Sabir S. MIRZOYEV, Ayten T. ALIYEVA

ON SOLVABILITY OF ONE CLASS OF BOUNDARY VALUE PROBLEM FOR A FOURTH ORDER OPERATOR-DIFFERENTIAL EQUATION

Abstract

The theorem about correct and univalent solvability of a class of boundary value problem for operator-differential equation with variable coefficients was obtained. These conditions are expressed only by the coefficients of the given equation.

In separable Hilbert space H consider the boundary value problem

$$P\left(\frac{d}{dt}\right)u \equiv \frac{d^4u(t)}{dt^4} + \rho(t)A^4u(t) + \sum_{j=0}^{4} A_{4-j}(t)\,u^{(j)}(t) = f(t),$$

$$t \in R_+ = (0,\infty), \tag{1}$$

$$u(0) = u'(0) = 0, \tag{2}$$

where $f(t)$, $u(t)$ are vector functions with values from H,

$$\rho(t) = \begin{cases} \alpha^4, t \in (0,1), \\ \beta^4, t \in (1,\infty), \end{cases}$$

$\alpha > 0$, $\beta > 0$ and operators A and $A_j(t)$ $\left(j = \overline{0,4}\right)$ satisfy the following conditions.

1. A is a normal reversible operator, whose spectrum is contained in angular sector $S_\varepsilon = \{\lambda : |\arg\lambda| \le \varepsilon\}$, $0 \le \varepsilon < \dfrac{\pi}{4}$;

2. The operators $B_j(t) = A_j(t)A^{-j}$ $\left(j = \overline{0,4}\right)$ are bounded in H and $B_j(t) \in L_\infty(R_+; L(H))$.

Here and later on the derivatives are understood in the sense of distributions, and $L(H)$ is a space of linear bounded operators acting in H.

From condition 1) it follows, that the operator A is represented in the form: $A = UC = CU$, where C is positive-definite self-adjoint operator, and U is a unitary operator in H. Let's consider the scale of Hilbert spaces generated by the operator C, i.e.

$$H_\gamma = D(C^\gamma), (x,y)_\gamma = (C^\gamma x, C^\gamma y), x, y \in H_\gamma, \gamma \ge 0.$$

Then, let's denote by $L_2(R_+; H)$ the Hilbert space of vector-functions $f(t)$, defined in R_+ with values from H for which

$$\|f\|_{L_2(R_+;H)} = \left(\int_0^\infty \|f(t)\|^2 dt\right)^2 < \infty.$$

Let's denote by $W_2^4(R_+;H)$ the Hilbert space (see [1])

$$W_2^4(R_+;H) = \{u : u^{(4)} \in L_2(R_+;H), A^4u \in L_2(R_+;H)\}$$

with norm

$$\|u\|_{W_2^4(R_+;H)} = \left(\left\| u^{(4)} \right\|_{L_2}^2 + \left\| A^4 u \right\|_{L_2}^2 \right)^{\frac{1}{2}}.$$

Let

$$\overset{\circ}{W}{}_2^4(R_+;H) = \{u : u \in W_2^4(R_+;H), u(0) = u'(0) = 0\}.$$

It follows from the theorem on traces [1], that $\overset{\circ}{W}{}_2^4(R_+;H)$ is a complete subspace of the space $W_2^4(R_+;H)$.

The spaces $L_2(R;H)$ and $W_2^4(R;H)$, where $R = (-\infty, \infty)$ are defined similarly.

Definition 1. *If at any $f(t) \in L_2(R_+;H)$ there exists the vector-function $u(t) \in W_2^4(R_+;H)$, satisfying the equation (1) almost everywhere, the boundary conditions (2) in the sense*

$$\lim_{t\to+0} \|u(t)\|_{7/2} = 0, \ \lim_{t\to+0} \left\|u'(t)\right\|_{5/2} = 0$$

and for which the estimate

$$\|u\|_{W_2^4} \leq const \, \|f\|_{L_2},$$

is true, then we'll call the problem (1), (2) regularly solvable.

Let's find the conditions of regular solvability of problem (1), (2) in the given work.

Let's note, that at $\rho(t) \equiv 1$ (i.e. $\alpha = \beta = 1$) this problem was investigated in the paper [2] and at $\alpha \neq \beta$ and A is a self-adjoint operator in [3].

Let's write the problem (1), (2) in the form of the equation

$$Pu = P_0 u + P_1 u = f,$$

where

$$f \in L_2(R_+;H), u \in \overset{\circ}{W}{}_2^2(R_+;H)$$

and

$$P_0 u = u^{(4)} + A^4 u, P_1 u = \sum_{j=0}^{4} A_{4-j}(t) \, u^{(j)}(t), \ u \in \overset{\circ}{W}{}_2^4(R_+;H).$$

It holds

Theorem 1. *Let the condition 1) be fulfilled then, the operator $P_0 : \overset{\circ}{W}{}_2^4(R_+;H) \to L_2(R_+;H)$ is isomorphism.*

Proof. It's easy to see, that the equation $P_0 u = 0$ has only a zero solution. Let's show that the image of the space operator P_0 coincides with the space $L_2(R;H)$. Evidently, the vector-functions

$$u_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\lambda^4 E + \alpha^4 A^4}\left(\int_0^{\infty} f(s)\, e^{-i\lambda(t-s)} ds\right) d\lambda$$

and

$$u_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\lambda^4 E + \beta^4 A^4}\left(\int_0^{\infty} f(s)\, e^{-i\lambda(t-s)} ds\right) d\lambda$$

satisfy respectively, the equation $u^{(4)} + \alpha^4 A^4 u = f$ and $u^{(4)} + \beta^4 A^4 = f$ almost everywhere in R_+. Let's show, that $u_1(t), u_2(t) \in W_2^4(R;H)$. By Plansharel theorem

$$\left\|u^{(4)}\right\|_{L_2(R;H)} = \left\|\lambda^4 \hat{u}_1(\lambda)\right\|_{L_2(R_+;H)} = \left\|\lambda^4\left(\lambda^4 E + \alpha^4 A^4\right)^{-1} \hat{f}(\lambda)\right\|_{L_2(R;H)} \leq$$

$$\leq \sup_{\lambda\in R}\left|\lambda^4\left(\lambda^4 + \alpha^4 A^4\right)^{-1}\right| \|f\|_{L_2(R;H)} \leq$$

$$\leq \sup_{\lambda\in R}\left(\sup_{\mu\in\sigma(A)}\left|\lambda^4\left(\lambda^4 + \alpha^4\mu^4\right)^{-1}\right|\right)\|f\|_{L_2(R;H)} \leq \|f\|_{L_2(R;H)}.$$

It's analogously proved, that $A^4 u_1 \in L_2(R;H)$, i.e. $u_1(t) \in W_2^4(R;H)$. By the same way it is proved, that $u_2(t) \in W_2^4(R;H)$. Let's denote the contractions of vector-functions $u_1(t)$ and $u_2(t)$ on $[0;1]$ and $(1;\infty)$, by $\psi_1(t)$ and $\psi_2(t)$, respectively. It is evident, that $\psi_1(t) \in W_2^4([0;1];H), \psi_2(t) \in W_{2.}^4((1;\infty);H)$, and $\psi_1(0) \in H_{7/2},\ \psi_1'(0) \in H_{5/2},\ \psi_1^{(j)}(1),\ \psi_2^j(1) \in H_{4-j-1/2}\left(j=\overline{0,3}\right)$.

Let's determine the vector-function

$$u(t) = \begin{cases} \xi_1(t) \equiv \psi_1(t) + e^{\alpha\omega_1 tA}\varphi_1 + e^{\alpha\omega_2 tA}\varphi_2 + e^{\alpha\omega_1(1-t)A}\varphi_3 + \\ +e^{\alpha\omega_2(1-t)A}\varphi_4,\ t\in[0;1), \\ \xi_2(t) \equiv \psi_2(t) + e^{\beta\omega_1(t-1)A}\varphi_5 + e^{\beta\omega_2(t-1)A}\varphi_6, \qquad t\in(1;\infty), \end{cases}$$

where $\omega_1 = -\frac{\sqrt{2}}{2}(1+i)$, $\omega_2 = -\frac{\sqrt{2}}{2}(1-i)$, and the unknown vectors $\varphi_j \in H_{7/2}\left(j=\overline{0,6}\right)$. It's easy to see, that vectors φ_j are identically defined from the condition $u \in \mathring{W}_2^4(R_+;H)$ $\left(\xi_1(0)=0,\ \xi_1'(0)=0,\ \xi_1^{(j)}(1) = \xi_2^{(j)}(1),\ j=\overline{0,3}\right)$. Thus, $u(t) \in \mathring{W}_2^4(R;H)$. Since at $u \in \mathring{W}_2^4(R_+;H)$

$$\|P_0 u\|_{L_2} \leq \sqrt{2}\max\left(1;\alpha^4;\beta^4\right)\|u\|_{W_2^4},$$

then approval of the theorem follows from Banach theorem on the inverse operator.

It follows from this theorem, that norms $\|P_0u\|_{L_2}$ and $\|u\|_{W_2^4}$ are equivalent in the space $\overset{\circ}{W}{}_2^4(R;H)$. Therefore, by the theorem on intermediate derivatives, the norms

$$\overset{\circ}{N}_j(R_+) = \sup_{0\neq u\in \overset{\circ}{W}{}_2^4(R_+;H)} \left\|A^{4-j}u^{(j)}\right\|_{L_2} \|P_0u\|_{L_2}^{-1}, \ j=\overline{0,4} \tag{3}$$

are finite. Let's prove the following lemma for estimation of these numbers.

Lemma 1. *Let the condition 1) be fulfilled, then the inequality*

$$\|P_0u\|_{L_2}^2 \geq \min\left(\alpha^4;\beta^4\right)\left(\left\|\rho^{-\frac{1}{2}}u^{(4)}\right\|_{L_2}^2 + \left\|\rho^{\frac{1}{2}}A^4u\right\|_{L_2}^2 + \right.$$
$$\left. + 2\cos 4\varepsilon \left\|A^2u''\right\|_{L_2}^2\right), \tag{4}$$

holds at any $u\in \overset{\circ}{W}{}_2^4(R_+;H)$.

Proof. Since

$$\left\|\rho^{-\frac{1}{2}}P_0u\right\|_{L_2}^2 = \left\|\rho^{-\frac{1}{2}}u^{(4)}+\rho^{\frac{1}{2}}A^4u\right\|_{L_2}^2 = \left\|\rho^{-\frac{1}{2}}u^{(4)}\right\|_{L_2}^2 +$$
$$+\left\|\rho^{\frac{1}{2}}A^4u\right\|_{L_2}^2 + 2\operatorname{Re}\left(u^{(4)},A^4u\right)_{L_2}. \tag{5}$$

Considering that $u\in \overset{\circ}{W}{}_2^4(R_+;H)$ $(u(0)=u'(0)=0)$ integrating by parts we get

$$\left(u^{(4)},A^4u\right)_{L_2} = \int_0^\infty \left(u^{(4)},A^4u\right)dt = \int_0^\infty \left(A^{*2}u'',A^2u''\right)dt = \left(A^{*2}u'',A^2u''\right)_{L_2},$$

i.e.

$$\operatorname{Re}\left(u^{(4)},A^4u\right)_{L_2} = \operatorname{Re}\left(A^{*2}u'',A^2u''\right)_{L_2} \geq \cos 4\varepsilon\left(A^2u'',A^2u''\right)_{L_2} =$$
$$=\cos 4\varepsilon\left\|A^2u''\right\|_{L_2}^2.$$

Thus it follows, from (5) that

$$\left\|\rho^{-\frac{1}{2}}P_0u\right\|_{L_2}^2 \geq \left\|\rho^{-\frac{1}{2}}u^{(4)}\right\|_{L_2}^2 + \left\|\rho^{\frac{1}{2}}A^4u\right\|_{L_2}^2 + 2\cos 4\varepsilon\left\|A^2u''\right\|_{L_2}^2. = \tag{6}$$

The approval of the lemma follows from inequality (6) subject to inequality

$$\left\|\rho^{-\frac{1}{2}}P_0u\right\|_{L_2}^2 \leq \max_t \rho^{-1}(t)\|P_0u\|_{L_2}^2 = \frac{1}{\min\left(\alpha^4,\beta^4\right)}\|P_0u\|_{L_2}^2.$$

The lemma is probed.

Lemma 2. *For numbers* $\overset{\circ}{N}_j(R_+)$ *the following estimations hold:*

$$\overset{\circ}{N}_j(R_+) \leq c_j(\alpha;\beta;\varepsilon), \ j=\overline{0,4},$$

where

$$c_0(\alpha;\beta;\varepsilon) = \frac{1}{\min\left(\alpha^4,\beta^4\right)}\begin{cases} 1, & 0\leq\varepsilon\leq\dfrac{\pi}{8}, \\ \dfrac{1}{\sqrt{2}\cos 2\varepsilon}, & \dfrac{\pi}{8}\leq\varepsilon<\dfrac{\pi}{4}, \end{cases} \tag{7}$$

$$c_1(\alpha;\beta;\varepsilon)=\frac{1}{\min\left(\alpha^3,\beta^3\right)}\begin{cases}\dfrac{1}{\sqrt{2\cos 2\varepsilon}}, & 0\le\varepsilon<\dfrac{\pi}{8},\\ \dfrac{1}{\sqrt[4]{8}\cos 2\varepsilon}, & \dfrac{\pi}{8}\le\varepsilon<\dfrac{\pi}{4},\end{cases} \tag{8}$$

$$c_2(\alpha;\beta;\varepsilon)=\frac{1}{2\cos 2\varepsilon\min\left(\alpha^2;\beta^2\right)}, 0\le\varepsilon<\frac{\pi}{4}, \tag{9}$$

$$c_4(\alpha;\beta;\varepsilon)=\frac{\max\left(\alpha^2;\beta^2\right)}{\min\left(\alpha^2;\beta^2\right)}\begin{cases}1, & 0\le\varepsilon<\dfrac{\pi}{8},\\ \dfrac{1}{\sqrt{2}\cos 2\varepsilon}, & \dfrac{\pi}{8}\le\varepsilon<\dfrac{\pi}{4}.\end{cases} \tag{10}$$

Proof. At $u\in \overset{\circ}{W}{}_2^4(R_+;H)$ $(u(0)=u'(0)=0)$ we have

$$\left\|A^2u''\right\|_{L_2}^2=\left\|C^2u''\right\|_{L_2}^2=\int_0^\infty\left(C^2u'',C^2u''\right)dt=\int_0^\infty\left(C^4u,u^{(4)}\right)dt=$$

$$=\left(\rho^{\frac12}C^4u,\rho^{-\frac12}u^{(4)}\right)_{L_2}\le\left\|\rho^{\frac12}C^4u\right\|_{L_2}\left\|\rho^{-\frac12}u^{(4)}\right\|_{L_2}=$$

$$=\left\|\rho^{\frac12}A^4u\right\|_{L_2}\left\|\rho^{-\frac12}u^{(4)}\right\|_{L_2}\le\frac12\left(\left\|\rho^{\frac12}A^4u\right\|_{L_2}^2+\left\|\rho^{-\frac12}u^{(4)}\right\|_{L_2}^2\right). \tag{11}$$

Taking into account inequality (4) in (11) we get:

$$\left\|A^2u''\right\|_{L_2}^2\le\frac12\left(\frac{1}{\min}\left(\alpha^4;\beta^4\right)\left\|P_0u\right\|_{L_2}^2-2\cos 4\varepsilon\left\|A^2u''\right\|_{L_2}^2\right)$$

or

$$\left\|A^2u''\right\|_{L_2}\le\frac{1}{2\cos 2\varepsilon}\frac{1}{\min\left(\alpha^2;\beta^2\right)}\left\|P_0u\right\|_{L_2}=c_2(\alpha;\beta;\varepsilon)\left\|P_0u\right\|_{L_2}, \tag{12}$$

i.e. $\overset{\circ}{N}_2(R_+)\le c_2(\alpha;\beta;\varepsilon)$.

At $0\le\varepsilon\le\dfrac{\pi}{8}$ $(\cos 4\varepsilon\ge 0)$ it follows from inequality (4), that

$$\left\|A^4u\right\|_{L_2}\le\max_t\rho^{-\frac12}(t)\left\|\rho^{\frac12}A^4u\right\|_{L_2}\le\frac{1}{\min\left(\alpha^4;\beta^4\right)}\left\|P_0u\right\|_{L_2}^2. \tag{13}$$

And at $\dfrac{\pi}{8}\le\varepsilon<\dfrac{\pi}{4}$ $(\cos 4\varepsilon\le 0)$ from inequality (4) with regard to (12) we get

$$\left\|P_0u\right\|_{L_2}^2\ge\min\left(\alpha^4;\beta^4\right)\left(\left\|\rho^{\frac12}A^4u\right\|_{L_2}^2+\frac{2\cos 4\varepsilon}{4\cos^2 2\varepsilon\min\left(\alpha^4;\beta^4\right)}\left\|P_0u\right\|_{L_2}^2\right)$$

or

$$\left\|\rho^{\frac12}A^4u\right\|_{L_2}^2\le\frac{1}{2\cos^2 2\varepsilon}\frac{1}{\min\left(\alpha^2;\beta^4\right)}\left\|P_0u\right\|_{L_2}^2.$$

Whence it follows, that

$$\left\|A^4u\right\|_{L_2}\le\frac{1}{\sqrt{2}\cos 2\varepsilon}\frac{1}{\min\left(\alpha^2;\beta^4\right)}\left\|P_0u\right\|_{L_2}. \tag{14}$$

It follows from inequality (13) and (14) that $\overset{\circ}{N}_0 \le c_0(\alpha;\beta;\varepsilon)$.

Now let's estimate the norms $\overset{\circ}{N}_1, \overset{\circ}{N}_3$ and $\overset{\circ}{N}_4$.

At $u \in \overset{\circ}{W}{}_2^4(R_+;H)$ we have:

$$\left\|A^3u'\right\|_{L_2}^2 = \left\|C^3u'\right\|_{L_2}^2 = \int_0^\infty \left(C^3u', C^3u'\right)dt = -\int_0^\infty \left(C^4u, C^2u''\right)dt =$$

$$= -\left(C^4u, C^2u''\right)_{L_2} \le \left\|C^4u\right\|_{L_2}\left\|C^2u''\right\|_{L_2} = \left\|A^4u\right\|_{L_2}\left\|A^2u''\right\|_{L_2}.$$

Hence taking into account the estimations proved for $\overset{\circ}{N}_0$ and $\overset{\circ}{N}_2$ we get, that

$$\left\|A^3u'\right\|_{L_2} \le c_0^{1/2}(\alpha;\beta;\varepsilon)\,c_2^{1/2}(\alpha;\beta;\varepsilon)\left\|P_0u\right\|_{L_2} = c_1(\alpha;\beta;\varepsilon)\left\|P_0u\right\|_{L_2},$$

i.e. $\overset{\circ}{N}_1 \le c_1(\alpha;\beta;\varepsilon)$.

Then at $0 \le \varepsilon < \dfrac{\pi}{8}$ $(\cos 2\varepsilon \ge 0)$ from inequalities (4) it follows, that

$$\left\|u^{(4)}\right\|_{L_2}^2 \le \max_t \rho(t)\left\|\rho^{-\frac{1}{2}}u^{(4)}\right\|_{L_2}^2 \le \frac{\max\left(\alpha^4;\beta^4\right)}{\min\left(\alpha^4;\beta^4\right)}\left\|P_0u\right\|_{L_2}^2,$$

i.e.

$$\left\|u^{(4)}\right\|_{L_2} \le \frac{\max\left(\alpha^2;\beta^2\right)}{\min\left(\alpha^2;\beta^2\right)}\left\|P_0u\right\|_{L_2}. \tag{15}$$

And at $\dfrac{\pi}{8} \le \varepsilon < \dfrac{\pi}{4}$, analogously to estimation of $\overset{\circ}{N}_0$ we get that

$$\left\|u^{(4)}\right\|_{L_2} \le \frac{1}{\sqrt{2}\cos 2\varepsilon}\frac{\max\left(\alpha^2;\beta^2\right)}{\min\left(\alpha^2;\beta^2\right)}\left\|P_0u\right\|_{L_2}. \tag{16}$$

It follows from (15) and (16), that $\overset{\circ}{N}_4 \le c_4(\alpha;\beta;\varepsilon)$. We use inequality for estimation of $\overset{\circ}{N}_3$

$$\left\|Au'''\right\|_{L_2}^2 \le 2\left\|A^2u''\right\|_{L_2}\left\|u^{(4)}\right\|_{L_2}, \tag{17}$$

which is obtained from the inequality

$$\left\|\xi C^2u'' + Cu''' + \frac{1}{\xi}u^{(4)}\right\|_{L_2}^2 = \xi^2\left\|C^2u''\right\|_{L_2}^2 + \frac{1}{\xi^2}\left\|u^{(4)}\right\|_{L_2}^2 -$$

$$-\left\|Cu'''\right\|_{L_2}^2 - \left\|\frac{1}{\sqrt{\xi}}C^{\frac{1}{2}}u'''(0) + \sqrt{\xi}C^{\frac{3}{2}}u''(0)\right\|^2$$

at $\xi = \left\|u^{(4)}\right\|_{L_2}^{1/2}\left\|C^2u''\right\|_{L_2}^{-1/2}$.

Thus, it follows from (17), that

$$\left\|Au'''\right\|_{L_2}^2 \le 2c_2(\alpha;\beta;\varepsilon)\,c_4(\alpha;\beta;\varepsilon)\left\|P_0u\right\|_{L_2}^2 = c_3^2(\alpha;\beta;\varepsilon)\left\|P_0u\right\|_{L_2}^2,$$

i.e.

$$\left\|Au'''\right\|_{L_2} \le c_3(\alpha;\beta;\varepsilon)\left\|P_0u\right\|_{L_2}.$$

So $\overset{\circ}{N}_3 \leq c_3(\alpha;\beta;\varepsilon)$.

The lemma is proved.

Now let's prove the main theorem.

Theorem. *Let the conditions 1), 2) be fulfilled and the following inequality hold*

$$\theta(\alpha;\beta;\varepsilon) = \sum_{j=0}^{4} c_j(\alpha;\beta;\varepsilon) \|B_{4-j}(t)\|_{L_\infty(R_+;L(H))} < 1.$$

Then the problem (1), (2) is regularly solvable.

Proof. Let's write the equation $Pu = f$ in the form

$$\upsilon + P_1 P_0^{-1} \upsilon = f,$$

where $\upsilon = P_0 u$. Since for any $\upsilon \in L_2(R_+;H)$

$$\left\| P_1 P_0^{-1} \upsilon \right\|_{L_2} = \|P_1 u\|_{L_2} \leq \sum_{j=0}^{4} \|B_{4-j}(t)\|_{L_\infty(R_+;L(H))} \left\| A^{4-j} u^{(j)} \right\|_{L_2} \leq$$

$$\leq \sum_{j=0}^{4} \|B_{4-j}(t)\|_{L_\infty(R_+;L(H))} c_j(\alpha;\beta;\varepsilon) \|P_0 u\|_{L_2} = \theta(\alpha;\beta;\varepsilon) \|\upsilon\|_{L_2}$$

and $\theta(\alpha;\beta;\varepsilon) < 1$, then $E + P_1 P_0^{-1}$ is invertible in $L_2(R_+H)$ and

$$u = P_0^{-1}\left(E + P_1 P_0^{-1}\right) f.$$

Hence we get, that

$$\|u\|_{W_2^4} \leq cons \|f\|_{L_2}.$$

The theorem is proved.

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Sabir S.Mirzoyev, Ayten T. Aliyeva

Institute of Mathematics and Mechanics of NAS of Azerbaijan.

9, F. Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 47 20 (off.)

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