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ON REPRESENTABILITY OF CAUCHY TYPE INTEGRALS BY THEIR BOUNDARY VALUES

Abstract

A problem on representability of the analytic function $F(z) = \frac{1}{2\pi i}\int\limits_T \frac{d\nu(t)}{t-z}$ *by the boundary values* $F(t)$ *for arbitrary finite complex measures, is considered.*

In [1] P.L. Ulyanov shows that each analytic function represented in a unit circle in the form of Cauchy $L-$integral is representable in the form of $A-$Cauchy integral, i.e. if ν is absolutely continuous finite measure with respect to Lebesgue measure on a unit circle $T = \{t \in C : |t| = 1\}$, we can represent the analytic function $F(z) = \frac{1}{2\pi i}\int\limits_T \frac{d\nu(t)}{t-z}$ in the form

$$F(z) = \frac{1}{2\pi i}\lim_{\lambda\to+\infty}\int\limits_{T_\lambda} \frac{F(t)}{t-z}dt \qquad (1)$$

where $F(t)$ is non-tangential limit value of the analytic function $F(z)$ as

$$z \to t = e^{i\theta} \quad \text{and} \quad T_\lambda = \left\{ t = e^{i\theta} : \left| \frac{F(t)}{t-z} \right| \le \lambda \right\}.$$

However, when measure ν is not absolutely continuous, relation (1) becomes invalid. For example, for the discrete measure

$$\nu(X) = \begin{cases} 2\pi i & \text{for} \quad 1 \in X \\ 0 & \text{for} \quad 1 \notin X \end{cases},$$

the analytic function $F(z)$ and its boundary values will be $F(z) = \frac{1}{1-z}$ and $F(t) = \frac{1}{1-t}$, respectively, but the limit

$$\frac{1}{2\pi i}\lim_{\lambda\to+\infty}\int\limits_{T_\lambda} \frac{F(t)}{t-z}dt = v.p.\frac{1}{2\pi i}\int\limits_T \frac{1}{1-t}\cdot\frac{1}{t-z}dt =$$

$$= v.p.\frac{1}{2\pi i}\int\limits_T \frac{1}{1-z}\left\{\frac{1}{1-t} + \frac{1}{t-z}\right\}dt =$$

$$= \frac{1}{1-z}v.p.\frac{1}{2\pi i}\int\limits_T \frac{1}{1-t}dt + \frac{1}{1-z}\cdot\frac{1}{2\pi i}\int\limits_T \frac{1}{t-z}dt =$$

$$= \frac{1}{1-z}\cdot\left(-\frac{1}{2}\right) + \frac{1}{1-z} = \frac{1}{2}\cdot\frac{1}{1-z} \ne F(z).$$

In the paper a problem on representation of the analytic function $F(z)=\dfrac{1}{2\pi i}\int\limits_T \dfrac{d\nu(t)}{t-z}$ by the boundary values $F(t)$ for arbitrary finite complex measures, is cousidered.

Theorem 1. *Let ν be a finite complex measure on T and $F(z)=\dfrac{1}{2\pi i}\int\limits_T \dfrac{d\nu(t)}{t-z}$, then it is true the equality*

$$F(z)=\frac{1}{2\pi i}\lim_{\lambda\to+\infty}\int\limits_{T_\lambda}\frac{F(t)}{t-z}dt+\frac{1}{4\pi i}\int\limits_T\frac{d\nu_s(t)}{t-z} \qquad (2)$$

where $F(t)$ is non-tangential limit value of the analytic function $F(z)$ as $z\to t=e^{i\theta}$, $T_\lambda=\left\{t=e^{i\theta}:\left|\dfrac{F(t)}{t-z}\right|\le\lambda\right\}$ and ν_s is a singular part of the measure ν.

We'll use three theorems, proved by the author [2], (theorem A and B) and by S.A. Vinogradov and S.V. Hruscev [3] (theorem C).

Theorem A [2]. *Let μ be an arbitrary finite complex Borel measure on the interval $T_0=[0;2\pi)$, $u\left(re^{i\varphi}\right)$ be its Poisson integral, $v\left(re^{i\varphi}\right)$ be a function harmonically conjugated with $u\left(re^{i\varphi}\right)$. Then*

$$F(z)=\frac{1}{2\pi i}\lim_{\lambda\to+\infty}\int\limits_{T_\lambda}\frac{F(t)}{t-z}dt+\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{e^{i\theta}}{e^{i\theta}-z}d\mu_s(\theta),\quad z=re^{i\varphi},$$

where $F(z)=u(z)+iv(z)$, $F(t)$ is non-tangential limit value of the function $F(z)$ as $z\to t=e^{i\theta}$, $T_\lambda=\left\{t=e^{i\theta}:\left|\dfrac{F(t)}{t-z}\right|\le\lambda\right\}$ and μ_s is a singular part of the measure μ.

Theorem B [2]. *Let μ be an arbitrary finite complex Borel measure on the interval $T_0=[0;2\pi)$, $u\left(re^{i\varphi}\right)$ be its Poisson integral, $v\left(re^{i\varphi}\right)$ be a function harmonically conjugated with $u\left(re^{i\varphi}\right)$. Then*

$$F(z)=\frac{1}{2\pi i}\lim_{\lambda\to+\infty}\int\limits_{T_\lambda}\frac{F(t)}{t-z}dt+\frac{1}{4i}\lim_{\lambda\to+\infty}\lambda\int\limits_0^{2\pi}r_{\lambda,\mathrm{Re}\,h}(\theta_0)\,d\theta_0+$$

$$+\frac{1}{4}\lim_{\lambda\to+\infty}\lambda\int\limits_0^{2\pi}r_{\lambda,\mathrm{Im}\,h}(\theta_0)\,d\theta_0,\quad z=re^{i\varphi},$$

where $F(z)=u(z)+iv(z)$, $h(\theta)=\dfrac{e^{i\theta}}{e^{i\theta}-z}F\left(e^{i\theta}\right)$, $F(t)$ is non-tangential limit value of the function $F(z)$ as $z\to t=e^{i\theta}$, $T_\lambda=\left\{t=e^{i\theta}:\left|\dfrac{F(t)}{t-z}\right|\le\lambda\right\}$.

Theorem C [3]. *For any finite complex Borel measure ν on a unit circle T it is valid the equality*

$$\lim_{\lambda\to+\infty}\lambda\cdot m\left\{\theta\in T_0:\left|\widetilde{\nu}\left(e^{i\theta}\right)\right|>\lambda\right\}=\frac{2}{\pi}\|\nu_s\|,$$

where m is a Lebesgue measure, $\widetilde{\nu}\left(e^{i\theta}\right)=\dfrac{1}{2\pi}\lim\limits_{\varepsilon\to 0+}\displaystyle\int\limits_{\{t\in T_0:|t-\theta|>\varepsilon\}} ctg\frac{\theta-t}{2}d\nu\left(e^{i\theta}\right)$ *is a function conjugated to the measure,* ν_s *is a singular part of the measure* ν*, a* $\|\nu_s\|$ *is a full variation of the measure* ν_s*.*

Proof of theorem 1. Let's consider the measure $d\mu(\theta)=\dfrac{1}{2i}e^{-i\theta}d\nu\left(e^{i\theta}\right)$, $\theta\in[0;2\pi)$ on the interval $[0;2\pi)$. Then

$$F(z)=\frac{1}{2\pi i}\int\limits_0^{2\pi}\frac{2ie^{i\theta}d\mu(\theta)}{e^{i\theta}-z}=\frac{1}{\pi}\int\limits_0^{2\pi}\frac{e^{i\theta}}{e^{i\theta}-z}d\mu(\theta)=\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta)+$$

$$+\frac{1}{2\pi}\int\limits_0^{2\pi}d\mu(\theta)=F_1(z)+c_0, \tag{3}$$

where

$$F_1(z)=\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta),\quad c_0=\frac{1}{2\pi}\int\limits_0^{2\pi}d\mu(\theta).$$

If $u\left(re^{i\varphi}\right)$ is an Poisson integral of the measure μ , and $v\left(re^{i\varphi}\right)$ is the function harmonically conjugated to $u\left(re^{i\varphi}\right)$, then $F_1(z)=u(z)+iv(z)$, $z=re^{i\varphi}$. Therefore, theorem A implies

$$F_1(z)=\frac{1}{2\pi i}\lim_{\lambda\to+\infty}\int\limits_{T_\lambda}\frac{F_1(t)}{t-z}dt+\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{e^{i\theta}}{e^{i\theta}-z}d\mu_s(\theta),\quad t=e^{i\theta}, \tag{4}$$

where $T'_\lambda=\left\{t=e^{i\theta}:\left|\dfrac{F_1(t)}{t-z}\right|\le\lambda\right\}$, and $F_1(t)$ is a non-tangential corner boundary value of the function $F_1(z)$.

We first prove the equality

$$\lim_{\lambda\to+\infty}\left(\int\limits_{T'_\lambda}\frac{F_1(t)}{t-z}dt-\int\limits_{T_\lambda}\frac{F_1(t)}{t-z}dt\right)=0. \tag{5}$$

Denote

$$c_1=\max_{t\in T}\left|\frac{c_0}{t-z}\right|.$$

Since

$$T_\lambda=\left\{t\in T:\left|\frac{F(t)}{t-z}\right|\le\lambda\right\}=\left\{t\in T:\left|\frac{F_1(t)+c}{t-z}\right|\le\lambda\right\}$$

then from the inequalities

$$\left|\frac{F_1(t)}{t-z}\right|\le\left|\frac{F_1(t)+c_0}{t-z}\right|+c_1\ ,\quad \left|\frac{F_1(t)}{t-z}\right|\ge\left|\frac{F_1(t)+c_0}{t-z}\right|-c_1$$

we get the inclusion

$$T'_\lambda \Delta T_\lambda \subset T^*_\lambda = \left\{ t \in T : \lambda - c_1 \leq \left| \frac{F_1(t)}{t-z} \right| \leq \lambda + c_1 \right\}.$$

Hence, it follows the inequality

$$\left| \int\limits_{T'_\lambda} \frac{F_1(t)}{t-z} dt - \int\limits_{T_\lambda} \frac{F_1(t)}{t-z} dt \right| \leq \int\limits_{T^*_\lambda} \left| \frac{F_1(t)}{t-z} \right| dt \leq (\lambda + c_1)\, m T^*_\lambda. \tag{6}$$

Let $d\mu(\theta) = f(\theta)\, d\theta + d\mu_s(\theta)$. It follows from [4] that

$$F_1(t) = F_1\left(e^{i\theta}\right) = f(\theta) + i\widetilde{\mu}(\theta). \tag{7}$$

Consider the measure $d\mu^*(\theta) = \dfrac{i}{e^{i\theta} - z} d\mu(\theta)$.

Then

$$\widetilde{\mu}^*(\theta) = \lim_{\varepsilon \to 0+} \frac{1}{2\pi} \int\limits_{\{t \in T_0 : |t-\theta| > \varepsilon\}} ctg \frac{t-\theta}{2} d\mu^*(t) =$$

$$= \lim_{\varepsilon \to 0+} \frac{1}{2\pi} \int\limits_{\{t \in T_0 : |t-\theta| > \varepsilon\}} ctg \frac{t-\theta}{2} \frac{i}{e^{it} - z} d\mu(t) =$$

$$= \frac{i}{e^{i\theta} - z} \widetilde{\mu}(\theta) + \frac{1}{2\pi} \lim_{\varepsilon \to 0+} \int\limits_{\{t \in T_0 : |t-\theta| > \varepsilon\}} ctg \frac{t-\theta}{2} \left(\frac{i}{e^{it} - z} - \frac{i}{e^{i\theta} - z} \right) d\mu(t) =$$

$$= \frac{i}{e^{i\theta} - z} \widetilde{\mu}(\theta) - \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{\cos \dfrac{\theta + t}{2} e^{i\frac{\theta+t}{2}}}{\left(e^{it} - z\right)\left(e^{i\theta} - z\right)} d\mu(t) = \frac{i}{e^{i\theta} - z} \widetilde{\mu}(\theta) + f_1(\theta), \tag{8}$$

where

$$f_1(\theta) = \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{\cos \dfrac{\theta + t}{2} e^{i\frac{\theta+t}{2}}}{\left(e^{it} - z\right)\left(e^{i\theta} - z\right)} d\mu(t) \in L_1([0; 2\pi)).$$

We get from (7) and (8)

$$\frac{F_1(t)}{t-z} = \frac{f(\theta)}{e^{i\theta} - z} + \frac{i\widetilde{\mu}(\theta)}{e^{i\theta} - z} = \frac{f(\theta)}{e^{i\theta} - z} + \widetilde{\mu}^*(\theta) - f_1(\theta) = \widetilde{\mu}^*(\theta) + f_2(\theta),$$

where $f_2(\theta) = \dfrac{f(\theta)}{e^{i\theta} - z} - f_1(\theta) \in L_1([0; 2\pi))$. It follows from the integrability of the function $f_2(\theta)$ that $\lim\limits_{\lambda \to +\infty} \lambda \cdot m\{t \in T_0 : |f_2(t)| > \lambda\} = 0$.

Since $\dfrac{F_1(t)}{t-z} = \widetilde{\mu}^*(\theta) + f_2(\theta)$ and by theorem C there exists a finite limit

$$\lim_{\lambda \to +\infty} \lambda \cdot m\{t \in T_0 : |\widetilde{\mu}^*(t)| > \lambda\} = \frac{2}{\pi} \|\widetilde{\mu}_s\|, \tag{9}$$

then $\forall\alpha > 0$ we get

$$m\left\{t \in T_0 : |\widetilde{\mu}^*(t)| > \lambda,\ \left|\frac{F_1(t)}{t-z}\right| \le \lambda\right\} \le m\left\{t \in T_0 : \lambda < |\widetilde{\mu}^*(t)| \le (1+\alpha)\lambda\right\} +$$

$$+m\left\{t \in T_0 : |\widetilde{\mu}^*(t)| > (1+\alpha)\lambda,\ \left|\frac{F_1(t)}{t-z}\right| \le \lambda\right\} \le m\left\{t \in T_0 : |\widetilde{\mu}^*(t)| > \lambda\right\} -$$

$$-m\left\{t \in T_0 : |\widetilde{\mu}^*(t)| > (1+\alpha)\lambda\right\} + m\left\{t \in T_0 : |f_2(t)| > \alpha\lambda\right\}.$$

Hence, it holds the inequality

$$\lim_{\lambda\to+\infty} m\left\{t \in T_0 : |\widetilde{\mu}^*(t)| > \lambda,\ \left|\frac{F_1(t)}{t-z}\right| \le \lambda\right\} \le \frac{2}{\pi}\frac{\alpha}{1+\alpha}\left\|\widetilde{\mu}_s\right\|.$$

From the arbitrariness of the number $\alpha > 0$ we get

$$\lim_{\lambda\to+\infty} m\left\{t \in T_0 : |\widetilde{\mu}^*(t)| > \lambda,\ \left|\frac{F_1(t)}{t-z}\right| \le \lambda\right\} = 0. \tag{10}$$

It is similarly proved that

$$\lim_{\lambda\to+\infty} m\left\{t \in T_0 : |\widetilde{\mu}^*(t)| \le \lambda,\ \left|\frac{F_1(t)}{t-z}\right| > \lambda\right\} = 0. \tag{11}$$

It follows from equalities $(9), (10)$ and (11) that there exists a finite limit

$$\lim_{\lambda\to+\infty} m\left\{t \in T_0 :\ \left|\frac{F_1(t)}{t-z}\right| > \lambda\right\} = \frac{2}{\pi}\left\|\mu_s^*\right\| = \alpha_0.$$

So

$$m\left\{t \in T_0 :\ \left|\frac{F_1(t)}{t-z}\right| > \lambda\right\} = \frac{\alpha_0}{\lambda} + \frac{\varepsilon(\lambda)}{\lambda},$$

where $\lim\limits_{\lambda\to+\infty}\varepsilon(\lambda) = 0$. Consequently

$$(\lambda + c_1)\, mT_\lambda^* = (\lambda + c_1)\left[\frac{\alpha_0}{\lambda + c_1} + \frac{\varepsilon(\lambda + c_1)}{\lambda + c_1} - \frac{\alpha_0}{\lambda - c_1} - \frac{\varepsilon(\lambda - c_1)}{\lambda - c_1}\right] =$$

$$= \frac{2c_1\alpha_0}{\lambda - c_1} + \varepsilon(\lambda + c_1) - \frac{\lambda + c_1}{\lambda - c_1}\varepsilon(\lambda - c_1) \to 0, \quad \mathit{for} \quad \lambda \to +\infty.$$

Equality (5) follows from ineguality (6). Now, let's prove equality (2). From $(3), (4), (5)$ and (6) we get

$$F(z) = F_1(z) + c_0 = \frac{1}{2\pi i}\lim_{\lambda\to+\infty}\int\limits_{T'_\lambda}\frac{F_1(t)}{t-z}dt + \frac{1}{2\pi}\int\limits_0^{2\pi}\frac{e^{i\theta}}{e^{i\theta}-z}d\mu_s(\theta) + c_0 =$$

$$= \frac{1}{2\pi i}\lim_{\lambda\to+\infty}\int\limits_{T_\lambda}\frac{F(t)-c_0}{t-z}dt + \frac{1}{2\pi}\int\limits_T\frac{e^{i\theta}}{e^{i\theta}-z}\frac{1}{2i}e^{-i\theta}d\nu_s\left(e^{i\theta}\right) + c_0 =$$

$$=\frac{1}{2\pi i}\lim_{\lambda\to+\infty}\int_{T_\lambda}\frac{F(t)}{t-z}dt+\frac{1}{4\pi i}\int_{T}\frac{d\nu_s(t)}{t-z}$$

The theorem is proved.

In future, we'll consider the functions determined on T_0 , as periodic functions determined on a real straightline with period 2π. For any real function f on T_0 we denote [2]

$$P(f\;;q\;;\theta_0\;;\theta)=\frac{\pi}{2}ctg\frac{\pi q^{n-1}}{2}f(\theta)$$

for $\theta\in\left(\theta_0+\pi q^n\;;\theta_0+\pi q^{n-1}\right)\cup\left(\theta_0-\pi q^{n-1}\;;\theta_0-\pi q^n\right)$,
$n\in N,\ 0<q<1,\ \ \theta_0\in T_0$,

$$P_1(f\;;\theta_0\;)=\lim_{q\to1-}\lim_{\alpha\to+\infty}\alpha m\{\theta\in(\theta_0;\theta_0+\pi):|P(f\;;q\;;\theta_0\;;\theta)|>\alpha\}\tag{12}$$

$$P_2(f\;;\theta_0\;)=\lim_{q\to1-}\lim_{\alpha\to+\infty}\alpha m\{\theta\in(\theta_0-\pi;\theta_0):|P(f\;;q\;;\theta_0\;;\theta)|>\alpha\}\tag{13}$$

$$r_{\lambda,f}(\theta_0)=\begin{cases} sign\,(P_2(f\;;\theta_0\;)-P_1(f\;;\theta_0\;)) & for\quad f\;(\theta_0)>\lambda\;,\\ 0 & for\quad |\,f\;(\theta_0)|\le\lambda\;,\\ sign\,(P_1(f\;;\theta_0\;)-P_2(f\;;\theta_0\;)) & for\quad f\;(\theta_0)<-\lambda\;,\end{cases}$$

if there exist limits at the right hand sides of equalities (12) and (13) almost for all $\theta_0\in[0;2\pi)$.

Notice that for any finite real Borel measure ν the function $r_{\lambda,\tilde\nu}(\theta_0)$ exists almost everywhere [2] .

Lemma 1. *Let the functions f , f_1 and f_2 be determined on T_0 . If $f(\theta)=f_1(\theta)+f_2(\theta)$, and $f_2(\theta)$ is a bounded function, then the existence of the function $r_{\lambda,f}(\theta_0)$ follows from the existence of the function $r_{\lambda,f_1}(\theta_0)$ and the equality*

$$r_{\lambda,f}(\theta_0)=r_{\lambda,f_1}(\theta_0)\tag{14}$$

is valid.

Proof. First we prove that if for some interval $(a;b)$ there exists a limit $\lim\limits_{\alpha\to+\infty}\alpha m\{\theta\in(a;b):|f_1(\theta)|>\alpha\}$. Then there exists a limit

$$\lim_{\alpha\to+\infty}\alpha m\{\theta\in(a;b):|f(\theta)|>\alpha\}$$

and the equality

$$\lim_{\alpha\to+\infty}\alpha m\{\theta\in(a;b):|f(\theta)|>\alpha\}=\lim_{\alpha\to+\infty}\alpha m\{\theta\in(a;b):|f_1(\theta)|>\alpha\}\tag{15}$$

is true.

Denote $\max\limits_{\theta}|f_2(\theta)|=c$, $\alpha_0=\lim\limits_{\alpha\to+\infty}\alpha m\{\theta\in(a;b):|f_1(\theta)|>\alpha\}$. From inclusions

$$\{\theta\in(a;b):|f_1(\theta)|>\alpha+c\}\subset\{\theta\in(a;b):|f(\theta)|>\alpha\}\subset$$

$$\subset\{\theta\in(a;b):|f_1(\theta)|>\alpha-c\}$$

we get

$$\alpha m\left\{\theta\in(a;b):\left|f_{1}(\theta)\right|>\alpha+c\right\}\leq\alpha m\left\{\theta\in(a;b):\left|f(\theta)\right|>\alpha\right\}\leq$$

$$\leq\alpha m\left\{\theta\in(a;b):\left|f_{1}(\theta)\right|>\alpha-c\right\}\qquad(16)$$

But the left and right hand sides of inequality (16) tend to the same limit α_0. So, there exists a limit $\lim\limits_{\alpha\to+\infty}\alpha m\{\theta\in(a;b):|f(\theta)|>\alpha\}$ and equality (15) is valid. Equality (14) follows from equality (15) and definition $r_{\lambda,f}(\theta_0)$. The lemma is proved.

Theorem 2. *Let* ν *be a finite complex measure on* T *and* $F(z)=\dfrac{1}{2\pi i}\displaystyle\int_T\frac{d\nu(t)}{t-z}$ *, then*

$$F(z)=\frac{1}{2\pi i}\lim_{\lambda\to+\infty}\int_{T_\lambda}\frac{F(t)}{t-z}dt+\frac{1}{4i}\lim_{\lambda\to+\infty}\lambda\int_0^{2\pi}r_{\lambda,\operatorname{Re}h}(\theta_0)\,d\theta_0+$$

$$+\frac{1}{4}\lim_{\lambda\to+\infty}\lambda\int_0^{2\pi}r_{\lambda,\operatorname{Im}h}(\theta_0)\,d\theta_0,\qquad(17)$$

where $h(\theta)=\dfrac{e^{i\theta}}{e^{i\theta}-z}F\left(e^{i\theta}\right)$, $F\left(e^{i\theta}\right)$ *is non-tangential boundary value of the function* $F(z)$ *as* $z\to t=e^{i\theta}$, $T_\lambda=\left\{t=e^{i\theta}:|h(\theta)|\leq\lambda\right\}$.

Proof. Denote $d\mu(\theta)=\dfrac{1}{2i}e^{-i\theta}d\nu\left(e^{i\theta}\right)$, $\theta_0\in[0;2\pi)$,
$F_1(z)=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta)$. Then from theorem B we get

$$\frac{1}{4\pi i}\int_T\frac{d\nu_s(t)}{t-z}=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}}{e^{i\theta}-z}d\mu_s(\theta)=\frac{1}{4i}\lim_{\lambda\to+\infty}\lambda\int_0^{2\pi}r_{\lambda,\operatorname{Re}h_1}(\theta_0)\,d\theta_0+$$

$$+\frac{1}{4}\lim_{\lambda\to+\infty}\lambda\int_0^{2\pi}r_{\lambda,\operatorname{Im}h_1}(\theta_0)\,d\theta_0,\qquad(18)$$

where $h_1(\theta)=\dfrac{e^{i\theta}}{e^{i\theta}-z}F_1\left(e^{i\theta}\right)$, $F_1\left(e^{i\theta}\right)=F\left(e^{i\theta}\right)-c_0$, $c_0=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}d\mu(\theta)$. But since (see (3)) $h_1(\theta)=h(\theta)-\dfrac{e^{i\theta}}{e^{i\theta}-z}c_0$ and the function $\dfrac{e^{i\theta}}{e^{i\theta}-z}c_0$ is bounded, then equality (17) follows from lemma 1 and equalities $(2),(18)$. The theorem is proved.

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