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RIEMANN BOUNDARY VALUE PROBLEM IN A CLASS OF GENERALIZED ANALYTIC FUNCTIONS

Abstract

Some boundary properties of generalized analytic functions are investigated and the analogy of Riemann boundary value problem in a class of generalized analytic functions is studied.

Riemann boundary value problem in a class of analytic functions is of particular significance. It was investigated in detail and it is investigated up till now.

It is enough to consider F.D.Qakhov's monograph (see [2]) and its bibliography, N.I.Muskheleshvili works (see [4]). A.A. Babayev (see for ex [5]), V.V. Salayev [6], F.K.Seyfullayev [7] and their followers were engaged in these questions in Azerbaijan.

The analogy of Riemann boundary value problem can be set in a class of generalized analytic functions to which this article is dedicated and has review character.

Let's consider a class of generalized analytic functions $U_{p,2}(A,B,G)$ in the sense of I.N. Vekua, i.e. a class of regular solutions of equation (see [1], p.147)

$$\partial_{\bar{z}} W(z) + A(z) W(z) + B(z) \bar{W}(z) = 0, \tag{1}$$

where $A(z), B(z) \in L_p(G), p > 2, \partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

Let G be a bounded domain bounded with rectifiable boundary Γ, $W(z)$ be a generalized analytic function given in all finite plane, excluding the points $t \in \Gamma$.

Let's call such function $W(z)$ a piecewise-generalized analytic function.

Let's denote by

$$W^+(t) = \lim_{\substack{z \to t \\ z \in G}} W(z), W^-(t) = \lim_{\substack{z \to t \\ z \bar{\in} \bar{G}}} W(z), t \in \Gamma. \tag{2}$$

We state the following problems:

Problem A. *Let functions $\mu(t) \in V(\Gamma)$ (with bounded variation on Γ), $q(t) \in L_1(\Gamma)$, $q(t) \neq 0$.*

We must find piecewise generalized analytic function $W(z)$, satisfying the conditions:

$W(\infty) = 0, W^+(t) = q(t) W^-(t) + \mu'(t)$ almost everywhere on Γ.

Problem B. *Let boundary of G be a smooth curve and the functions $q(t), g(t) \in Lip_\alpha \Gamma, 0 < \alpha < 1, q(t) \neq 0$ be given.*

We must find a piecewise generalized analytic function $W(z)$, satisfying the conditions:

$W(\infty) = 0, W^+(t) = q(t) W^-(t) + g(t)$ for any $t \in \Gamma$.

Let's begin to investigate these problems from studying some boundary properties of generalized analytic functions. Let G be simply connected finite domain with bounded Jordan rectifiable boundary Γ.

Let's establish the complex-valued function $\phi(t)$ on Γ with bounded variation on Γ. We can consider the function $\phi(s)$ as a function of arc lengths s on contour

Γ considering it a function from $s, t = t(s)$. The function obtained from s we will denote by $\phi(s) = \phi(t(s))$.

Definition 1. *Let's call integral of Cauchy-Stielties type the expression*

$$W(z) = \frac{1}{2\pi i}\int_{\Gamma} \Omega_1(z,t)\, d\phi(t) - \Omega_2(z,t)\, \overline{d\phi(t)}, z\bar{\in}\Gamma, \qquad (3)$$

where $\Omega_1(z,t)$ and $\Omega_2(z,t)$ are the normalized kernels of the class $U_{p,2}(A,B,G)$.

The integral (3) is a regular solution of equation (1) out of Γ (it will be proved below), i.e.

$$W(z) = \frac{1}{2\pi i}\int_{\Gamma} \Omega_1(z,t)\, d\phi(t) - \Omega_2(z,t)\, \overline{d\phi(t)} \in U_{p,2}(A,B,E\backslash\Gamma),$$

where E is all finite plane.

In special case when $\phi(t)$ is absolutely continuous on Γ and $d\phi(t) = \phi'(t)\, dt = \varphi(t)\, dt$, we'll call the corresponding integral

$$W(z) = \frac{1}{2\pi i}\int_{\Gamma} \Omega_1(z,t)\, \varphi(t) - \Omega_2(z,t)\, \overline{\varphi(t)}\ \overline{dt}, t\bar{\in}\Gamma \qquad (4)$$

the Cauchy-Lebesgue type integral.

Definition 2. *Let the point $t_0 \in \Gamma$, and s_0 be the value of the arc corresponding to t_0. Let's denote by Γ_ε a part of line Γ, remained after moving off from Γ that arc, whose ends are the points $t(s_0-\varepsilon), t(s_0+\varepsilon)$ and which contains t_0.*

The ultimate limit (if it exists) of the expression,

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \Omega_1(t_0,t)\, d\phi(t) - \Omega_2(t_0,t)\, \overline{d\phi(t)},$$

is said to be a particular integral and in this case we write

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \Omega_1(t_0,t) - \Omega_2(t_0,t)\, \overline{d\phi(t)} =$$

$$= \frac{1}{2\pi i}\int_{\Gamma} \Omega_1(t_0,t)\, d\phi(t) - \Omega_2(t_0,t)\, \overline{d\phi(t)}. \qquad (5)$$

Let's consider the following difference:

$$\frac{1}{2\pi i}\left[\int_{\Gamma} \Omega_1(z,t)\, d\phi(t) - \Omega_2(z,t)\, \overline{d\phi(t)} - \right.$$

$$\left. - \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \Omega_1(t_0,t)\, d\phi(t) - \Omega_2(t_0,t)\, \overline{d\phi(t)}\right]. \qquad (6)$$

Considering that at the point t_0 there is a tangent to Γ, we draw a normal to Γ at this point and we'll take on the straight line $\overline{zt_0}$ the point z, running at angle

$\psi_0, 0 < \psi_0 < \dfrac{\pi}{2}$ to a normal. The point z can be situated both inside and outside of G. Let the distance $\rho(z, t_0) = \varepsilon$.

If φ_0 is an angle between positive directions of abscissa axes and tangent (at the point t_o), then $z = t_0 \pm i\varepsilon e^{i(\varphi_0+\psi_0)}$.

Theorem 1. *If the curve Γ has a tangent at the point $t_0 = t\ (s_0)$, the functions $\phi(s)$ and its variation $V(s)$ has finite derivatives, then the expression (6) tends to the limit*

$$\pm\frac{1}{2}\phi'(t_0)\ \left(+\frac{1}{2}\phi'(t_0)\ \text{if } z \to t_0 \text{ in } G, -\frac{1}{2}\phi'(t_0)\,,\ \text{if } z \to t_0 \text{ from without } G\right).$$

Convergence to limit is uniform with respect to $\psi_0\,|\psi_0| \leq \dfrac{\pi}{2}\theta, \theta < 1$. As $\phi'(s)$ and $V'(s)$ exist everywhere on Γ and the curve Γ, being rectifiable has tangent at almost all points where $\left|\dfrac{ds}{dt}\right|_{s=s_0} = 1$, then the presence of limit $\pm\dfrac{1}{2}\phi'(t_0)$ in (6) holds everywhere on Γ.

This theorem is the analogy of known well for analytic functions of so-called "I.I. Privalov main lemma" (see [3] p. 183) for case of generalized analytic functions. The proof of the theorem is cited in [8] and therefore we'll not cite it.

If in (6) $\Omega_1(z,t) = \dfrac{1}{t-z}$, $\Omega_1(z,t) = 0$ (it comes out in case $A(z) = B(z) = 0$), then the affirmation of our theorem coincides with the mentioned "I.I. Privalov main lemma" for Cauchy-Stielties type ordinary integrals.

Corollary 1. *Let $z = t_0 + i\varepsilon e^{i(\varphi_0+\psi_0)}, z^* = t_0 - i\varepsilon e^{i(\varphi_0+\psi_0)}$. The difference of values of Cauchy–Stielties type integrals (3) at the points z and z^*, located inside and outside Γ is*

$$J = \frac{1}{2\pi i}\left[\int_\Gamma \Omega_1(z,t)\,d\phi(t) - \Omega_2(z,t)\,\overline{d\phi(t)} - \right.$$

$$\left. - \int_\Gamma \Omega_1(z^*,t)\,d\phi(t) - \Omega_2(z^*,t)\,\overline{d\phi(t)}\right] \to \phi'(t_0)\,,$$

when $\varepsilon \to 0$ for all points t_0 of the line Γ, may be except the point set of measure zero.

Evidently $\phi'(t_0) = \varphi(t_0)$ will be in (6) in case of Cauchy–Lebesgue type integral and described above results will be reformed by an evident way. Let's cite, for example: the formulating of corollary (1) for the case of Cauchy–Lebesgue type integral.

Corollary 2. *Let $z = t_0 + i\varepsilon e^{i(\varphi_0+\psi_0)}$, $z^* = t_0 - i\varepsilon e^{i(\varphi_0+\psi_0)}$. The difference of values of Cauchy-Lebesgue type integral (4) inside and outside Γ is*

$$\frac{1}{2\pi i}\left[\int_\Gamma \Omega_1(z,t)\,\varphi(t) - \Omega_2(z,t)\,\overline{\varphi(t)}\overline{dt} - \right.$$

$$\left. - \int_\Gamma \Omega_1(z^*,t)\,\varphi(t)\,dt - \Omega_2(z^*,t)\,\overline{\varphi(t)}\overline{dt}\right] \to \varphi(t_0)\,,$$

when $\varepsilon \to 0$ *for all points* t_0 *of line* Γ *may be except the point set of measure zero. Let's pass to the solution of Riemann boundary value problem.*

Let's first consider the case $q(t) = 1$, i.e. the analogy of the jump problem.

Theorem 2. *Let* G *a bounded simply connected domain, the boundary* Γ *be a rectifiable Jordan curve,* $q(t) = 1$, *and* $\mu(t) \in V(\Gamma)$ *(the function with bounded variation on* Γ*).*

Then the solution of problem A is a generalized Cauchy-Stielties type integral with measure $\mu(t)$:

$$F(z) = \frac{1}{2\pi i}\int_\Gamma \Omega_1(z,t)\, d\mu(t) - \Omega_2(z,t)\,\overline{d\mu(t)}.$$

Proof. Let's prove first, that $F(z)$ satisfies the equation (1), i.e. it is a generalized analytic function from the class $U_{p,2}(A,B,G)$.

As it's known, (see [1] p. 177) generalized kernels of class $U_{p,2}(A,B,G)$ $\Omega_1(z,t)$ and $\Omega_2(z,t)$ satisfy the equations:

$$\partial_{\bar z}\Omega_1(z,t) = -A(z)\,\Omega_1(z,t) - B(z)\,\overline{\Omega_2(z,t)},$$

$$\partial_{\bar z}\Omega_2(z,t) = -A(z)\,\Omega_2(z,t) - B(z)\,\overline{\Omega_1(z,t)}.$$

We have

$$\partial_{\bar z}F(z) = \frac{1}{2\pi i}\int_\Gamma \partial_{\bar z}\Omega_1(z,t)\, d\mu(t) - \partial_{\bar z}\Omega_2(z,t)\,\overline{d\mu(t)} =$$

$$= \frac{1}{2\pi i}\int_\Gamma \left[-A(z)\,\Omega_1(z,t) - B(z)\,\overline{\Omega_2(z,t)}\right] d\mu(t) -$$

$$- \left[-A(z)\,\Omega_2(z,t) - B(z)\,\overline{\Omega_1(z,t)}\right]\overline{d\mu(t)}\Big] =$$

$$= -\frac{1}{2\pi i}\left[\int_\Gamma A(z)\,\Omega_1(z,t) - A(z)\,\Omega_2(z,t)\,\overline{d\mu(t)}\right] +$$

$$+\frac{1}{2\pi i}\left[\int_\Gamma B(z)\,\overline{\Omega_1(z,t)\, d\mu(t)} - B(z)\,\Omega_2(z,t)\, d\mu(t)\right] =$$

$$= -A(z)\,\frac{1}{2\pi i}\int_\Gamma \Omega_1(z,t)\, d\mu(t) - \Omega_2(z,t)\,\overline{d\mu(t)} -$$

$$-B(z)\left[\overline{\frac{1}{2\pi i}\int_\Gamma \Omega_1(z,t)\, d\mu(t) - \Omega_2(z,t)\,\overline{d\mu(t)}}\right] =$$

$$= -A(z)\,F(z) - B(z)\,\overline{F(z)}.$$

Now we'll prove, that $F(\infty) = 0$.

It is also known, that (see [1], p.179)

$$\Omega_1(z,t) = \frac{e^{\omega_1(z,t)} + e^{\omega_2(z,t)}}{t-z}, \ \Omega_1(z,t) = \frac{e^{\omega_1(z,t)} + e^{\omega_2(z,t)}}{t-z},$$

where the functions $\omega_1(z,t)$ and $\omega_2(z,t)$ are bounded in all plane and continuous (see [1], p.178).

We have:

$$|F(z)| = \left| \frac{1}{2\pi i} \int_\Gamma \Omega_1(z,t)\, d\mu(t) - \Omega_2(z,t)\, \overline{d\mu(t)} \right| \leq$$

$$\leq \frac{1}{\pi} \int_\Gamma \frac{\left|e^{\omega_1(z,t)}\right| + \left|e^{\omega_2(z,t)}\right|}{|t-z|} |d\mu(t)| \leq C \int_\Gamma \frac{|d\mu(t)|}{|t-z|} \to 0,$$

as $z \to \infty$, i.e $F(\infty) = 0$ (here it was taken into account that $\mu(t) \in V(\Gamma)$, and $|d\mu(t)| = \left|\overline{d\mu(t)}\right|$).

It is clear, that corollary 1 in denotation W^+ and W^- is written in the form $F^+(t) - F^-(t) = \mu'(t)$. By this we finish the proof of the theorem. In this case from the fact that under the condition on Γ we require almost everywhere and the function $\mu(t)$ is under the differential, the uniqueness of the solution of problem A can't be affirmed.

As above in theorem 1, using corollary 2 analogically we'll get the following theorem:

Theorem 3. *Let G be a bounded simply-connected domain with smooth boundary Γ. $q(t) = 1$ and $g(t)$ be the given function from class $Lip_\alpha\Gamma, 0 < \alpha < 1$.*

Then the solution of problem B is generalized Cauchy–Lebesgue type integral with density $g(t)$:

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \Omega_1(z,t)\, g(t)\, dt - \Omega_2(z,t)\, \overline{g(t)} dt.$$

Here unlike the problem A smoothness of the curve Γ and belongingness of $g(t) \in Lip_\alpha\Gamma, 0 < \alpha < 1$ provide the uniqueness of problems solution.

In fact, if there are even if two functions $F_1(z)$ and $F_2(z)$ with given conditions, then

$$\left. \begin{array}{l} F_1^+(t) - F_1^-(t) = g(t) \\ F_2^+(t) - F_2^-(t) = g(t) \end{array} \right\} \Longrightarrow \left[F_1^+(t) - F_1^-(t)\right] - \left[F_2^+(t) - F_2^-(t)\right] = 0.$$

If follows from the theorem of uniqueness (see [1] p.158) of generalized analytic functions, that $F_1(z) = F_2(z)$.

By solving Riemann boundary value problem in the second step we consider the homogeneous problem

$$W^+(t) = q(t)\, W^-(t).$$

This expression is lagarithmized

$$\ln W^+(t) = \ln W^-(t) + \ln q(t). \tag{7}$$

In the analytic case in (7) $\ln W^{+}(t)$ and $\ln W^{-}(t)$ are again analytic functions, taking their one-valued branch it is solved as a jump problem with the given function $\ln q(t)$ and by this way the function $\ln W(z)$ is found and hence the function $W(z)$ itself.

In the generalized analytic case the logarithming of functions takes out this function from the given class and the problem looses its sense.

We can meet the same situation at full giving of data $q(t)$ and $\mu(t)$.

That's why now we were satisfied with considering the component part of the Riemann problem, the analogy of the jump problem.

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