Gulbahar A. SALIMOVA

ON A BOUNDARY VALUE PROBLEM FOR A FOURTH ORDER PARTIAL DIFFERENTIAL EQUATION

Abstract

The existence of a classic solution of a boundary value problem for a fourth order partial differential equation is proved.

Recently, there is a great interest to the problems of wave propagation in stratified liquids in connection with needs of oceanology and applied geophysics.

In the paper [1] it was derived the main equation of dynamics of compressible exponentially stratified liquid:

$$\frac{\partial^2}{\partial t^2}\left[\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}-\Delta_3 u+\beta^2 u\right]-\omega_0^2\Delta_2 u=0, \tag{*}$$

here Δ_3 and Δ_2 are Laplace operators with respect to variables (x_1, x_2, x_3) and (x_1, x_2), respectively. The fourth order differential equation

$$\frac{\partial^2}{\partial t^2}\left[\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}-u_{xx}+\beta^2 u\right]-\omega_0^2 u_{xx}=0, \tag{**}$$

being one-dimensional analogy of general equation (*) was considered in [2]. Parameter c denotes sound velocity in the considered compressible exponentially stratified liquid; the quantity 2β equals index in the exponent characterizing density distribution of liquid and in accepted approximation may be considered to be constant; main frequency parameter of the considered liquid called Weysel-Brandtl frequency is denoted by ω_0. Its quantity is determined by the equality

$$\omega_0^2=2\beta g-g^2/c^2,$$

where g is free fall acceleration.

For convenience of further considerations we pass to non-dimensional independent variables $\beta x, \omega_0 t$ and keep previous denotation x and t. The equation (**) takes the form:

$$\varepsilon^2\frac{\partial^4 u}{\partial t^4}-\frac{\partial^2}{\partial t^2}\left[u_{xx}-u\right]-u_{xx}=0 \tag{***}$$

where $\varepsilon=\omega_0/(\beta c)\le 1$. Fundamental solution of equation (***) is studied in [2].

Now, let's consider the boundary-value problem:

$$u_{tttt}(x,t)-u_{ttxx}(x,t)+u_{tt}(x,t)-u_{xx}(x,t)=F\left(x,t,u(x,t),u_t(x,t)\right),$$

$$(x,t)\in D_T\equiv\left\{(x,t): 0\le x\le 1, 0\le t\le T\right\}, \tag{1}$$

$$u(0,t)=0,\quad u_x(1,t)=0,\quad 0\le t\le T, \tag{2}$$

$$u(x,0)=\varphi_0(x),\quad u_t(x,0)=\varphi_1(x),$$

$$u_{tt}(x,0)=\varphi_2(x), \quad u_{ttt}(x,0)=\varphi_3(x), \quad 0\le x\le 1, \tag{3}$$

where

$$F(x,t,u(x,t),u_t(x,t))=p_0(t)\,u(x,t)+p_1(t)\,u_t(x,t)+f(x,t), \tag{4}$$

$f(x,t), \varphi_0(x), \varphi_1(x), \ \varphi_2(x), \varphi_3(x), p_0(t), p_1(t)$ are the given functions, $u(x,t)$ is the desired function.

Under the classic solution of problem (1)-(3) we understand the functions $u(x,t)$, continuous in closed domain D_T together with its derivatives contained in equation (1) and satisfying all conditions of (1)-(3) in an ordinary sense.

By $B_{2,T}^{\alpha,\beta}$ ([3]) we denote aggregate of all functions of the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\lambda_k x, \ \ \lambda_k=\frac{\pi}{2}(2k-1),$$

considered in D_T where each of the functions $u_k(t)$ is continuously differentiable on [0,T], and

$$J(u)\equiv\left\{\sum_{k=1}^{\infty}\left(\lambda_k^{\alpha}\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^{\beta}\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}<+\infty$$

moreover $\alpha\ge 0, \ \beta\ge 0$.

In this set we define the norm as follows:

$$\left\|u(x,t)\right\|_{B_{2,T}^{\alpha,\beta}}=J(u).$$

It is known that [3] $B_{2,T}^{\alpha,\beta}$ are Banach spaces.

Since the system $\{\sin\lambda_k x\}_{k=1}^{\infty}$ is complete in $L_2(0,1)$, obviously each classic solution $u(x,t)$ of problem (1)-(3) is of the form:

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\lambda_k x, \tag{5}$$

where

$$u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx.$$

Now we apply Fourier method and get from (1)-(3):

$$u_k^{(4)}(t)+\left(1+\lambda_k^2\right)u_k''(t)+\lambda_k^2 u_k(t)=F_k(t;u), \tag{6}$$

$$u_k(0)=\varphi_{0k}, \quad u_k'(0)=\varphi_{1k}, \quad u_k''(0)=\varphi_{2k}, \quad u_k'''(0)=\varphi_{3k} \tag{7}$$

where

$$F_k(t;u)\equiv p_0(t)\,u_k(t)+p_1(t)\,u_k'(t)+f_k(t),$$

$$f_k(t)=2\int_0^1 f(x,t)\sin\lambda_k x dx,$$

$$\varphi_{ik} = 2\int_0^1 \varphi_i(x)\sin\lambda_k x dx \quad \left(i=\overline{0,3}\right) \tag{8}$$

The roots of the characteristic function

$$\mu_{4k} + \left(1+\lambda_k^2\right)\mu_k^2 + \lambda_k^2 = 0$$

corresponding to (6) are determined by the relations:

$$\mu_{jk} = (-1)^j\sqrt{-1} \quad (j=1,2),$$

$$\mu_{jk} = (-1)^j\lambda_k\sqrt{-1} \quad (j=3,4).$$

Then, after applying the method of variation of parameters we reduce the solution of problem (6), (7) to the solution of the following denumerable system:

$$u_k(t) = \frac{1}{\lambda_k^2-1}\left[\left(\lambda_k^2\cos t - \cos\lambda_k t\right)\varphi_{0k} + \left(\lambda_k^2\sin t - \frac{\sin\lambda_k t}{\lambda_k}\right)\varphi_{1k} + \right.$$
$$+\left(\cos t - \cos\lambda_k t\right)\varphi_{2k} + \left(\sin t - \frac{\sin\lambda_k t}{\lambda_k}\right)\varphi_{3k} +$$
$$\left. +\frac{1}{\lambda_k}\int_0^t F_k(\tau;u)\left(\lambda_k\sin(t-\tau) - \sin\lambda_k(t-\tau)\right)d\tau\right]. \tag{9}$$

Substituting $u_k(t)$ form (9) to the representation of function (5) we get:

$$u(x,t) = \sum_{k=1}^{\infty}\left\{\frac{1}{\lambda_k^2-1}\left[\left(\lambda_k^2\cos t - \cos\lambda_k t\right)\varphi_{0k} + \left(\lambda_k^2\sin t - \frac{\sin\lambda_k t}{\lambda_k}\right)\varphi_{1k} + \right.\right.$$
$$+\left(\cos t - \cos\lambda_k t\right)\varphi_{2k} + \left(\sin t - \frac{\sin\lambda_k t}{\lambda_k}\right)\varphi_{3k} +$$
$$\left.\left. +\frac{1}{\lambda_k}\int_0^t F_k(\tau;u)\left(\lambda_k\sin(t-\tau) - \sin\lambda_k(t-\tau)\right)d\tau\right]\right\}\sin\lambda_k t. \tag{10}$$

From (9) we have:

$$u_k'(t) = \frac{1}{\lambda_k^2-1}\left[\left(-\lambda_k^2\sin t + \lambda_k\sin\lambda_k t\right)\varphi_{0k} + \left(\lambda_k^2\cos t - \cos\lambda_k t\right)\varphi_{1k} + \right.$$
$$+\left(-\sin t + \lambda_k\sin\lambda_k t\right)\varphi_{2k} + \left(\cos t - \cos\lambda_k\right)\varphi_{3k} +$$
$$\left. +\int_0^t F_k(\tau;u)\left(\cos(t-\tau) - \cos\lambda_k(t-\tau)\right)d\tau\right], \tag{11}$$

$$u_k''(t) = \frac{1}{\lambda_k^2-1}\left[\left(-\lambda_k^2\cos t + \lambda_k^2\cos\lambda_k t\right)\varphi_{0k} + \left(-\lambda_k^2\sin t + \lambda_k\sin\lambda_k t\right)\varphi_{1k} + \right.$$
$$+\left(-\cos t + \lambda_k^2\sin\lambda_k t\right)\varphi_{2k} + \left(-\sin t + \lambda_k\sin\lambda_k t\right)\varphi_{3k} +$$

$$+\int_0^t F_k(\tau;u)\left(-\sin(t-\tau)+\lambda_k\sin\lambda_k(t-\tau)\right)d\tau\Bigg], \qquad (12)$$

$$u_k'''(t)\,\frac{1}{\lambda_k^2-1}\left[\left(\lambda_k^2\sin t-\lambda_k^3\sin\lambda_k t\right)\varphi_{0k}+\left(\lambda_k^2\cos t+\lambda_k^2\cos\lambda_k t\right)\varphi_{1k}+\right.$$

$$+\left(\sin t-\lambda_k^3\sin\lambda_k t\right)\varphi_{2k}+\left(-\cos t+\lambda_k^2\cos\lambda_k(t-\tau)\right)\varphi_{3k}+$$

$$\left.+\int_0^t F_k(\tau;u)\left(-\cos(t-\tau)+\lambda_k^2\cos\lambda_k(t-\tau)\right)d\tau\right], \qquad (13)$$

$$u_k^4(t)=\frac{1}{\lambda_k^2-1}\left[\left(\lambda_k^2\cos t-\lambda_k^4\cos\lambda_k t\right)\varphi_{0k}+\left(\lambda_k^2\sin t-\lambda_k^3\sin\lambda_k t\right)\varphi_{1k}+\right.$$

$$+\left(\cos t-\lambda_k^4\cos\lambda_k t\right)\varphi_{2k}+\left(\sin t-\lambda_k^3\sin\lambda_k\right)\varphi_{3k}+$$

$$\left.+\int_0^t F_k(\tau;u)\left(\sin(t-\tau)-\lambda_k^3\sin\lambda_k(t-\tau)\right)d\tau\right]+F_k(t;u). \qquad (14)$$

Proceeding from the determination of a classic solution of problem (1)-(3) we easily solve the following

Lemma. *If* $u(x,t)$ *is any classic solution of problem (1)-(3), the functions*

$$u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k x\,dx \quad (k=1,2,...)$$

satisfy system (9) on $[0,T]$.

We have from (9), (11)-(14):

$$|u_k(t)|\le\frac{2}{\lambda_k^2}\left(1+\lambda_k^2\right)|\varphi_{0k}|+\frac{2}{\lambda_k^3}\left(1+\lambda_k^3\right)|\varphi_{1k}|+\frac{4}{\lambda_k^2}|\varphi_{2k}|+$$

$$+\frac{2}{\lambda_k^3}(1+\lambda_k)|\varphi_{3k}|+\frac{2}{\lambda_k^3}(1+\lambda_k)\sqrt{T}\left(\int_0^T|F_k(\tau;u)|^2d\tau\right)^{\frac{1}{2}},$$

$$\left|u_k'(t)\right|\le\frac{2}{\lambda_k^2}\left(\lambda_k+\lambda_k^2\right)|\varphi_{0k}|+\frac{2}{\lambda_k^2}\left(1+\lambda_k^2\right)|\varphi_{1k}|+\frac{2}{\lambda_k^2}(1+\lambda_k)|\varphi_{2k}|+$$

$$+\frac{4}{\lambda_k^2}|\varphi_{3k}|+\frac{4}{\lambda_k^2}\sqrt{T}\left(\int_0^T|F_k(\tau;u)|^2d\tau\right)^{\frac{1}{2}},$$

$$\left|u_k''(t)\right|\le 4|\varphi_{0k}|+\frac{2}{\lambda_k^2}\left(\lambda_k+\lambda_k^2\right)|\varphi_{1k}|+\frac{2}{\lambda_k^2}\left(1+\lambda_k^2\right)|\varphi_{2k}|+$$

$$+\frac{2}{\lambda_k^2}(1+\lambda_k)|\varphi_{3k}|+\frac{2}{\lambda_k^2}(1+\lambda_k)\sqrt{T}\left(\int_0^T|F_k(\tau;u)|^2d\tau\right)^{\frac{1}{2}},$$

$$\left|u_k'''(t)\right| \le 2(1+\lambda_k)|\varphi_{0k}| + 4|\varphi_{1k}| + \frac{2}{\lambda_k^2}(1+\lambda_k)|\varphi_{2k}| +$$

$$+\frac{2}{\lambda_k^2}(1+\lambda_k)|\varphi_{3k}| + \frac{2}{\lambda_k^2}(1+\lambda_k)\sqrt{T}\left(\int_0^T |F_k(\tau;u)|^2 d\tau\right)^{\frac{1}{2}},$$

$$\left|u_k^{(4)}(t)\right| \le 2\left(1+\lambda_k^2\right)|\varphi_{0k}| + 2(1+\lambda_k)|\varphi_{1k}| + \frac{2}{\lambda_k^2}\left(1+\lambda_k^4\right)|\varphi_{2k}| +$$

$$+\frac{2}{\lambda_k^2}\left(1+\lambda_k^3\right)|\varphi_{3k}| + \frac{2}{\lambda_k^2}(1+\lambda_k)\sqrt{T}\left(\int_0^T |F_k(\tau;u)|^2 d\tau\right)^{\frac{1}{2}} + |F_k(t;u)|.$$

Hence we have:

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3 \|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le$$

$$\le 4\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\varphi_{0k}|\right)^2\right)^{\frac{1}{2}} + 4\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\varphi_{1k}|\right)^2\right)^{\frac{1}{2}} +$$

$$+4\left(\sum_{k=1}^{\infty}\left(\lambda_k|\varphi_{2k}|\right)^2\right)^{\frac{1}{2}} + 4\left(\sum_{k=1}^{\infty}\left(\lambda_k|\varphi_{3k}|\right)^2\right)^{\frac{1}{2}} +$$

$$+4T\|p_0(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} +$$

$$+4T\|p_1(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\|u_k'(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} +$$

$$+4\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k|f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}; \tag{15}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3 \|u_k'(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le$$

$$\le 4\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\varphi_{0k}|\right)^2\right)^{\frac{1}{2}} + 4\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\varphi_{1k}|\right)^2\right)^{\frac{1}{2}} +$$

$$+4\left(\sum_{k=1}^{\infty}\left(\lambda_k^2|\varphi_{2k}|\right)^2\right)^{\frac{1}{2}} + 4\left(\sum_{k=1}^{\infty}\left(\lambda_k|\varphi_{3k}|\right)^2\right)^{\frac{1}{2}} +$$

$$+4T\|p_0(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} +$$

$$+4T\left\|p_1(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$+4\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\left|f_k(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}; \qquad (16)$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k''(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq 4\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{0k}\right|\right)^2\right)^{\frac{1}{2}}+4\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{1k}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+4\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{2k}\right|\right)^2\right)^{\frac{1}{2}}+4\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\varphi_{3k}\right|\right)\right)^2+$$

$$+4T\left\|p_0(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$+4T\left\|p_1(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left\|u_k'(t)\right\|_{C[0,T]}\right)\right)^2+$$

$$+4\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^2\left|f_k(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}. \qquad (17)$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k'''(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq$$

$$\leq 4\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\varphi_{0k}\right|\right)^2\right)^{\frac{1}{2}}+4\left(\sum_{k=1}^{\infty}\left(\lambda_k\left|\varphi_{1k}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+4\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\varphi_{2k}\right|\right)^2\right)^{\frac{1}{2}}+4\left(\sum_{k=1}^{\infty}\left(\lambda_k\left|\varphi_{3k}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+4T\left\|p_0(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$+4T\left\|p_1(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$+4\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\left|f_k(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}. \qquad (18)$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k^{(4)}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq$$

$$\leq 4\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{0k}\right|\right)\right)^2+4\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\varphi_{1k}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+4\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{2k}\right|\right)^2\right)^{\frac{1}{2}}+4\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\varphi_{3k}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+4T\left\|p_0(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$+4T\left\|p_1(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+4\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\lambda_k^2\left|f_k(\tau)\right|^2d\tau\right)^{\frac{1}{2}}+$$

$$+\left\|p_0(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$+\left\|p_1(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|f_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}. \quad (19)$$

Assume that the problem's data satisfy the conditions:

1. $\varphi_i(x)\in C^2[0,1]$, $\varphi_i'''(x)\in L_2(0,1)$ and
$\varphi_i(0)=\varphi_i'(1)=\varphi_i''(0)=0 \quad (i=0,1,2)$.
2. $\varphi_3(x)\in C^1[0,1]$, $\varphi_3'''(x)\in L_2(0,1)$ and
$\varphi_3(0)=\varphi_3'(1)=0$.
3. $f(x,t)\in C_{x,t}^{1,0}(D_T)$ $f_{xx}(x,t)\in L_2(D_T)$ and
$f(0,t)=f_x(1,t)=0 \quad (0\leq t\leq T).,.$
4. $p_0(t),\ p_1(t)\in C[0,T]$.

Then from (15)-(19) we get:

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq 4\sqrt{2}\left\|\varphi_0'''(x)\right\|_{L_2(0,1)}+4\sqrt{2}\left\|\varphi_1'''(x)\right\|_{L_2(0,1)}+$$

$$+4\sqrt{2}\left\|\varphi_2''(x)\right\|_{L_2(0,1)}+4\sqrt{2}\left\|\varphi_3'(x)\right\|_{L_2(0,1)}+4\sqrt{2T}\left\|f_x(x,t)\right\|_{L_2(D_T)}+$$

$$+4T\left(\left\|p_0(t)\right\|_{C[0,T]}+\left\|p_1(t)\right\|_{C[0,T]}\right)\left\|u\right\|_{B_{2,T}^{3,3}}; \quad (20)$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq 4\sqrt{2}\left\|\varphi_0'''(x)\right\|_{L_2(0,1)}+4\sqrt{2}\left\|\varphi_1'''(x)\right\|_{L_2(0,1)}+$$

$$+4\sqrt{2}\left\|\varphi_2''(x)\right\|_{L_2(0,1)}+4\sqrt{2}\left\|\varphi_3'(x)\right\|_{L_2(0,1)}+4\sqrt{2T}\left\|f_x(x,t)\right\|_{L_2(D_T)}+$$

$$+4T\left(\left\|p_0(t)\right\|_{C[0,T]}+\left\|p_1(t)\right\|_{C[0,T]}\right)\left\|u\right\|_{B_{2,T}^{3,3}}; \quad (21)$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k''(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq 4\sqrt{2}\left\|\varphi_0'''(x)\right\|_{L_2(0,1)}+4\sqrt{2}\left\|\varphi_1'''(x)\right\|_{L_2(0,1)}+$$

$$+4\sqrt{2}\left\|\varphi_2'''(x)\right\|_{L_2(0,1)}+4\sqrt{2}\left\|\varphi_3''(x)\right\|_{L_2(0,1)}+4\sqrt{2T}\left\|f_{xx}(x,t)\right\|_{L_2(D_T)}+$$

$$+4T\left(\left\|p_0(t)\right\|_{C[0,T]}+\left\|p_1(t)\right\|_{C[0,T]}\right)\left\|u\right\|_{B^{3,3}_{2,T}}; \tag{22}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k'''(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq 4\sqrt{2}\left\|\varphi_0''(x)\right\|_{L_2(0,1)}+4\sqrt{2}\left\|\varphi_1'(x)\right\|_{L_2(0,1)}+$$

$$+4\sqrt{2}\left\|\varphi_2''(x)\right\|_{L_2(0,1)}+4\sqrt{2}\left\|\varphi_3'(x)\right\|_{L_2(0,1)}+4\sqrt{2T}\left\|f_x(x,t)\right\|_{L_2(D_T)}+$$

$$+4T\left(\left\|p_0(t)\right\|_{C[0,T]}+\left\|p_1(t)\right\|_{C[0,T]}\right)\left\|u\right\|_{B^{3,3}_{2,T}}; \tag{23}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k^{(4)}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq 4\sqrt{2}\left\|\varphi_0'''(x)\right\|_{L_2(0,1)}+$$

$$+4\sqrt{2}\left\|\varphi_1''(x)\right\|_{L_2(0,1)}+4\sqrt{2}\left\|\varphi_2'''(x)\right\|_{L_2(0,1)}+4\sqrt{2}\left\|\varphi_3''(x)\right\|_{L_2(0,1)}+$$

$$+\left(4\sqrt{T}+1\right)\sqrt{2}\left\|f_{xx}(x,t)\right\|_{L_2(D_T)}+$$

$$+(4T+1)\left(\left\|p_0(t)\right\|_{C[0,T]}+\left\|p_1(t)\right\|_{C[0,T]}\right)\left\|u\right\|_{B^{3,3}_{2,T}}; \tag{24}$$

Now we deduce from inequalities (20) and (21):

$$\left\|u\right\|_{B^{3,3}_{2,T}}\equiv\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq A_1(T)+A_2(T)\,T\left\|u\right\|_{B^{3,3}_{2,T}}, \tag{25}$$

where

$$A_1(T)=8\sqrt{2}\left\|\varphi_0'''(x)\right\|_{L_2(0,1)}+8\sqrt{2}\left\|\varphi_1'''(x)\right\|_{L_2(0,1)}+8\sqrt{2}\left\|\varphi_2''(x)\right\|_{L_2(0,1)}+$$

$$+8\sqrt{2}\left\|\varphi_3'(x)\right\|_{L_2(0,1)}+8\sqrt{2}\sqrt{T}\left\|f_x(x,t)\right\|_{L_2(D_T)},$$

$$A_2(T)=8\left(\left\|p_0(t)\right\|_{C[0,T]}+\left\|p_1(t)\right\|_{C[0,T]}\right). \tag{26}$$

So, we can prove the following theorem.

Theorem. *Let conditions 1-4 be fulfilled and* $q\equiv A_2(T)\cdot T\leq 1$ *where the numbers* $A_2(T)$ *is determined by relation (26). Then problem (1)-(3) has a classic solution.*

Proof. Write equation (10) in the form

$$u=\Phi u, \tag{27}$$

where the operator Φ is defined by the right hand side of (10).

Let's consider the operator Φ fixed in the ball $K=K_R\left(\left\|u\right\|_{B^{3,3}_{2,T}}\leq R\right)$ where R is such that

$$A_1(T)+A_2(T)\cdot T\leq R,$$

from the space $B^{3,3}_{2,T}$, it is seen from (25) that under the conditions of the theorem for any $u \in K_R$ it holds the inequality

$$\|\Phi u\|_{B^{3,3}_{2,T}} \le A_1(T) + A_2(T)\,TR \le R \tag{28}$$

and for any $u_1, u_2 \in K_R$ we have

$$\|\Phi u_1 - \Phi u_2\|_{B^{3,3}_{2,T}} \le A_2(T)\,T\,\|u_1 - u_2\|_{B^{3,3}_{2,T}} \tag{29}$$

moreover, by conditions of the theorem, $q < 1$.

It follows from inequalities (28) and (29) that under the conditions of the theorem the operator Φ acts in the ball $K = K_R$ and it is contractive. Therefore, in the ball $K = K_R$ the operator Φ has a unique fixed point $\{U\}$ and this point is the solution of (27).

The function $u(x,t)$ as an element of the space $B^{3,3}_{2,T}$ has continuous derivatives $u_x(x,t)$, $u_{xx}(x,t)$, $u_t(x,t)$, $u_{tx}(x,t)$, $u_{txx}(x,t)$.

Obviously,

$$|u_{ttxx}(x,t)| \le \left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \left\|u_k''(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}, \tag{30}$$

$$|u_{ttt}(x,t)| \le \left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \left(\lambda_k \left\|u_k'''(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}, \tag{31}$$

$$|u_{tttt}(x,t)| \le \left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \left(\lambda_k \left\|u_k^{(4)}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}. \tag{32}$$

It follows from (31)-(32) that allowing for (22)-(24) the functions $u_{tt}(x,t)$, $u_{ttxx}(x,t)$, $u_{ttt}(x,t)$, $u_{tttt}(x,t)$ are continuous in D_T.

It is easily verified that equation (1) and conditions (2) and (3) are satisfied in an ordinary sense. So $u(x,t)$ is a classic solution of problem (1)-(3). The theorem is proved.

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Gulbahar A. Salimova
Baku State University
23, Z.I. Khalilov str., AZ1148, Baku, Azerbaijan
Tel.: (99412) 510 32 42 (off.)

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Translated by Nazirova S.H.