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REMOVABLE SETS OF THE SOLUTIONS OF THE SECOND ORDER BOUNDARY-VALUE PROBLEM FOR DEGENERATED PARABOLIC EQUATIONS

Abstract

In the paper we establish sufficient removability condition of a compact with respect to the second boundary-value problem for degenerated parabolic equations in the space of Hölder functions.

1. Let $Q_T = \Omega \times (0,T)$ be a cylindrical domain lying in $R^{n+1}, \Omega \subset R^n$ be a bounded domain with the boundary $\partial\Omega$. $S_T = \partial\Omega \times (0,T)$,

$Q_0 = \{(x,t) : x \in \Omega,\ t = 0\}$. In Q_T we consider the parabolic equation

$$Lu = \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) = f(x,t), \tag{1}$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\Gamma(Q_T)} = 0, \tag{2}$$

where $\frac{\partial}{\partial \nu}$ denotes a derivative by conorms, i.e. $\frac{\partial}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial u}{\partial x_j} n_i$, $(t,x) \in \Gamma(Q_T)$ is a parabolic boundary and n_i is an external unit normal, to the surface of Γ.

Let E be some compact set lying on Γ. The compact E is said to be removable with respect to the second boundary-value problem for equation (1) in $C^{0,\lambda}(Q_T)$, $0 < \lambda < 1$, if it follows form

$$Lu = 0, x \in Q_T, \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma(Q_T)} = 0, \left.u\right|_{t=0} = 0,\ u(x,t) \in C^{0,\lambda}(Q_T) \tag{3}$$

that $u(x,t) \equiv 0$ in Q_T.

With respect to the coefficients we suppose that for all $(x,t) \in Q_T$ and $\xi \in R^n$ the condition

$$\gamma\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi \le \gamma^{-1}\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2, \tag{4}$$

is fulfilled, where $\gamma \in (0,1]$ is a constant, $\lambda_i(x,t) = \left(|x|_\alpha + \sqrt{|t|}\right)^{\alpha_i}$, $|x|_\alpha = \sum_{i=1}^{n} |x_i|^{\bar{\alpha}_i}$, $\bar{\alpha}_i = \frac{2}{2+a_i}$, $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_i \ge 0$, $i = 1, ..., n$, $0 \le \alpha_i < \frac{2}{n-1}$. With respect to the right hand side we suppose, that $f(x,t) \in L_2(Q_T)$.

By $m_H^s(A)$ we denote Hausdorff measure of the set A of order $s>0$.

In case of the Neumann problem for the Laplace equation in piecewise smooth domains the removability problem has been investigated in [1], [2].

The removability problems for the solutions of the first boundary-value problem for elliptic and parabolic equations for the second boundary-value problem set have been investigated in the paper [3]. The removability problems for uniformly degenerated elliptic equations have been investigated in the paper [4], for non-uniformly degenerated parabolic equations in the paper [5].

Theorem 1. *Let Q_T be a cylindrical domain in R^{n+1}, $E\subset Q_T$ be some compact. The condition (4) is fulfilled with respect to the coefficients, the right part $f(x,t)\in L_2(Q_T)$. Then for removability of the compact E with respect to problem (3) it suffices, that*

$$m_H^{\frac{n+\lambda}{2}}(E)=0. \tag{5}$$

Proof. Let's choose some simply connected domain $D\subset\{(\tau,\xi)\in R^{n+1};\ \tau\neq 0,T\}$ such that $E\subset D$. Denote

$$\Pi_r^{(t,x)}=\left\{(\tau,\xi)\in R^{n+1}, t-\frac{r^2}{2}\le\tau\le t+\frac{r^2}{2}; x_i-\frac{r}{2}\le\xi_i\le x_i+\frac{r}{2}\right\},$$

$$i=1,...,m.$$

Let's fix arbitrary $\varepsilon>0$ and cover the set E by the final system $\{\Pi_{r_n}^{(t_n,x_n)}\}$, such that $\bigcup\limits_n\Pi_{4r_n}^{(t_n,x_n)}\subset D$. Denote $\Pi_n=\Pi_{r_n}^{(t_n,x_n)}$, $\Pi_n(\alpha)=\Pi_{\alpha r_n}^{(t_n,x_n)}$, $\sum(a)=\bigcup\limits_n\Pi_n(\alpha)$, $\sigma(\alpha)=\partial\sum(\alpha)\cap Q_T$, $\sigma_n(\alpha)=\sigma(\alpha)\cap\partial\Pi_T(\alpha)$, and E strictly is in $\sum(\alpha)$, $1\le\alpha\le 4$. Let's consider the following function

$$\varphi(\alpha)=\int\limits_{Q_T\setminus\bar\Sigma(\alpha)}\sum_{i,j=1}^n a_{ij}(t,x)\,u_{x_i}u_{x_j}dxdt.$$

By condition (4) we have

$$\varphi(\alpha)\ge\gamma\int\limits_{Q_T\setminus\bar\Sigma(\alpha)}\sum_{i=1}^n\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2dxdt\ge 0.$$

By $\gamma=(\gamma_1,...,\gamma_n,\gamma_t)$ we denote external unit normal to the surface $\sigma(\alpha)$. Then we get

$$-\varphi(\alpha)+\int\limits_{\Gamma\setminus\bar\Sigma(\alpha)}u\left(\sum_{i,j=1}^n a_{ij}(t,x)\,u_{x_j}n_i\right)ds+\sum_{i,j=1}^n\int\limits_{\sigma(\alpha)}ua_{ij}(t,x)\,u_{x_j}\gamma_i ds=$$

$$= \frac{1}{2} \int\limits_{\sigma(\alpha)} u^2 \gamma_t ds + \int\limits_{Q_T \setminus \bar{\Sigma}(\alpha)} f(x,t)\, dxdt.$$

Therefore

$$\varphi(\alpha) \le \sum_{i,j=1}^{n} \int\limits_{\sigma(\alpha)} u \cdot a_{ij}(t,x)\, u_{x_j} \gamma_i ds + \int\limits_{\sigma(\alpha)} u^2 \gamma_t ds + \int\limits_{Q_T \setminus \bar{\Sigma}(\alpha)} f(x,t)\, dxdt. \qquad (6)$$

Further we take into account that

$$\tilde{a}_{ij}(x,t) = \frac{a_{ij}(x,t)}{\sqrt{\lambda_i(x,t)\, \lambda_j(x,t)}} \in C\left(\bar{Q}_T\right) \ \ i,j = 1,...,n \ \ \text{and}$$

$$|\tilde{a}_{ij}(x,t)| \le a_0, i, j = 1, ..., n,$$

where a_0 is a positive constant.

Let's fix the first integral from the right in (6) with the help of Cauchy inequality with $\beta > 0$

$$\sum_{i,j=1}^{n} \int\limits_{\sigma(\alpha)} u \cdot a_{ij}(x,t)\, u_{x_j} \gamma_i ds \le \sum_{i,j=1}^{n} \int\limits_{\sigma(\alpha)} u \frac{a_{ij}}{\sqrt{\lambda_i(x,t)\, \lambda_j(x,t)}} \lambda_i(x,t)\, u_{x_j} \gamma_i ds \le$$

$$\le \sum_{i,j=1}^{n} a_0 \int\limits_{\sigma(\alpha)} u \cdot \lambda_i(x,t)\, u_{x_j} \gamma_i ds \le a_0 \cdot \beta \int\limits_{\sigma(\alpha)} \sum_{i=1}^{n} (u_{x_i})^2 \lambda_i(x,t)\, \gamma_i ds +$$

$$+ \frac{1}{\beta} a_0 \int\limits_{\sigma(\alpha)} \sum_{i=1}^{n} u^2 \lambda_i(x,t)\, \gamma_i ds \le a_0 \cdot \beta \int\limits_{Q_T \setminus \bar{\Sigma}(\alpha)} \sum_{i=1}^{n} \lambda_i(x,t)\, (u_{x_i})^2\, dxdt +$$

$$+ \frac{1}{\beta} a_0 \int\limits_{\sigma(\alpha)} \sum_{i=1}^{n} u^2 \lambda_i(x,t)\, \gamma_i ds.$$

As $\int\limits_{\sigma(\alpha)} |\gamma_i|\, ds \le \sum_{m=1}^{M_0} \int\limits_{\sigma_m(\alpha)} |\gamma_i|\, ds \le K \cdot \sum_{m=1}^{M_0} r_m^{n+1} \le K \cdot \sum_{m=1}^{M_0} r_m^{n+\alpha} \le K \cdot \varepsilon$, where K is a constant and M_0 is the number of parallelepipeds.

Then

$$\int\limits_1^4 \varphi(\alpha)\, d\alpha \le K \cdot \varepsilon.$$

Hence, by virtue of arbitrariness of $\varepsilon > 0$, we conclude, that

$$\int\limits_{\sigma(\alpha)} \sum_{i=1}^{n} \lambda_i(x,t) \left(\frac{\partial u}{\partial x_i} \right)^2 dxdt = 0,$$

almost everywhere in Q_T, and since $\lambda_i(x,t) > 0$ a.e., then $u(x,t) \equiv 0$. The theorem is proved.

2. In Q_T we consider the parabolic equation

$$Lu = \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} +$$

$$+c(x,t)u = 0 \quad \text{in } Q_T \tag{7}$$

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial u}{\partial x_i}n_i + \frac{1}{2}\sum_{i=1}^{n} b_i(t,x)un_i = 0, \quad (t,x) \in \Gamma(Q_T). \tag{8}$$

With respect to the coefficients the condition (4) is fulfilled and also

$$|b_i(t,x)| \le b_0, \ -b_0 \le c(t,x) < 0. \tag{9}$$

Theorem 2. *Let Q_T be a cylindrical domain in R^{n+1}, $E \subset Q_T$ be some compact. Conditions (4), (9) are fulfilled with respect to the coefficients. Then for removability of the compact E with respect to problem (7), (8), it suffices, that*

$$m_H^{\frac{n+\lambda}{2}}(E) = 0. \tag{10}$$

Proof. Let's choose some simply connected domain $D \subset \{(\tau,\xi) \in R^{n+1};\ \tau \ne 0,T\}$ such that $E \subset D$. Let's fix arbitrary $\varepsilon > 0$ and cover the set E by the final system $\{\Pi_{z_n}^{(t_n,x_n)}\}$ such that $\bigcup\limits_{n} \Pi_{4r_n}^{(t_n,x_n)} \subset D$. Let's show this process.

We cover the set E by no more than countable system $\{\Pi_{h_m}^{(\eta_m,y_m)}\}$, for which $\sum\limits_m h_m^{\frac{n+\lambda}{2}} < \varepsilon$ and choose final subcovering from M elements, each of them intersects E. Then $E \subset \bigcup\limits_{m=1}^{M} \Pi_{H_m}^{(\theta_m,y_m)}$, where $\theta_m = \eta_m - \frac{1}{2}, H_m = C_1 h_m^{1/2}$ and $\sum\limits_m H_m^{n+\lambda} < C_1^{n+\alpha}\varepsilon$, and E is strictly contained in this unification. Let's denote $\overline{M} = \inf \sum\limits_m H_m^{n+\alpha} \le C_1^{n+\alpha}\varepsilon$, where inf is taken on all coverings, consisting of no more than M parallelepipeds. Then there exists the system $\Pi_{r_n}^{(t_n,x_m)}$ consisting of M_0 parallelepipeds, for which $M_0 \le M, E$ is strictly contained in $\bigcup\limits_{m=1}^{M_0} \Pi_{r_m}^{(t_m,x_m)}$, $\sum\limits_{m=1}^{M_0} r_m^{n+\lambda} \le \left(C_1^{n+\alpha}+1\right)\varepsilon$, $r_m < \delta_0 < 1$, $\sum\limits_{m=1}^{M_0} r_m^{n+\lambda} < \sum\limits_{\tau} h_\tau^{n+\alpha} + \frac{\varepsilon}{M}$, for any covering $\left\{\Pi_{h_\tau}^{(\eta_\tau,y_\tau)}\right\}$ consisting of no more than of M elements. Now suppose $\Pi_m = \Pi_{r_m}^{(t_m,x_m)}; \Pi_m(\alpha) = \Pi_{\alpha r_m}^{(t_m,x_m)};\ 1 \le \alpha \le 4;\ \Sigma(\alpha) = \bigcup\limits_{m=1}^{M_0} \Pi_m(\alpha);\ \sigma(\alpha) = \partial\Sigma(\alpha); \sigma_m(\alpha) = \sigma(\alpha) \cap \partial\Pi_m(\alpha)$. Then it is evident that E is strictly contained in $\Sigma(\alpha)$, $1 \le \alpha \le 4$, $\sigma(\alpha) = \sum\limits_{m=1}^{M_0} \sigma(\alpha)$, $1 \le \alpha \le 4$. Further acting as in theorem 1 we obtain

$$-\varphi(\alpha) + \int\limits_{\Gamma\backslash\bar{\Sigma}(\alpha)} u \left(\sum_{i,j=1}^{n} a_{ij}(t,x)\, u_{x_j} n_i \right) ds + \sum_{i,j=1}^{n} \int\limits_{\sigma(\alpha)} u a_{ij}(t,x)\, u_{x_j} \gamma_i ds +$$

$$+\frac{1}{2} \int\limits_{\Gamma\backslash\bar{\Sigma}(\alpha)} u \cdot \sum_{i=1}^{n} b_i(t,x)\, u n_i ds - \frac{1}{2} \int\limits_{Q_T\backslash\bar{\Sigma}(\alpha)} u^2 \cdot \sum_{i=1}^{n} \frac{\partial b_1}{\partial x_i} dxdt +$$

$$+ \int\limits_{Q_T\backslash\bar{\Sigma}(\alpha)} c(x,t)\, u^2 dxdt + \frac{1}{2} \sum_{i=1}^{n} \int\limits_{\sigma(\alpha)} b_i(t,x)\, u^2 \gamma_i ds = \frac{1}{2} \int\limits_{\sigma(\theta)} u^2 \gamma_t ds.$$

Further making estimations close to ones in theorem 1 and estimating the members with $b_i(t,x)$ and with $c(t,x)$ subject to conditions (9), we get:

$$\int\limits_1^4 \varphi(\alpha)\, d\alpha - \int\limits_1^4 d\alpha \int\limits_{Q_T\backslash\bar{\Sigma}(\alpha)} c(t,x)\, u^2 dxdt \le K \cdot \varepsilon,$$

where K is a positive constant. Hence, by arbitrariness of $\varepsilon > 0$ we obtain, that $u(x,t) \equiv 0$.

The theorem is proved.

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