leq 1$.
Then taking all the above mentioned into account we get from (35)

$$I_1 \geq (1 - SD(T)) \int_Q \left(\sum_{i,j=1}^n u_{ij}^2 + u_t^2 + \psi^2 u_{tt}^2 + \psi \sum_{i=1}^n u_{it}^2 \right) dxdt +$$

$$+\left(\mu'\right)^2\int\limits_Q u^2dxdt-\mu' C_{13}\alpha D\int\limits_Q u_t^2dxdt-\frac{\mu'}{\alpha}\int\limits_Q u^2dxdt\ .$$

If we put $\alpha=\dfrac{1}{\mu'}$, then

$$I_1\geq\left(1-SD\left(T\right)\right)\int\limits_Q\left(\sum_{i,j=1}^n u_{ij}^2+u_t^2+\psi^2\left(x,t\right)u_{tt}^2+\psi\left(x,t\right)\sum_{i=1}^n u_{it}^2\right)dxdt-$$

$$-C_{13}D\left(T\right)\int\limits_Q u_t^2dxdt=\left(1-S_1D\left(T\right)\right)\int\limits_Q\left(\sum_{i,j=1}^n u_{ij}^2+u_t^2+\psi^2u_{tt}^2+\psi\sum_{i=1}^n u_{it}^2\right)dxdt$$

where $S_1=S+C_{13}$.

Whence,

$$\int\limits_Q\left(\sum_{i,j=1}^n u_{ij}^2+u_t^2+\psi^2u_{tt}^2+\psi u_{it}^2\right)dxdt\leq$$

$$\leq\frac{1}{1-S_1D\left(T\right)}I_1=I_1+\frac{S_1D\left(T\right)}{1-S_1D\left(T\right)}I_1.$$

Let T_5 be so small, that $S_1D\left(T\right)\leq\dfrac{1}{2}$. Then

$$\int\limits_Q\left(\sum_{i,j=1}^n u_{ij}^2+u_t^2+\psi^2u_{tt}^2+\psi u_{it}^2\right)dxdt\leq\left(1+2S_1D\left(T\right)\right)I_1=\left(1+S_2D\left(T\right)\right)I_1.$$

So we get the needed estimate (34).

Lemma has been proved.

Lemma 4. *If coefficients of the operator $\mathcal{L}$ satisfy the conditions (3)-(7), then for any fucntion $u\left(x,t\right)\in C^\infty\left(\bar{Q}_T\right)$, $u|_{\Gamma(Q_T)}=0$ at $T\leq T_6\left(\gamma,\sigma,\psi,n,\Omega\right)$ and any $\tau\in[0,1]$ the estimate is true*

$$\|u\|_{W^{2,2}_{2,\psi}(Q_T)}\leq C_{14}\left(\gamma,\sigma,\psi,n\right)\left\|\mathcal{L}u-\frac{\tau}{T}u\right\|_{L_2(Q_T)}$$

Proof is similar to the proof of coercive estimate for the operator $\mathcal{L}$ in the work [16].

Further we will denote the operators $\mathcal{L}_0-\mu$ and $\mathcal{L}_\varepsilon-\mu$ by M_0 and M_ε respectively. Let's denote $\min\left\{T_9,T_6\right\}$ by T^0.

Theorem 1. *If the fucntion $\varphi\left(z\right)$ satisfies the conditions (7), then at $T\leq T^0$ the first boundary value problem*

$$M_0u=f\left(x,t\right),\quad\left(x,t\right)\in Q_T\tag{38}$$

$$u|_{\Gamma(Q_T)}=0\tag{39}$$

has a unique strong solution in the space $\dot{W}^{2,2}_{2,\varphi}\left(Q_T\right)$ for any fucntion $f\left(x,t\right)\in L_2\left(Q_T\right)$.

Proof. First assume that f$(x,t)\in C^\infty\left(\bar{Q}_T\right)$. Let $v\left(x,t\right)$ be classical solution of the first boundary-value problem

$$\Delta v-v_t=f\left(x,t\right),\quad\left(x,t\right)\in Q_T$$

$$v|_{\Gamma(Q_T)}=0$$

It's clear that this solution exists and owing to [17] $v\left(x,t\right)\in W_2^{2,2}\left(Q_T\right)$, and

$$\|u\|_{W_2^{2,2}(Q)}\leq C_{15}\left(n,\Omega,f\right),\tag{40}$$

where $W_2^{2,2}(Q_T)$ is a Banach space of functions given on Q_T with finite norms (8), where $\varphi \equiv 1$. As at $\varepsilon \in (0,T)$ the fucntion $\varphi_\varepsilon(z) \leq 1$, we conclude from (40) that

$$\|v\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)} \leq C_{15} \tag{41}$$

We denote by $\dot{W}_2^{2,2}(Q_T)$ the completion of a set of all functions from $C^\infty\left(\bar{Q}_T\right)$ vanishing on ∂Q_T with respect to the norm of the space $W_2^{2,2}(Q_T)$; by $u^\varepsilon(x,t)$ - the strong (almost everywhere) solution of Dirichlet problem

$$M_\varepsilon u^\varepsilon = f(x,t), \quad (x,t) \in Q_T$$

$$(u^\varepsilon(x,t) - v(x,t)) \in \dot{W}_2^{2,2}(Q_T).$$

This solution exists at every $\varepsilon > 0$ owing to [18]. It's clear, that $(u^\varepsilon(x,t) - v(x,t)) \in \dot{W}^{2,2}_{2,\varphi_\varepsilon}(Q_T)$. Taking into account that $v|_{\Gamma(Q_T)} = 0$ and the inequality (9), we get that $u^\varepsilon(x,t) \in \dot{W}^{2,2}_{2,\varphi_\varepsilon}(Q_T)$.

Besides, for $F_\varepsilon(x,t) = M_\varepsilon v$ taking into account (41), we have

$$\|F_\varepsilon\|_{L_2(Q_T)} \leq C_{16}(n,\Omega,T,f) \tag{42}$$

From lemma 2 it follows that

$$\|u^\varepsilon - v\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)} \leq C_1\left(\|f\|_{L_2(Q_T)} + \|F_\varepsilon\|_{L_2(Q_T)}\right).$$

Then from (41), (42) and (9) we conclude

$$\|u^\varepsilon\|_{W^{2,2}_{2,\varphi}(Q_T)} \leq C_{17}\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)} \leq C_{18}(\varphi,n,\Omega,T,f) \tag{43}$$

Hence, a family of functions $\{u^\varepsilon(x,t)\}$ is bounded by norm of the space $\dot{W}^{2,2}_{2,\varphi}(Q_T)$ uniformly with respect to ε. So this family is weakly compact in $\dot{W}^{2,2}_{2,\varphi}(Q_T)$.

And this, in particular, means that there exist such sequences of positive numbers $\{\varepsilon_k\}$, $\lim\limits_{k\to\infty}\varepsilon_k = 0$ and a fucntion $u_0(x,t) \in \dot{W}^{2,2}_{2,\varphi}(Q_T)$, that for any $h(x,t) \in C^\infty\left(\bar{Q}_T\right)$

$$\lim_{k\to\infty}(M_0 u^{\varepsilon_k}, h) = (M_0 u_0, h), \tag{44}$$

where $(a,b) = \int\limits_{Q_T} ab\ dx\ dt$. But

$$(M_0 u^{\varepsilon_k}, h) = ((M_0 - M_{\varepsilon_k})u^{\varepsilon_k}, h) + (M_{\varepsilon_k}u^{\varepsilon_k}, h) = ((M_0 - M_{\varepsilon_k})u^{\varepsilon_k}, h) + (f,h). \tag{45}$$

Besides, taking into account (9) and (43) we have

$$J(k) = |(M_0 - M_{\varepsilon_k})u^{\varepsilon_k}, h| \leq \left\|\left(\varphi - \varphi_{\varepsilon_k}\right)u^{\varepsilon_k}_{tt}\right\|_{L_2(Q(\varepsilon_k))} \times$$

$$\times\|h\|_{L_2(Q(\varepsilon_k))} \leq 3\|u^{\varepsilon_k}\|_{W^{2,2}_{2,\varphi_{\varepsilon_k}}(Q_T)}\|h\|_{L_2(Q(\varepsilon_k))} \leq 3C_{18}\|h\|_{L_2(Q(\varepsilon_k))},$$

where $Q(\varepsilon) = \Omega \times (T-\varepsilon, T)$. Thus, we have that

$$J(k) \underset{k\to\infty}{\longrightarrow} 0 \tag{46}$$

From (44)-(46) it follows that $(M_0 u_0, h) = (f,h)$, and $M_0 u_0 = f(x,t)$ almost everywhere in Q_T.

Now let $f(x,t) \in L_2(Q_T)$. In this case such a sequence $\{f_m(x,t)\}$, $m = 1,2,...$ exists, that $f_m(x,t) \in C^\infty\left(\bar{Q}_T\right)$ and $\lim\limits_{m\to\infty}\|f_m - f\|_{L_2(Q_T)} = 0$. For all natural m we will consider a sequence $\{u_m(x,t)\}$ of strong solutions of the first boundary value problems

$$M_0 u_m = f_m(x,t), \quad (x,t) \in Q_T;$$

$$u_m|_{\Gamma(Q_t)} = 0.$$

On the base of above mentioned we say that for any m the fucntion $u_m(x,t)$ exists. Using the estimate of the previous lemma for the operator $\mathcal{L}_0$ at $\tau = 1$ we get

$$\|u_m\|_{W^{2,2}_{2,\varphi}(Q_T)} \le C_{20} \|f_m\|_{L_2(Q_T)} \le C_{19}(\varphi, n, \Omega, f) \tag{47}$$

Thus, the sequence $\{u_m(x,t)\}$ is weakly compact in $\dot{W}^{2,2}_{2,\varphi}(Q_T)$, i.e. there exist such a subsequence $\{m_k\} \in \mathbf{N}$, $\lim\limits_{k\to\infty} m_k = \infty$, and a fucntion $u(x,t) \in \dot{W}^{2,2}_{2,\varphi}(Q_T)$, that for any $h(x,t) \in C^\infty(\bar{Q}_T)$ $\lim\limits_{k\to\infty}(M_0 u_{m_k}, h) = (M_0 u, h)$.

But

$$\lim_{k\to\infty}(M_0 u_{m_k}, h) = \lim_{k\to\infty}(f_{m_k}, h) = (f, h).$$

That is why $(M_0 u, h) = (f, h)$, and $M_0 u = f(x,t)$ almost everywhere in Q_T. Therefore, the existence of strong solution of the problem (38)-(39) has been proved. The uniqueness of the solution follows from lemma 4.

Theorem has been proved.

3^0. Strong solvability of the first boundary value problem.

Let's replace the condition (3) by a weaker one

$$\inf_{Q_T} \sum_{i=1}^{n} a_{ii}(x,t) = \gamma' > 0 \tag{3'}$$

Theorem 2. *If coefficients of the operator $\mathcal{L}$ satisfy the conditions (3') and (4)-(7), then at $T \le T^0$ the first boundary value problem (1)-(2) has a unique strong solution in the space $\dot{W}^{2,2}_{2,\psi}(Q_T)$ for any $f(x,t) \in L_2(Q_T)$. And the following estimate is true for the solution $u(x,t)$*

$$\|u\|_{W^{2,2}_{2,\psi}(Q_T)} \le C_{20} \|f\|_{L_2(Q_T)}. \tag{48}$$

Proof. The estimate (48) and uniqueness of the solution follow from the coercive estimate in the work [16]. Let's prove the existence of the solution. We will consider a family of operators $L(\tau) = (1-\tau) M_0 + \tau\mathcal{L}$ for $\tau \in [0,1]$. Let's show that the set E of points τ for which the problem

$$L^{(\tau)} u = f(x,t), \ (x,t) \in Q_T, \tag{49}$$

$$u|_{\Gamma(Q_T)} = 0, \tag{50}$$

has a unique strong solution in $\dot{W}^{2,2}_{2,\psi}(Q_T)$ for any $f(x,t) \in L_2(Q_T)$, is nonempty and simultaneously open and closed with respect to $[0,1]$. Whence, $E = [0,1]$ and, in particular, the problem (49)-(50) is solvable at $\tau = 1$, i.e. when $\mathcal{L}^{(1)} = \mathcal{L}$.

The nonemptiness of the set E follows directly from theorem 1. Let's prove its openness. Let $\tau_0 \in E$, and $\varepsilon > 0$ will be chosen later. We will show that the problem (49)-(50) is solvable for all $\tau \in [0,1]$ such that $|\tau - \tau_0| < \varepsilon$. The problem (49)-(50) can be rewritten in the equivalent form

$$\begin{gathered} \mathcal{L}^{(\tau_0)} u = f(x,t) - \left(\mathcal{L}^{(\tau)} - \mathcal{L}^{(\tau_0)}\right) u, \quad (x,t) \in Q_T \\ u(x,t) \in \dot{W}^{2,2}_{2,\psi}(Q_T) \end{gathered} \tag{51}$$

Let's consider an arbitrary fucntion $v(x,t) \in \dot{W}^{2,2}_{2,\psi}(Q_T)$ and the first boundary value problem

$$\mathcal{L}^{(\tau_0)}u = f(x,t) - \left(\mathcal{L}^{(\tau)} - \mathcal{L}^{(\tau_0)}\right) v(x,t), \quad (x,t) \in Q_T$$

$$u(x,t) \in \dot{W}^{2,2}_{2,\psi}(Q_T). \tag{52}$$

It's clear, that $\left(\mathcal{L}^{(\tau)} - \mathcal{L}^{(\tau_0)}\right) v \ \ (x,t) \in L_2(Q_T)$. Let's note that for all operators $\mathcal{L}^{(\tau)}$ the conditions (3') and (4) are fulfilled with constants $\gamma'_{(\tau)} \geq \min\{\gamma', n\}$ and $\sigma_{(\tau)} \leq \sigma$ respectively.

Let's prove it. We denote coefficients of senior derivatives of the operator $\mathcal{L}^{(\tau)}$ with respect to space variables by $a^{(\tau)}_{ij}(x,t), \ \ i,j=\overline{1,n}$. Let

$$\bar{r} = \sup_{Q_T} \frac{\sum\limits_{i,j=1}^{n} a^2_{ij}(x,t)}{g^2(x,t)}, \quad r^{(\tau)} = \frac{\sum\limits_{i,j=1}^{n} \left[a^{(\tau)}_{ij}(x,t)\right]^2}{\left[\sum\limits_{i=1}^{n} a^{(\tau)}_{ii}(x,t)\right]^2}$$

$$\bar{r}^{(\tau)} = \sup_{Q_T} r^{(\tau)}(x,t), \quad \text{where} \quad g(x,t) = \sum_{i=1}^{n} a_{ii}(x,t).$$

Taking into account (4) and the fact, that for any operator of $\mathcal{L}$-type the inequality $\bar{r} \geq \frac{1}{n}$ is true, we conclude

$$r^{(\tau)}(x,t) = \frac{n(1-\tau)^2 + 2\sigma(1-\tau)g(x,t) + \tau^2 \sum\limits_{i,j=1}^{n} a^2_{ij}(x,t)}{n^2(1-\tau)^2 + 2\tau(1-\tau)ng(x,t) + \tau^2 g^2(x,t)} \leq$$

$$\leq \frac{1}{n} + \frac{\tau^2\left(\bar{r} - \frac{1}{n}\right)g^2(x,t)}{n^2(1-\tau)^2 + 2\tau(1-\tau)ng(x,t) + \tau^2 g(x,t)} \leq$$

$$\leq \frac{1}{n} + \frac{\tau^2\left(\bar{r} - \frac{1}{n}\right)g^2(x,t)}{\tau^2 g^2(x,t)} = \bar{r}. \tag{53}$$

Let $\lambda^- = \inf_{Q_T} g(x,t), \ \ \lambda^+ = \sup_{Q_T} g(x,t), \ \bar{\lambda}(\tau) = \inf_{Q_T} \sum\limits_{i=1}^{n} a^{(\tau)}_{ii}(x,t) / \sup_{Q_T} \sum\limits_{i=1}^{n} a^{(\tau)}_{ii}(x,t)$.

It's easy to see that $\bar{\lambda}(\tau) = \dfrac{(1-\tau)n + \tau\lambda^-}{(1-\tau)n + \tau\lambda^+}$.

But on the other hand

$$\bar{\lambda}'(\tau) = \frac{\lambda^- - \lambda^+}{\left[(1-\tau)n + \tau\lambda^+\right]^2} \leq 0$$

That's why

$$\bar{\lambda}(\tau) \geq \bar{\lambda}(1) = \lambda. \tag{54}$$

From (53) and (54) it follows that

$$\sigma_{(\tau)} = \bar{r}^{(\tau)} - \frac{1}{n - \bar{\lambda}^2(\tau)} \leq \bar{r} - \frac{1}{n - \lambda^2} = \sigma,$$

and our statement has been proved.

Now let's note that from the above mentioned and lemma 4 it follows that at $T \leq T^0$ for any $\tau \in [0,1]$ and any fucntion $u(x,t) \in \dot{W}^{2,2}_{2,\psi}(Q_T)$ the estimate is true

$$\|u\|_{W^{2,2}_{2,\psi}(Q_T)} \leq C_{20} \left\| \mathcal{L}^{(\tau)} u \right\|_{L_2(Q_T)}. \tag{55}$$

On the base of the made assumption the boundary value problem (52) has a strong solution $u(x,t)$ for any $v(x,t) \in \dot{W}^{2,2}_{2,\psi}(Q_T)$. Thus, the operator $\mathcal{P}$ from $\dot{W}^{2,2}_{2,\psi}(Q_T)$ into $\dot{W}^{2,2}_{2,\psi}(Q_T)$ is defined, and $u = \mathcal{P}v$. It is compressing when ε is chosen in the corresponding way. We will show that. Let $v^{(i)}(x,t) \in \dot{W}^{2,2}_{2,\psi}(Q_T)$, $u^{(i)} = \mathcal{P}v^{(i)}$, $i = 1,2$.

Then, taking into account the equality $\mathcal{L}^{(\tau)} - \mathcal{L}^{(\tau_0)} = (\tau - \tau_0)(\mathcal{L} - M_0)$ we conclude that $u^{(1)}(x,t) - u^{(2)}(x,t)$ is a strong solution of the first boundary value problem

$$\mathcal{L}^{(\tau_0)} \left(u^{(1)} - u^{(2)} \right) = (\tau - \tau_0)(\mathcal{L} - M_0) \left(v^{(1)} - v^{(2)} \right);$$

$$\left(u^{(1)} - u^{(2)} \right) \in \dot{W}^{2,2}_{2,\psi}(Q_T).$$

Using (55) we get

$$\left\| u^{(1)} - u^{(2)} \right\|_{W^{2,2}_{2,\psi}(Q_T)} \leq C_{20} |\tau - \tau_0| \left\| (\mathcal{L} - M_0) \left(v^{(1)} - v^{(2)} \right) \right\|_{L_2(Q_T)}. \tag{56}$$

On the other hand

$$\left\| (\mathcal{L} - M_0) \left(v^{(1)} - v^{(2)} \right) \right\|_{L_2(Q_T)} \leq C_{21}(\mathcal{L}, n, \Omega, T) \left\| v^{(1)} - v^{(2)} \right\|_{W^{2,2}_{2,\psi}(Q_T)}$$

Thus,

$$\left\| u^{(1)} - u^{(2)} \right\|_{W^{2,2}_{2,\psi}(Q_T)} \leq C_{20} C_{21} \varepsilon \left\| v^{(1)} - v^{(2)} \right\|_{W^{2,2}_{2,\psi}(Q_T)}.$$

Now choosing $\varepsilon = \dfrac{1}{2C_{20}C_{21}}$ we prove that the operator $\mathcal{P}$ is compressing. From here it follows that it has a stationary point $u = \mathcal{P}u$, that is a strong solution of the boundary value problem (51), and, consequently, of (49)-(50). Therefore, the openness of the set E has been proved. Now let's show that the set E is closed. Let $\tau_k \in E$, $k = 1,2,...$; $\lim\limits_{k\to\infty} \tau_k = \tau$. For natural k we denote by $u_{[k]}(x,t)$ a strong solution of the first boundary value problem

$$\mathcal{L}^{(\tau_k)} u_{[k]} = f(x,t), \quad (x,t) \in Q_T; \quad u_{[k]}\big|_{\Gamma(Q_T)} = 0.$$

According to (55) we have

$$\left\| u_{[k_l]} \right\|_{W^{2,2}_{2,\psi}(Q_T)} \leq C_{20} \|f\|_{L_2(Q_T)}. \tag{57}$$

So, the family of functions $\left\{ u_{[k]}(x,t) \right\}$ is weakly compact in $\dot{W}^{2,2}_{2,\psi}(Q_T)$, i.e. there exists such a subsequence of natural numbers $\{k_l\}$, $\lim\limits_{l\to\infty} k_l = \infty$, and a fucntion $u(x,t) \in \dot{W}^{2,2}_{2,\psi}(Q_T)$, that for any $\psi(x,t) \in C^\infty\left(\bar{Q}_T\right)$

$$\lim_{l\to\infty} \left(\mathcal{L}^{(\tau_{k_l})} u_{[k_l]}, \psi \right) = \left(\mathcal{L}^{(\tau)} u, \psi \right). \tag{58}$$

But

$$\left(\mathcal{L}^{(\tau)} u_{[k_l]}, \psi \right) = \left(\left(\mathcal{L}^{(\tau)} - \mathcal{L}^{(\tau_{k_l})} \right) u_{[k_l]}, \psi \right) + (f, \psi) = J_1(l) + (f, \psi). \tag{59}$$

Moreover, taking into account (56) and (57) we have

$$|J_1(l)| \leq |\tau - \tau_{k_l}| \; \left|\left(\mathcal{L} - M_0\right) u_{[k_l]}, \psi\right| \leq |\tau - \tau_{k_l}| \, C_{21} \left\|u_{[k_l]}\right\|_{W^{2,2}_{2,\psi}(Q_T)} \times$$

$$\times \|\psi\|_{L_2(Q_T)} \leq C_{20} C_{21} \, |\tau - \tau_{k_l}| \; \|f\|_{L_2(Q_T)} \, \|\psi\|_{L_2(Q_T)} \tag{60}$$

From (60) it follows that $\lim\limits_{l\to\infty} J_1(l) = 0$. Now from (58) and (59) we can conclude that $\left(\mathcal{L}^{(\tau)} u, \psi\right) = (f, \psi)$, i.e. $\mathcal{L}^{(\tau)} u = f(x,t)$ almost everywhere in Q_T. Thereby it is shown that $\tau \in E$, i.e. the set E is closed.

Theorem has been proved.

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