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ON A INVERSE BOUNDARY-VALUE PROBLEM FOR PSEUDOPARABOLIC EQUATION OF THE THIRD ORDER WITH NON SELF-ADJOINT BOUNDARY CONDITIONS

Abstract

In the present paper is proved the existence and uniqueness of classical solution of one non self-adjoint one-dimensional inverse boundary-value problem for pseudoparabolic equation of the third order.

In the domain $Q_T = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le T\}$ consider the following equation:

$$u_t(x,t) - bu_{txx}(x,t) - a(t)\,u_{xx}(x,t) = c_0(t)\,u(x,t) + c_1(t)\,g(x,t) + f(x,t),$$

$$(x,t) \in Q_T = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le T\}, \tag{1}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le 1, \tag{2}$$

$$u(0,t) = 0,\ u_x(0,t) = u_x(1,t), \quad 0 \le t \le T, \tag{3}$$

$$\alpha_i u(1,t) + \beta_i \int_0^1 u(x,t)\,dx = h_i(t) + \delta_i \int_0^t \chi(\tau)\,u(1,\tau)\,d\tau,\ i = 0,1, \quad 0 \le t \le T, \tag{4}$$

where $b > 0,\ \alpha_i,\ \beta_i,\ \delta_i\ (i = 0,1)$ are constants,

$a(t) > 0,\ \varphi(x),\ g(x,t),\ f(x,t),\ \chi(t),\ h_i(t)\ (i = 0,1)$ are given functions, and $u(x,t),\ c_0(t),\ c_1(t)$ are unknown functions, moreover under the classical solution of problem (1)-(4) we understand the following

Definition. *Classical solution of inverse boundary-value problem (1)-(4) we'll call the three* $\{u(x,t),\ c_0(t),\ c_1(t)\}$ *of the functions* $u(x,t),\ c_0(t),\ c_1(t)$ *satisfying the following conditions:*

a) the function $u(x,t)$ *is continuous in* Q_T *together with all of its derivatives, entering in equation (1);*

b) the functions $c_i(t)\ (i = 0,1)$ *are continuous on* $[0,T]$*;*

c) equation (1) and conditions (2)-(4) are satisfied in ordinary sense.

The following assertion is true

Lemma 1. *Let* $\varphi(x) \in C[0,1],\ a(t) \in C[0,T],\ f(x,t) \in C(Q_T),\ h_i(t) \in C^1[0,T],\ i = 0,1$ *and the fitting condition*

$$\alpha_i \varphi(1) + \beta_i \int_0^1 \varphi(x)\,dx = h_i(0), \quad i = 0,1, \tag{5}$$

be fulfilled, and also

$$h_0(t)\left(\alpha_1 g(1,t) + \beta_1 \int_0^1 g(x,t)\,dx\right) -$$

$$-h_1(t)\left(\alpha_0 g(1,t)+\beta_0\int_0^1 g(x,t)\,dx\right)\neq 0,\quad t\in[0,T]. \tag{6}$$

Then problem (1)-(4) is equivalent to the problem on determination of the functions $u(x,t)$, $c_0(t)$, $c_1(t)$ from (1)-(3) and

$$c_0(t)\left(h_i(t)+\delta_i\int_0^t \chi(\tau)\,u(1,\tau)\,d\tau\right)+c_1(t)\left(\alpha_i g(1,t)+\beta_i\int_0^1 g(x,t)\,dx\right)=$$

$$=h_i'(t)+\delta_i\chi(t)\,u(1,t)-\alpha_i b u_{txx}(1,t)-\alpha_i a(t)\,u_{xx}(1,t)-\alpha_i f(1,t)-$$

$$-\beta_i\int_0^1 f(x,t)\,dx,\quad t\in[0,T],\quad i=0,1. \tag{7}$$

From [1] we can get that the sequences of the functions

$$X_0(x)=x,\ X_{2k-1}(x)=x\cos\lambda_k x,\ X_{2k}(x)=\sin\lambda_k x,$$

$$\lambda_k=2\pi k,\ k=1,2,..., \tag{8}$$

$$Y_0(x)=2,\ Y_{2k-1}(x)=4\cos\lambda_k x,\ Y_{2k}(x)=4(1-x)\sin\lambda_k x,$$

$$\lambda_k=2\pi k,\quad k=1,2,..., \tag{9}$$

form in $L_2(0,1)$ biorthogonal system and system (8) form the basis in $L_2(0,1)$. Then the arbitrary function $\psi(x)\in L_2(0,1)$ is expanded in biorthogonal series:

$$\psi(x)=\sum_{k=0}^{\infty}\psi_k X_k(x),\quad \text{where}\ \ \psi_k=\int_0^1\psi(x)\,Y_k(x)\,dx\qquad (k=0,1,2,...).$$

Besides, for any function $\psi(x)\in L_2(0,1)$ the following estimation is true

$$\frac{3}{4}\|\psi(x)\|^2_{L_2(0,1)}\le\sum_{k=0}^{\infty}\psi_k^2\le 16\|\psi(x)\|^2_{L_2(0,1)}. \tag{10}$$

Let $\psi(x)\in C^{2i-1}[0;1]$, $\psi^{(2i)}(x)\in L_2(0,1)$ and $\psi^{(2s)}(0)=0$, $\psi^{(2s+1)}(0)=\psi^{(2s+1)}(1)$, $\left(s=\overline{0,i-1}\right)$, $i\ge 1$, then we can show that

$$\sum_{k=1}^{\infty}\left(k^{2i}\psi_{2k-1}\right)^2\le\frac{8}{(2\pi)^{4i}}\left\|\psi^{(2i)}(x)\right\|^2_{L_2(0,1)},\quad i\ge 1,$$

$$\sum_{k=1}^{\infty}\left(k^{2i}\psi_{2k}\right)^2\le\frac{8}{(2\pi)^{4i}}\left\|\psi^{(2i)}(x)(1-x)-2i\psi^{(2i-1)}(x)\right\|^2_{L_2(0,1)},\qquad i\ge 1. \tag{11}$$

and at assumptions

$$\psi(x)\in C^{2i}[0,1],\ \psi^{(2i+1)}(x)\in L_2(0,1),$$

$$\psi^{(2s)}(0)=0,\ \psi^{(2s-1)}(0)=\psi^{(2s-1)}(1),\quad \left(s=\overline{1,i}\right)$$

we can establish the truth of the estimations:

$$\sum_{k=1}^{\infty}\left(k^{2i+1}\psi_{2k-1}\right)^2 \leq \frac{8}{(2\pi)^{2(2i+1)}}\left\|\psi^{(2i+1)}(x)\right\|^2_{L_2(0,1)}, \quad i \geq 1,$$

$$\sum_{k=1}^{\infty}\left(k^{2i+1}\psi_{2k}\right)^2 \leq \frac{8}{(2\pi)^{2(2i+1)}}\left\|\psi^{(2i+1)}(x)(1-x)-(2i+1)\psi^{(2i)}(x)\right\|^2_{L_2(0,1)},$$

$$i \geq 1. \tag{12}$$

We can search the first component $u(x,t)$ of the classical solution $\{u(x,t), c_0(t), c_1(t)\}$ of problem (1)-(4) in the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) X_k(x), \tag{13}$$

where

$$u_k(t) = \int_0^1 u(x,t) Y_k(x)\, dx \quad (k = 0,1,2,...)$$

is a solution of the following problem:

$$u_0'(t) = F_0(t), \tag{14}$$

$$\left(1+b\lambda_k^2\right)u_{2k-1}'(t) + a(t)\lambda_k^2 u_{2k-1}(t) = F_{2k-1}(t), \quad k = 1,2,..., \tag{15}$$

$$\left(1+b\lambda_k^2\right)u_{2k}'(t) + a(t)\lambda_k^2 u_{2k}(t) = F_{2k}(t) - 2b\lambda_k u_{2k-1}'(t) - 2a(t)\lambda_k u_{2k-1}(t),$$

$$k = 1,2,..., \tag{16}$$

$$u_k(0) = \varphi_k \qquad (k = 0,1,2,...), \tag{17}$$

moreover

$$\varphi_k = \int_0^1 \varphi(x) Y_k(x)\, dx, \quad F_k(t) = c_0(t) u_k(t) + c_1(t) g_k(t) + f_k(t),$$

$$f_k(t) = \int_0^1 f(x,t) Y_k(x)\, dx, \quad g_k(t) = \int_0^1 g(x,t) Y_k(x)\, dx \quad (k = 0,1,2,...).$$

Now from (14)-(16) subject to (17) we find:

$$u_0(t) = \varphi_0 + \int_0^t F_0(\tau)\, d\tau, \tag{18}$$

$$u_{2k-1}(t) = \varphi_{2k-1} e^{-\int_0^t \frac{a(s)\lambda_k^2 ds}{1+b\lambda_k^2}} + \int_0^t \frac{F_{2k-1}(\tau)}{1+b\lambda_k^2} e^{-\int_\tau^t \frac{a(s)\lambda_k^2 ds}{1+b\lambda_k^2}} d\tau, \quad k = 1,2,..., \tag{19}$$

$$u_{2k}(t)=\varphi_{2k}\cdot e^{-\int\limits_0^t\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^2}}+\int\limits_0^t\frac{F_{2k}(\tau)}{1+b\lambda_k^2}e^{-\int\limits_\tau^t\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^2}}d\tau-$$

$$-\frac{2\lambda_k}{\left(1+b\lambda_k^2\right)^2}\varphi_{2k-1}e^{-\int\limits_0^t\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^2}}\times$$

$$\times\int\limits_0^t a(\tau)\,d\tau-\frac{2\lambda_k}{\left(1+b\lambda_k^2\right)^3}\int\limits_0^t a(\tau)\int\limits_0^\tau F_{2k-1}(\xi)\,e^{-\int\limits_\xi^t\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^2}}d\xi d\tau-$$

$$-\frac{2b\lambda_k}{\left(1+b\lambda_k^2\right)^2}\int\limits_0^t F_{2k-1}(\tau)\,e^{-\int\limits_\tau^t\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^2}}d\tau,\quad k=1,2,\ldots\,. \tag{20}$$

Proceeding from the definition of the classical solution of problem (1)-(4) the following lemma is proved.

Lemma 2. *If $\{u(x,t),\ c_0(t),\ c_1(t)\}$ is any classical solution of problem (1)-(4) then the functions $u_k(t)$ $(k=0,1,2,\ldots)$ on $[0,T]$ satisfy countable system (18)-(20) of nonlinear integro-differential equations.*

Thus, for determination of the component $u(x,t)$ of solution of problem (1)-(4) we have:

$$u(x,t)=\left(\varphi_0+\int\limits_0^t F_0(\tau)\,d\tau\right)X_0(x)+$$

$$+\sum_{k=1}^{\infty}\left(\varphi_{2k-1}e^{-\int\limits_0^t\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^2}}+\int\limits_0^t\frac{F_{2k-1}(\tau)}{1+b\lambda_k^2}e^{-\int\limits_\tau^t\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^2}}d\tau\right)\times$$

$$\times X_{2k-1}(x)+\sum_{k=1}^{\infty}\left(\varphi_{2k}\cdot e^{-\int\limits_0^t\frac{a(s)\lambda_k^2}{1+b\lambda_k^2}ds}+\int\limits_0^t\frac{F_{2k}(\tau)}{1+b\lambda_k^2}e^{-\int\limits_\tau^t\frac{a(s)\lambda_k^2}{1+b\lambda_k^2}ds}d\tau-\frac{2\lambda_k}{\left(1+b\lambda_k^2\right)^2}\times\right.$$

$$\times\varphi_{2k-1}e^{-\int\limits_0^t\frac{a(s)\lambda_k^2}{1+b\lambda_k^2}ds}\int\limits_0^t a(\tau)\,d\tau-\frac{2\lambda_k}{\left(1+b\lambda_k^2\right)^3}\int\limits_0^t a(\tau)\int\limits_0^\tau F_{2k-1}(\xi)\,e^{-\int\limits_\xi^t\frac{a(s)\lambda_k^2}{1+b\lambda_k^2}ds}d\xi d\tau-$$

$$\left.-\frac{2b\lambda_k}{\left(1+b\lambda_k^2\right)^2}\int\limits_0^t F_{2k-1}(\tau)\,e^{-\int\limits_\tau^t\frac{a(s)\lambda_k^2}{1+b\lambda_k^2}ds}d\tau\right)X_{2k}(x),\quad k=1,2,\ldots\,. \tag{21}$$

Now, we'll choose $c_0(t),c_1(t)$ such that condition (7) be also fulfilled. To determine the components $c_0(t),c_1(t)$ of the classical solution $\{u(x,t),c_0(t),c_1(t)\}$ from (7) subject to (6),(19) we'll obtain:

$$c_0(t)=h^{-1}(t)\left\{h_0'(t)\,q_1(t)-h_1'(t)\,q_0(t)+p_1(t)\,q_0(t)-p_0(t)\,q_1(t)+\right.$$

$$+\left(\delta_0q_1(t)-\delta_1q_0(t)\right)\left[\chi(t)\left(\varphi_0+\int\limits_0^t F_0(\tau)\,d\tau+\right.\right.$$

$$+\sum_{k=1}^{\infty}\left(\varphi_{2k-1}e^{-\int\limits_0^t\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^2}}+\int\limits_0^t\frac{F_{2k-1}(\tau)}{1+b\lambda_k^2}e^{-\int\limits_\tau^t\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^2}}d\tau\right)\Bigg)-$$

$$-c_0(t)\int\limits_0^t\chi(\tau)\left(\varphi_0+\int\limits_0^\tau F_0(\xi)\,d\xi+\right.$$

$$\left.+\sum_{k=1}^{\infty}\left(\varphi_{2k-1}e^{-\int\limits_0^\tau\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^2}}+\int\limits_0^\tau\frac{F_{2k-1}(\xi)}{1+b\lambda_k^2}e^{-\int\limits_\xi^\tau\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^2}}d\xi\right)\right)d\tau\Bigg]+$$

$$+(\alpha_0q_1(t)-\alpha_1q_0(t))\sum_{k=1}^{\infty}\left[\frac{a(t)\lambda_k^2}{1+b\lambda_k^2}\left(\varphi_{2k-1}e^{-\int\limits_0^t\frac{a(\tau)\lambda_k^2d\tau}{1+b\lambda_k^2}}+\right.\right.$$

$$\left.\left.+\int\limits_0^t\frac{F_{2k-1}(\tau)}{1+b\lambda_k^2}e^{-\int\limits_\tau^t\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^5}}d\tau\right)+\frac{b\lambda_k^2}{1+b\lambda_k^2}F_{2k-1}(t\right)\Bigg]\},\tag{22}$$

$$c_1(t)=h^{-1}(t)\Big\{h_0(t)h_1'(t)-h_1(t)h_0'(t)+h_1(t)p_0(t)-h_0(t)p_1(t)+$$

$$+(h_0(t)\delta_1-h_1(t)\delta_0)\left[\chi(t)\left(\varphi_0+\int\limits_0^t F_0(\tau)\,d\tau+.\right.\right.$$

$$\left.+\sum_{k=1}^{\infty}\left(\varphi_{2k-1}e^{-t\int\limits_0^t\frac{a(\tau)\lambda_k^2d\tau}{1+b\lambda_k^2}}+\int\limits_0^t\frac{F_{2k-1}(\tau)}{1+b\lambda_k^2}e^{-\int\limits_\tau^t\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^3}}d\tau\right)\right)-c_0(t)\times$$

$$\times\int\limits_0^t\chi()\left(\varphi_0+\int\limits_0^\tau F_0(\xi)\,d\xi+\right.$$

$$\left.+\sum_{k=1}^{\infty}\left(\varphi_{2k-7}e^{-\int\limits_0^\tau\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^2}}+\int\limits_6^\tau\frac{F_{2k-1}(\xi)}{2+b\lambda_k^2}e^{-\int\limits_\xi^\tau\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^0}}d\xi\right)\right)d\tau\Bigg]+(h_0(t)\alpha_2-$$

$$-h_1(t)\alpha_0)\sum_{k=9}^{\infty}\left[\frac{a(t)\lambda_k^2}{1+b\lambda_k^5}\left(\varphi_{9k-1}e^{-\int\limits_0^t\frac{a(\tau)\lambda_k^2d\tau}{1+b\lambda_k^2}}+\int\limits_0^t\frac{F_{2k-1}(\tau)}{1+b\lambda_k^2}e^{-\int\limits_\tau^t\frac{a(s)\lambda_k^2ds}{1+b\lambda_k^2}}d\tau\right)+\right.$$

$$\left.+\frac{b\lambda_k^2}{3+b\lambda_k^2}F_{2k-1}(t)\right]\Big\},\tag{20}$$

where

$$h(t)=h_0(t)q_1(t)-h_1(t)q_5(t),\quad q_i(t)=\alpha_ig(1,t)+\beta_i1\int\limits_5 g(x,t)\,dx\qquad(i=0,1),$$

$$p_i(t)=\alpha_if(1,t)+\beta_i\int\limits_0^1 f(x,t)\,dx\qquad(i=0,1).$$