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THE LIMIT THEOREMS FOR ONE CLASS OF MOMENTS OF FIRST INTERSECTION OF NONLINEAR BOUNDARY BY MULTIDIMENSIONAL WALK

Abstract

In the present paper the moment of first intersection of nonlinear boundary by the random process, described by nonlinear functions of multidimensional walk is considered. The integral limit theorems for one class of n on-linear moments of the first intersection are proved.

1. Introduction.

Let $\xi_n,\ n \geq 1$ be a sequence of independent equally distributed random vectors in R^k, $k \geq 1$, and let the numerical Borel function $\Delta(x)$, $x \in R^k$ be given.

Assume at $n \geq 1$

$$S_n = \sum_{k=1}^{n} \xi_k, \qquad T_n = n\Delta(S_n/n).$$

Consider the first passage time of the process T_n crosses the nonlinear boundary

$$\tau = \tau_a = \inf\{n \geq 1 : T_n \geq f_a(n)\},$$

where $f_a(t),\ a > 0,\ t > 0$ is some family of nonlinear boundaries.

Many papers (see, for example, the monograph [1] and thesis [3]) have been devoted to study of asymptotic properties of the Markov moment τ_c in one-dimensional case $(k = 1)$ at $f_a(t) \equiv a$.

In the papers [1] and [3] the integral limit theorems (ILT) for τ_a are studied, under which it is got any statement that at some conditions there exist the constants $A(a)$, $B(a) > 0$ and the variate η such that

$$\frac{\tau_a - A(a)}{B(a)} \Longrightarrow \eta, \quad a \longrightarrow \infty, \tag{1}$$

where the sign $\Longrightarrow$ means convergence in distribution.

Recently, there is a great interest to study of boundary-value problems for multidimensional random walk [4-7].

As it was noted in [7] the multidimensional case (i.e. the case $k > 1$) has been studied substantially smaller. It is connected therewith, that the method of study of boundary-value problems in multi-dimensional case has been developed less systematically, than one-dimensional case. A series of particular results, relating to asymptotic properties of the distribution τ_a are in the papers [2, 5, 6, 7]. In [2] some problems of sequential analysis are investigated, in which arises the moment of stoppage of the form τ_c and $\Delta(x) = \|x\|^2$ and $f_a(t) \equiv a$, where $\|\cdot\|$ is an ordinary Euclidean norm in R^k. In the paper [5], the ILT have been studied for τ_a in the case of linear boundary $(f_a(t) \equiv a)$.

The purpose of the present paper is the further study of ILT in multidimensional case for boundary crossing time τ_a $(f_a(t) \neq a)$.

2. Conditions and denotation.

We'll assume, that the random vector ξ_1 has finite mathematical expectation $\nu = E\xi_1$. Let's denote by H a class of the functions $\Delta(x)$, $x \in R^k$, for which the following conditions are fulfilled: $\Delta(x)$ has continuous partial derivatives $\Delta'_{x_i}(x)$, $i = \overline{1,k}$ at some neighborhood of the point $x = \nu$, at that $\Delta(\nu) > 0$ and $\Delta'_{x_i}(\nu) \neq 0$ at least for one $i = \overline{1,k}$.

With respect to nonlinear boundary $f_a(t)$ we'll assume, that it satisfies the following conditions:

1) For each a the function $f_a(t)$ monotonically increases, continuously differentiable at $t > 0$, and $f_a(1) \uparrow \infty$, $a \to \infty$.

2) $n = n(a) \to \infty$ as $a \to \infty$, such that $\frac{1}{n} f(n) \to \mu = \Delta(\nu) > 0$ and $f'_a(n) \to \theta$ for some $\theta \in [0, \mu)$.

3) For each a the function $f'_a(t)$ weakly oscillates at the infinity, i.e.

$$\frac{f'_a(n)}{f'_a(m)} \to 1 \ as \quad \frac{n}{m} \to 1, \quad n \to \infty.$$

Let's denote by W a class of family of boundaries, satisfying conditions 1)-3), and by $N_a = N_a(\mu)$ - the solution of the equation $f_a(n) = n\mu$, which exists for sufficiently large a [4].

(x, y) below means ordinary scalar product of the vectors $x, y \in R^k$.

3. Statement of the basic results.

Theorem 1. *Let ξ_n, $n \geq 1$ be a sequence of the independent identically distributed random vectors in R^k with the mean value $\nu = E\xi_1$ and matrix of covariation B. Moreover, let $\Delta(x) \in H$ and $f_a(t) \in W$. Then*

$$\frac{\tau_a - N_a}{\sqrt{N_a}} \Longrightarrow \frac{(\eta, \lambda)}{\mu - \theta},$$

where $\lambda = \left(\Delta'_{x_1}(\nu), ..., \Delta'_{x_k}(\nu)\right)$, and $\eta = (\eta_1, ..., \eta_k)$ is a k-dimensional normal random vector with a zero vector of mathematical expectations and matrix of covariation B.

Note, that it follows from the well known properties of the Gauss variate that the random variable (η, λ) has the normal distribution with the parameters

$$\left(0, \sum_{i,j=1}^{k} cov\left(\eta_i, \eta_j\right) \Delta'_{x_i}(\nu) \Delta'_{x_j}(\nu)\right).$$

Theorem 1 permits the following generalization for random vectors, belonging to the gravitation domain of multidimensional stable distribution in the sense of Levi-Feldheym with the characteristical exponent $\alpha \in (1, 2]$ ([9], [10]).

Theorem 2. *Let ξ_n, $n \geq 1$ be a sequence of independent identically distributed random vectors in R^k with a finite vector of mathematical expectations $\nu = E\xi_1$* and *$\Delta(x) \in H$, $f_a(t) \in W$.*

Suppose, that there is the sequence $A(n) > 0$ and k-dimensional random vector J such that

$$\frac{S_n - n\nu}{A(n)} \Longrightarrow J.$$

Then

$$\frac{\tau_c - N_a}{A([N_a])} \Longrightarrow -\frac{(J, \lambda)}{\mu - \theta},$$

where [] is a sign of the whole part.

Note, that here the random vector J has k-dimensional stable distribution in R^k with the characteristical exponent $\alpha \in (1,2]$ and as a $A(n)$ we can take $A(n) = n^{1/\alpha} L(n)$, where $L(x)$, $x > 0$ is a slowly varying function [10].

4. The proof of basic results.

First of all, let's remark, that under the given conditions, concerning the function $\Delta(x)$, we have

$$T_n = Z_n + \varepsilon_n \tag{2}$$

where

$$Z_n = \sum_{k=1}^{n} X_i, \quad X_i = \Delta(\nu) + (\lambda, \xi_i - \nu),$$

$$\varepsilon_n = n\left(\frac{1}{n} S_n - \nu, \lambda_n - \lambda\right),$$

$$\lambda_n = \left(\Delta'_{x_1}(\nu_n), ..., \Delta'_{x_k}(\nu_n)\right)$$

and ν_n, $n \geq 1$ is some sequence of the random points from the neighbourhood of ν, at that $\nu_n \overset{?????}{\longrightarrow} \nu$, $n \to \infty$.

As it is seen, Z_n, $n \geq 1$ is a one-dimensional random walk with $EZ_1 = \Delta(\nu) > 0$.

So, the first passage time τ_a takes the following form

$$\tau_a \inf\{n \geq 1 : Z_n + \varepsilon_n \geq f_a(t)\},$$

which allows to apply analytical methods of the papers [1] and [4].

The proof of the stated theorems is based on the following auxiliary lemmas.

Lemma 1. *Let the random vector ξ_1 have the finite mathematical expectation $\nu = E\xi_1$ and $\Delta(x) \in H$, $f_a(t) \in W$. Then*

$$1) \;\; \tau_a \overset{a.c}{\longrightarrow} \infty, \; a \to \infty;$$

$$2) \;\; \frac{\tau_a}{N_a} \overset{a.c.}{\longrightarrow} 1, \; a \to \infty;$$

$$3) \;\; \frac{A(\tau_a)}{A([N_a])} \overset{P}{\longrightarrow} 1, \; a \to \infty.$$

Proof. From expansion (2) we have

$$\frac{T_n}{n} = \frac{Z_n}{n} + \left(\frac{S_n}{n} - \nu, \lambda_n - \lambda\right).$$

By virtue of strong law of large numbers

$$\left(\frac{S_n}{n} - \nu, \lambda_n - \lambda\right) \overset{a.c.}{\longrightarrow} 0, \quad n \to \infty.$$

Therefore,

$$\frac{T_n}{n} \overset{a.c.}{\longrightarrow} \Delta(\nu) > 0, \quad n \to \infty.$$

From here, it follows, that $\sup_n T_n = \infty$.

According to

$$P(\tau_a > n) = P\left(\max_{1\le k\le n}(T_k - f_a(k)) \le 0\right) \ge P\left(\max_{1\le k\le n} T_k \le f_a(1)\right),$$

we obtain statement 1).

Let us prove statement 2). By definition of τ we have

$$\frac{T_{\tau-1}}{\tau} \le \frac{f_a(\tau)}{\tau} \le \frac{T_\tau}{\tau}$$

or

$$\frac{Z_{\tau-1}+\varepsilon_{\tau-1}}{\tau} \le \frac{f_a(\tau)}{\tau} \le \frac{Z_\tau+\varepsilon_\tau}{\tau}.$$

By virtue of strong law of large numbers

$$\frac{Z_n}{n} \xrightarrow{a.c.} \mu \quad \text{and} \quad \frac{\varepsilon_n}{n} \xrightarrow{a.c.} 0, \quad n\to\infty.$$

Therefore, from the first part of lemma 2 and from the Richter lemma [11] we obtain, that

$$\frac{f_a(\tau_a)}{\tau_a} \xrightarrow{a.c.} \mu \quad \text{as} \quad a\to\infty.$$

Then

$$\Delta(a) = \frac{f_a(\tau_a)}{\tau_a} - \frac{f_a(N_a)}{N_a} = \frac{\lambda_a(\nu_a)}{\nu_a}\frac{N_a-\tau_a}{\nu_a},$$

where

$$\lambda_a(t) = f_a(t) - tf'_a(t).$$

Taking into account, that $\Delta(a) \xrightarrow{a.c.} 0$ as $a\to\infty$ from the last correlation we obtain statement 2) of lemma 2.

Statement 3) follows from statement 2 and from the fact, that the convergence

$$\frac{L(tx)}{L(x)} \to 1, \quad x\to\infty$$

is uniformly fulfilled with respect to t from the bounded set in $(0,\infty)$ [10].

Lemma 2. *Let* $\xi_n = (\xi_{n1},\xi_{n2},...,\xi_{nk})$, $n\ge 1$ *be a sequence of* k*-dimensional random vectors and* $\xi = (\xi_1,\xi_2,...,\xi_k)$ *be a* k*-dimensional random vector such that* $\xi_n \Longrightarrow \xi$. *Then*

$$\sum_{i=1}^{k}\xi_{ni} \Longrightarrow \sum_{i=1}^{k}\xi_i, \quad n\to\infty.$$

Proof. Denote by $f(t)$ and $f(t)$, $t=(t_1,...,t_k)\in R^k$ the characteristical functions of the random vectors ξ_n and ξ respectively. From the condition of the lemma we have

$$f_n(t)\to f(t), \quad n\to\infty$$

for each $t\in R^k$.

The characteristical functions of the sums $\sum_{i=1}^{k}\xi_{ni}$ and $\sum_{i=1}^{k}\xi_i$ are equal to

$$\varphi_{nk}(s) = f_n(t)\,|_{t_1=t_2=...=t_k=S} \quad \text{and} \quad \varphi_k(s) = f(t)\,|_{t_1=t_2=...=t_k=S}\ .$$

respectively.

Hence, we obtain that for each $s \in R$

$$\varphi_{nk}(s) \to \varphi_k(s) \quad \text{as} \quad n \to \infty,$$

that proves the statement of lemma 2.

Now let us prove the theorems.

Proof of theorem 1. We have

$$Z_n^* = \frac{Z_n - n\Delta(\nu)}{\sqrt{n}} = \left(\lambda, \frac{S_n - n\nu}{\sqrt{n}}\right), \tag{3}$$

According to multidimensional central limit theorem

$$S_n^* = \frac{S_n - n\nu}{\sqrt{n}} \Longrightarrow \eta, \qquad n \to \infty.$$

Then from (3) and lemma 2 we find

$$Z_n^* \Longrightarrow (\lambda, \eta), \quad n \to \infty. \tag{4}$$

It is well known (see [1]), that the sequence of normalized sums Z_n^*, ≥ 1 of independent equally distributed variates is uniformly continuous in probability (see also [3]). Hence, by virtue of (4) and statement (2) of lemma 1 the Anskombe theorem is applicable to it, according to which

$$\frac{Z_\tau - \tau\Delta(\nu)}{\sqrt{\tau}} \Longrightarrow (\lambda, \eta), \quad a \to \infty. \tag{5}$$

Then, by the definition $\tau = \tau_a$ and $\chi_a = T_\tau - f_a(\tau)$ we have

$$\frac{Z_\tau - \mu\tau}{\sqrt{\tau}} = \frac{f_a(\tau) - \mu\tau}{\sqrt{\tau}} + \frac{\chi_a - \varepsilon_\tau}{\sqrt{\tau}} =$$

$$= \frac{f_a(N_a) - \mu\tau}{\sqrt{\tau}} + \frac{f_a(\tau) - f_a(N_a)}{\sqrt{\tau}} + \frac{\chi_a - \varepsilon_\tau}{\sqrt{\tau}} =$$

$$= -\mu\tau^* + f_a'(\nu)\tau^* + \frac{\chi_a - \varepsilon_\tau}{\sqrt{\tau}} = \tau^*\left(f_a'(\nu_a) - \mu\right) + \frac{\chi_a - \varepsilon_\tau}{\sqrt{\tau}}, \tag{6}$$

where $\tau^* = \dfrac{\tau - N_a}{\sqrt{N_a}}$ and ν_a is some intermediate point between τ_a and N_a.

Let's show that

$$\frac{\chi_a - \varepsilon_\tau}{\sqrt{\tau}} \xrightarrow{P} 0, \qquad a \to \infty. \tag{7}$$

Really, we have

$$0 \leq \chi_a \leq T_\tau - T_{\tau-1} \leq X_\tau + \varepsilon_\tau - \varepsilon_{\tau-1},$$

$$\frac{\varepsilon_n}{\sqrt{n}} = \left(\lambda_n - \lambda, \frac{S_n - n\nu}{\sqrt{n}}\right)$$

and

$$\frac{X_n}{\sqrt{n}} = \frac{\Delta(\nu)}{\sqrt{n}} + \left(\lambda, \frac{\xi_n - \nu}{\sqrt{n}}\right).$$

It is easy to see, that $\lambda_n \xrightarrow{nh} \lambda$ and $\dfrac{X_n}{\sqrt{n}} \xrightarrow{P} 0$ as $n \to \infty$.

With the help of discussions, realized in the papers [4] and [5] we can show, that

$$\frac{\chi_\tau}{\sqrt{\tau}} \xrightarrow{P} 0 \quad \text{and} \quad \frac{\varepsilon_\tau}{\sqrt{\tau}} \xrightarrow{P} 0, \qquad a \to \infty \tag{8}$$

(7) follows from (8).

Now the statement of the proved theorem follows from (5) and (7).

Proof of theorem 2. This theorem is proved by the scheme of the proof of theorem 1. At that it suffices to show, that the sequence

$$Z_n^* = \frac{Z_n - n\mu}{A(n)}, \quad n \geq 1$$

is uniformly continuous in probability and this follows from the fact, that the normalizing constants $A(n) = n^{1/\alpha} L(n)$ satisfy the condition of the paper [6] and thus we can apply the Anskombe theorem. Other distributions are realized with the corresponding computations, beginning with equality (3), where instead of $\sqrt{n}$ we are to take $A(n)$.

References

[1]. Woodroofe M. *Nonlinear renewal theory in sequential analysis.* SIAM, 1982.

[2]. Siegmund D. *Sequential analysis and confidence interval.* New-York, Springer-Verlag, 1985.

[3]. Ragimov F.G. *Limit theorems for moments of first going out of processes with independent increments.* Cand. Thesis, M., 1985, p.127. (Russian)

[4]. Ragimov F.G. *Integral limit theorems for time of intersection of nonlinear boundaries, by perturbed random walk.* Doklady NANA, 2003, No3,4, p.3-8.

[5]. Ragimov F.G. *On asymptotic behavior of one class of moments of the first going out beyond the boundary in multidimentional random walk.* Teoria veroyatnostey I ee primenenie. 2005 (to appear). (Russian)

[6]. Aras C, Woodroofe M. *Asymptotic expansions for the moments of randomly stopped average.* Annal of statistics. 1993, v.21, No1, pp.503-519.

[7]. Borovkov A.A., Mogulsky A.A. *Limit theorems in the problem of boundary achievement by multidimensional walk.* Sib. mat. zhurn., 2001, v.42, No2, pp.289-318. (Russian)

[8]. Mogyoridi J. *On limiting distributions for sums of random number of independent random.* Variables-Mogyar tud. Akad. Mati. Kutato int. Kozl, 1961, A6, No3, pp.365-372.

[9]. Rvacheva Ye.L. *On the gravitation domains of multidimensional stable distributions.* Uch. Zapiski Lvovsk. Universiteta, ser. mech-math., 1954, v.XXIX, No1, pp.5-44. (Russian)

[10]. Zolotarev V.M. *One-dimensional stable distributions.* M., 1983. (Russian)

[11]. Richter W. *Ubertoigung Von geren zanssagen für folgen von zufallingen intizes.* Teor. veroyatn. I ee primen. 1965, v.X, No1, pp.82-93.

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