Elshan N. MAMEDOV

ON A FOURTH ORDER OPERATOR-DIFFERENTIAL EQUATION IN HILBERT SPACE

Abstract

In this paper algebraic conditions, providing existence and uniqueness of a regular solution of a class of elliptic type fourth order operator-differential equations, are obtained

Consider the following fourth order equation in separable Hilbert space H:

$$P\left(d/dt\right)u=\frac{d^4u}{dt^4}+A^4u+\sum_{j=0}^{4}A_{4-j}u^{(j)}=f\left(t\right),\qquad t\in R=\left(-\infty,\infty\right). \tag{1}$$

Here $f\left(t\right)$, $u\left(t\right)$ are vector-valued functions with the values in H, A and A_j $\left(j=\overline{0,4}\right)$ are linear operators in H.

Let A be a self-adjoined positive operator in H, i.e. $A=A^*\geq\mu_0>0$. Determine the following Hilbert spaces [1-3] for $\gamma\in\left(-\infty,\infty\right)$

$$L_{2,\gamma}\left(R;H\right)=\left\{f|\ \ \|f\|_{L_{2,\gamma}}=\left(\int_{-\infty}^{\infty}\|f\left(t\right)\|^2e^{-2\gamma t}dt\right)^{1/2}<\infty\right\}$$

and

$$W_{2,\gamma}^4\left(R;H\right)=\left\{u|\frac{d^4u}{dt^4},\ A^4u\in L_{2,\gamma}\left(R;H\right),\right.$$

$$\left.\|u\|_{W_{2,\gamma}}=\left(\left\|A^4u\right\|_{L_{2,\gamma}}^2+\left\|u^{(4)}\right\|_{L_{2,\gamma}}^2\right)^{1/2}\right\}.$$

For $\gamma=0$ we'll assume, that $L_{2,0}\left(R;H\right)=L_2\left(R;H\right)$, $W_{2,0}^4\left(R;H\right)=W_2^4\left(R;H\right)$.

Definition. *If for $f\in L_{2,\gamma}\left(R;H\right)$ there exists a vector-function $u\in W_{2,\gamma}^4\left(R;H\right)$ which satisfies equation (1) almost everywhere in R, we call it regular solution of equation (1), in addition, if the following inequality holds:*

$$\|u\|_{W_{2,\gamma}}\leq const\,\|f\|_{L_{2,\gamma}},$$

then, equation (1) will be called regularly solvable.

In the given paper we find condition on coefficients of equation (1), that provide regular solvability of equation (1).

Note, that for A an elliptical operator with discrete spectrum, and operators $A_j=a_j$ constant scalar numbers of equation (1) in some weight spaces are investigated in the paper [3]. For $\gamma=0$ and A_j some unbounded operators of equation (1) are investigated in [4], and for some conditions on resolvents $P^{-1}\left(\lambda\right)$, are considered in the papers [1], [5].

In the given paper solvability conditions of equation (1) are expressed by some properties of operator coefficients A and A_j $\left(j=\overline{0,4}\right)$ and therefore easy verifiable in concrete problems.

Denote by

$$P_0 u=\frac{d^4u}{dt^4}+A^4u,\quad P_1 u=\sum_{j=0}^{4}A_{4-j}u^{(j)},\qquad u\in W^4_{2,\gamma}\left(R;H\right).$$

It holds the following

Theorem 1. *Let A be a positive-definite self-adjoint operator and* $\inf\sigma\left(A\right)=\mu_0>0$. *Then for* $|\gamma|<\frac{1}{\sqrt[4]{8}}\mu_0$ *the following equation*

$$P_0 u=\frac{d^4u}{dt^4}+A^4u=f \tag{2}$$

is regularly solvable.

Proof. Let $u\left(t\right)=\vartheta\left(t\right)e^{-\gamma t}$, $g\left(t\right)=f\left(t\right)e^{-\gamma t}$, then equation (2) takes the following form

$$P_{0,\gamma}\vartheta=\left(\frac{d}{dt}+\gamma\right)^4\vartheta+A^4\vartheta=g, \tag{3}$$

where $\vartheta\in W^4_2\left(R;H\right)$, $g\in L_2\left(R;H\right)$. Since the roots of the characteristic equation $\left(\lambda+\gamma\right)^4+\mu^4=0$ $\left(\mu\in\sigma\left(A\right)\right)$ has the form $\lambda_i=-\gamma+\omega_i\mu$, where ω_i are roots of equation $z^4+1=0$, two roots of characteristic equation lie in half-plane $\operatorname{Re}\lambda<-\gamma+\operatorname{Re}\omega_i\mu_0=-\gamma-\frac{1}{\sqrt{2}}\mu_0<0$, and two of them lie in half plane $\operatorname{Re}\lambda>-\gamma+\frac{1}{\sqrt{2}}\mu_0>0$. Therefor for any $\xi\in R=\left(-\infty,\infty\right)$ the operator bundle $P_0\left(-i\xi\right)=\left(-i\xi+\gamma\right)^4E+A^4$ has a bounded inverse $P^{-1}_{0,\gamma}\left(-i\xi\right)$. Then denote by

$$\hat{\vartheta}\left(\xi\right)=P^{-1}_{0,\gamma}\left(-i\xi\right)\hat{g}\left(\xi\right), \tag{4}$$

where $\hat{g}\left(\xi\right)$ is a Fourier transformation of the vector-function $g\left(t\right)$ and show that

$$\vartheta\left(t\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}P^{-1}_{0,\gamma}\left(-i\xi\right)\hat{g}\left(\xi\right)\bar{e}^{i\xi t}d\xi \tag{5}$$

is a regular solution of equation (2) for $|\gamma|<\frac{1}{\sqrt[4]{8}}$.

Show, that for $|\gamma|<\frac{1}{\sqrt[4]{8}}\mu_0$ a vector-function $v\in W^4_2\left(R;H\right)$. From (2) and (3) it follows, that it sufficies to prove, that $A^4\vartheta\in L_2\left(R;H\right)$. So, by Plansharel theorem it sufficies to show, that $A^4\hat{\vartheta}\left(\xi\right)\in L_2\left(R;H\right)$. From (4) it follows, that

$$\left\|A^4\hat{\vartheta}\right\|_{L_2}=\left\|A^4P^{-1}_0\left(-i\xi\right)\hat{g}\left(\xi\right)\right\|_{L_2}\le\sup_{\xi\in R}\left\|A^4P^{-1}_0\left(-i\xi\right)\right\|\left\|\hat{g}\right\|_{L_2}. \tag{6}$$

On the other hand for $\xi\in R$ the following inequalities hold

$$\left\|A^4P^{-1}_0\left(-i\xi\right)\right\|\le$$

$$\leq \sup_{\mu\geq\mu_0}\left|\mu^4\left(\left(\xi^4-6\xi^2\gamma^2+\gamma^4+\mu^4\right)^2+16\xi^2\gamma^2\left(\xi^2+\gamma^2\right)\right)^{-1/2}\right|\leq$$

$$\leq \sup_{\mu\geq\mu_0}\left|\mu^4\left(\xi^4-6\xi^2\gamma^2+\gamma^4+\mu^4\right)^{-1}\right|\leq$$

$$\leq \sup_{\mu\geq\mu_0}\left|\mu^4\left(\left(\xi^2-3\gamma^2\right)^2+\mu^4-8\gamma^4\right)^{-1}\right|\leq$$

$$\leq \sup_{\mu\geq\mu_0}\left|\mu^4\left(\mu^4-8\gamma^4\right)^{-1}\right|\leq\frac{\mu_0^4}{\mu_0^4-8\gamma^4}>0.$$

Thus,

$$\left\|A^4\vartheta\right\|_{L_2}=\left\|A^4\hat{\vartheta}\right\|_{L_2}\leq\frac{\mu_0^4}{\mu_0^4-8\gamma^4}\left\|\hat{g}\right\|_{L_2}=\frac{\mu_0^4}{\mu_0^4-8\gamma^4}\left\|g\right\|_{L_2}. \qquad (7)$$

Further, obviously, that $\vartheta(t)$ satisfies equation (2) almost everywhere in R. Then vector-function $u(t)=\vartheta(t)e^{\gamma t}\in W_{2,\gamma}^4(R;H)$ satisfy equation (2) almost everywhere in R and from (2) and (7) it follows, that

$$\|u\|_{W_{2,\gamma}}\leq const\,\|f\|_{L_{2,\gamma}}.$$

The theorem is proved.

Theorem 2. *Let conditions of theorem 1 be fulfilled. Then for any* $u\in W_{2,\gamma}^4(R;H)$ *the following inequalities hold:*

$$\left\|A^{4-j}u^{(j)}\right\|_{L_{2,\gamma}}\leq c_j(\gamma;\mu_0)\left\|P_0u\right\|_{L_{2,\gamma}}\qquad j=\overline{0,4}, \qquad (8)$$

where

$$c_0(\gamma;\mu_0)=\frac{\mu_0^4}{\mu_0^4-8\gamma^4} \qquad (9)$$

$$c_1(\gamma;\mu_0)=c_3(\gamma;\mu_0)=\frac{3^{3/4}}{4}\left(1+\frac{32\mu_0^4}{\mu_0^4-8\gamma^4}+\frac{4\gamma^2}{\sqrt{\mu_0^4-8\gamma^4}}\right) \qquad (10)$$

$$c_2(\gamma;\mu_0)=\frac{1}{2}\left(1+\frac{8\gamma^4+2\gamma^2\mu_0^2}{\mu_0^4-8\gamma^4}\right) \qquad (11)$$

$$c_4(\gamma;\mu_0)=1+\frac{24\gamma^4}{\mu_0^4-8\gamma^4}+\frac{4\gamma^2}{\sqrt{\mu_0^4-8\gamma^4}}. \qquad (12)$$

Proof. Write inequality (9) in the equivalent form

$$\left\|A^{4-j}\left(\frac{d}{dt}+\gamma\right)^j\vartheta\right\|_{L_2}\leq c_j(\gamma;\mu_0)\left\|P_{0,\gamma}\vartheta\right\|_{L_2},\qquad j=\overline{0,4}, \qquad (13)$$

where $\vartheta(t)=u(t)e^{-\gamma t}\in W_2^4(R;H)$ and $P_{0,\gamma}\vartheta$ is determined from (3).

Note, that inequality (13) for $j=0$ follows from inequality (7). Prove the other inequalities. Using equality (4) we get, that

$$\left\|A^{4-j}(-i\xi+\gamma)^j\hat{\vartheta}(\xi)\right\|_{L_2}=\left\|A^{4-j}(-i\xi+\gamma)^jP_{0,\gamma}^{-1}(-i\xi)\hat{g}(\xi)\right\|_{L_2}\leq$$

$$\leq \sup_{\xi \in R} \left\| A^{4-j} \left(-i\xi + \gamma\right)^{j} P_{0,\gamma}^{-1} \left(-i\xi\right) \right\| \|\hat{g}\|_{L_2}.$$

Denote by

$$\varphi_j \left(\xi; \gamma; \mu\right) = \frac{\mu^{4-j} \left(\xi^2 + \gamma^2\right)^{1/2}}{\xi^4 - 6\xi^2\gamma^2 + \gamma^4 + \mu^4}, \quad \xi \in R, \quad \mu \geq \mu_0, \quad |\gamma| < \frac{1}{\sqrt[4]{8}}\mu_0, \tag{14}$$

$j = \overline{1,4}$. Then by spectral decomposition of operator A we have:

$$\left\| A^{4-j} \left(-i\xi + \gamma\right)^{j} \hat{\vartheta} \left(\xi\right) \right\|_{L_2} \leq \sup_{\xi \in R} \sup_{\mu \in R} \varphi_j \left(\xi; \gamma; \mu\right) \|\hat{g}\|_{L_2}. \tag{15}$$

So, estimate functions $\varphi_j \left(\xi; \gamma; \mu\right)$ for $j = \overline{1,4}$. For $j = 1$ we have:

$$\varphi_1 \left(\xi; \gamma; \mu\right) = \frac{\mu^3 \left(\xi^2 + \gamma^2\right)^{1/2}}{\xi^4 - 6\xi^2\gamma^2 + \gamma^4 + \mu^4}, \quad \xi^2 \geq 0, \ \mu \geq \mu_0 > 0, \ \ |\gamma| < \frac{1}{\sqrt[4]{8}}\mu_0.$$

Let $\delta > 0$. Then, by Young inequality we have:

$$\varphi_1 \left(\xi; \gamma; \mu\right) = \frac{\left(\delta \left(\xi^2 + \gamma^2\right)\right)^{1/4} \left(\delta^{-1/3}\mu^4\right)^{3/4}}{\left(\xi^2 - 3\gamma^2\right)^2 + \left(\mu^4 - 8\gamma^4\right)} \leq \frac{\frac{1}{4}\delta \left(\xi^2 + \gamma^2\right)^2 + \frac{3}{4}\delta^{-1/3}\mu^4}{\left(\xi^2 - 3\gamma^2\right)^2 + \left(\mu^4 - 8\gamma^4\right)}.$$

Let $\frac{1}{4}\delta = \frac{3}{4}\delta^{-1/3}$, i.e. $\delta = 3^{3/4}$. Then

$$\varphi_1 \left(\xi; \gamma; \mu\right) = \frac{3^{3/4}}{4} \frac{\xi^4 + \gamma^4 + 2\xi^2\gamma^2 + \mu^4}{\left(\xi^2 - 3\gamma^2\right)^2 + \left(\mu^4 - 8\gamma^4\right)} \leq$$

$$\leq \frac{3^{3/4}}{4} \left(1 + \frac{8\xi^2\gamma^2 + 8\gamma^4}{\left(\xi^2 - 3\gamma^2\right)^2 + \left(\mu^4 - 8\gamma^4\right)}\right) \leq$$

$$\leq \frac{3^{3/4}}{4} \left(1 + 8\gamma^2 \frac{\left(\xi^2 - 3\gamma^2\right) + 4\gamma^2}{\left(\xi^2 - 3\gamma^2\right) + \mu^4 - 8\gamma^4}\right) \leq$$

$$\leq \frac{3^{3/4}}{4} \left(1 + \frac{32\gamma^4}{\mu_0^4 - 8\gamma^4} + 8\gamma^2 \frac{\xi^2 - 3\gamma^2}{\left(\xi^2 - 3\gamma^2\right)^2 + \mu_0^4 - 8\gamma^4}\right) \leq$$

$$\leq \frac{3^{3/4}}{4} \left(1 + \frac{32\gamma^4}{\mu_0^4 - 8\gamma^4} + \frac{4\gamma^2}{\sqrt{\mu_0^4 - 8\gamma^4}}\right) = c_1 \left(\gamma; \mu_0\right).$$

Analogously we prove that

$$\varphi_3 \left(\xi; \gamma; \mu\right) = \frac{\mu \left(\xi^2 + \gamma^2\right)^{3/2}}{\xi^4 - 6\xi^2\gamma^2 + \gamma^4 + \mu^4} \leq c_1 \left(\gamma; \mu_0\right) = c_3 \left(\gamma; \mu_0\right).$$

Let $j = 2$. Then

$$\varphi_2 \left(\xi; \gamma; \mu\right) = \frac{\mu^2 \left(\xi^2 + \gamma^2\right)}{\xi^4 - 6\xi^2\gamma^2 + \gamma^4 + \mu^4} = \frac{\left(\xi^2 - 3\gamma^2\right)\mu^2 + 4\gamma^2\mu^2}{\left(\xi^2 - 3\gamma^2\right)^2 + \mu^4 - 8\gamma^4} \leq$$

$$\leq \frac{1}{2}\left(\frac{\left(\xi^2-3\gamma^2\right)^2+\mu^2+2\gamma^2\mu^2}{\left(\xi^2-3\gamma^2\right)^2+\mu^4-8\gamma^4}\right)=\frac{1}{2}\left(1+\frac{8\gamma^4+2\gamma^2\mu^2}{\left(\xi^2-3\gamma^2\right)+\mu^4-8\gamma^4}\right)\leq$$

$$\leq \frac{1}{2}\left(1+\frac{8\gamma^4+2\gamma^2\mu_0^2}{\mu_0^4-8\gamma^4}\right)=C_2\left(\gamma;\mu_0\right).$$

For $j=4$ it is easy to see, that

$$\varphi_4\left(\xi;\gamma;\mu\right)=1+\frac{8\xi^2\gamma^2-\mu^4}{\left(\xi^2-3\gamma^2\right)^2+\mu^4-8\gamma^4}\leq 1+\frac{8\left(\xi^2-3\gamma^2\right)\gamma^2+24\gamma^4}{\left(\xi^2-3\gamma^2\right)^2+\mu^4-8\gamma^4}\leq$$

$$\leq 1+\frac{24\gamma^4}{\mu_0^4-8\gamma^4}+8\gamma^2\frac{\xi^2-3\gamma^2}{\left(\xi^2-3\gamma^2\right)^2+\mu_0^4-8\gamma^4}\leq$$

$$\leq 1+\frac{24\gamma^4}{\mu_0^4-8\gamma^4}+\frac{4\gamma^2}{\sqrt{\mu_0^4-8\gamma^4}}=c_4\left(\gamma;\mu_0\right)$$

Thus, from the inequality (15) it follows, that

$$\left\|A^{4-j}\left(-i\xi+\gamma\right)^j\hat{\vartheta}\left(\xi\right)\right\|_{L_2}\leq c_j\left(\gamma;\mu_0\right)\left\|\hat{g}\right\|_{L_2},\quad j=\overline{1,4}.$$

By Plansharel theorem

$$\left\|A^{4-j}\left(\frac{d}{dt}+\gamma\right)^j\vartheta\right\|_{L_2}\leq c_j\left(\gamma;\mu_0\right)\left\|g\right\|_{L_2}=c_j\left(\gamma;\mu_0\right)\left\|P_{0,\gamma}\left(d/dt\right)\vartheta\right\|_{L_2}$$

or

$$\left\|A^{4-j}u^j\right\|_{L_{2,\gamma}}\leq c_j\left(\gamma;\mu_0\right)\left\|P_0u\right\|_{L_{2,\gamma}},\quad j=\overline{1,4}.$$

The theorem is proved.

Note, that for $\gamma=0$ the constants of inequality (8) are exact [4].

Now prove a theorem on regular solvability of equation (1).

Theorem 3. *Let conditions of theorem 1 be fulfilled, operators* $B_j=A_jA^{-j}$ $\left(j=\overline{0,4}\right)$ *be bounded in* H, $|\gamma|<\frac{1}{\sqrt[4]{8}}\mu_0$ *and*

$$\alpha\left(\gamma;\mu_0\right)=\sum_{j=0}^{4}c_j\left(\gamma;\mu_0\right)\left\|B_{4-j}\right\|<1,$$

where numbers $c_j\left(\gamma;\mu_0\right)$ $\left(j=\overline{0,4}\right)$ *are determined from (9)-(12). Then equation (1) is regularly solvable.*

Proof. By theorem 1 operator $P_0^{-1}:L_{2,\gamma}\left(R;H\right)\to W_{2,\gamma}^4\left(R;H\right)$ is bounded. Then, after substitution of $u=P_0^{-1}\omega$, where $\omega\in L_{2,\gamma}\left(R;H\right)$ we get equation

$$\omega+P_1P_0^{-1}\omega=f$$

in space $L_{2,\gamma}\left(R;H\right)$. Since for any $\omega\in L_{2,\gamma}\left(R;H\right)$ the following estimations are true (see theorem 2)

$$\left\|P_1P_0^{-1}\omega\right\|L_{2,\gamma}=\left\|P_1u\right\|_{L_{2,\gamma}}\leq\sum_{j=0}^{4}\left\|A_{4-j}u^{(j)}\right\|_{L_{2,\gamma}}\leq$$

$$\leq \sum_{j=0}^{4} \|B_{4-j}\| \left\|A_{4-j}u^{(j)}\right\|_{L_{2,\gamma}} \leq \sum_{j=0}^{4} \|B_{4-j}\| c_j (\gamma; \mu_0) \|P_1 u\|_{L_{2,\gamma}} =$$

$$= \left(\sum_{j=0}^{4} c_j (\gamma; \mu_0) \|B_{4-j}\| \right) \|\omega\|_{L_{2,\gamma}} = \alpha (\gamma; \mu_0) \|\omega\|_{L_{2,\gamma}} .$$

Since $\alpha (\gamma; \mu_0) < 1$, then operator $E + P_1 P_0^{-1}$ is reversible in $L_{2,\gamma} (R; H)$, we can find ω:

$$\omega = \left(E + P_1 P_0^{-1}\right)^{-1} f.$$

Hence we get

$$u = P_0^{-1} \left(E + P_1 P_0^{-1}\right) f$$

and

$$\|u\|_{W_{2,\gamma}} \leq const \|f\|_{L_{2,\gamma}} .$$

The theorem is proved.

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Elshan N. Mamedov
Nakhichevan State University
AZ7000, Nakhchivan, Azerbaijan

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