Nurali M. MAHMUDOV

NECESSARY OPTIMALITY CONDITION IN AN OPTIMAL CONTROL PROBLEM FOR SCHRODINGER EQUATION WITH PURE IMAGINARY COEFFICIENT IN THE NONLINEAR PART OF THIS EQUATION

Abstract

In the present paper we consider an optimal control problem for Schrodinger nonlinear equation with pure imaginary coefficient in the nonlinear part.

The optimal control problems for Schrodinger nonlinear equation often arise in quantum mechanics, nuclear physics, nonlinear optics, theory of superconductivity and in other fields of up-to-date physics and engineering.

In the present paper we consider an optimal control problem for Schrodinger nonlinear equation with pure imaginary coefficient in the nonlinear part. It should be noted that such problems for Schrodinger nonlinear eqaution in another statements were investigated in the papers [1,2] and others.

Let $l > 0$, $T > 0$ be the given numbers, $x \in (0,l)$, $t \in (0,T)$, $\Omega_t = (0,l) \times \times (0,t)$, $\Omega = \Omega_t$.

It is required to minimize the functional

$$J_\alpha(\nu) = \int_\Omega |\psi_1(x,t) - \psi_2(x,t)|^2 \, dxdt + \alpha \|\nu - \omega\|_H^2 \tag{1}$$

on the set $V \equiv \left\{ \nu = \nu(x) : \nu \in W_2^1(0,l), \|\nu\|_{W_2^1(0,l)} \le b \right\}$ under conditions:

$$i\frac{\partial \psi_k}{\partial t} + a_0 \frac{\partial^2 \psi_k}{\partial x^2} - a(x)\psi_k - \nu(x)\psi_k + ia_1 |\psi_k|^2 \psi_k = f_k(x,t), \qquad (x,t) \in \Omega, \tag{2}$$

$$\psi_k(x,0) = \varphi_k(x), \ k = 1,2, \qquad x \in (0,l) \tag{3}$$

$$\psi_1(0,t) = \psi_1(l,t) = 0, \qquad t \in (0,T), \tag{4}$$

$$\frac{\partial \psi_2(0,t)}{\partial x} = \frac{\partial \psi_2(l,t)}{\partial x} = 0, \qquad t \in (0,T) \tag{5}$$

where $i^2 = -1$, $a_0 > 0$, $a_1 > 0$, $b > 0$ are the given numbers, $a = a(x)$ is a bounded measurable function satisfying the condition

$$0 < \mu_0 \le a(x) \le \mu_1, \ \left| \frac{da(x)}{dx} \right| \le \mu_2, \ \overset{\circ}{\forall} x \in (0,l), \ \mu_0, \mu_1, \mu_2 = const > 0, \tag{6}$$

and the functions $\varphi_k(x)$, $f_k(x,t)$, $k = 1,2$ satisfy the conditions:

$$\varphi_1 \in \overset{\circ}{W}{}_2^2(0,l), \ \varphi_2 \in W_2^2(0,l), \ \frac{d\varphi_2(0)}{dx} = \frac{d\varphi_2(l)}{dx} = 0, \tag{7}$$

$$f_1 \in \overset{\circ}{W}{}_2^{1,1}(\Omega), \quad f_2 \in W_2^{1,1}(\Omega) \tag{8}$$

The problem on definition of functions $\psi_k = \psi_k(x,t)$, $k=1,2$ from the condition (2)-(5) for the given $\nu \in V$ is said to be a reduced problem. Under the solution of this problem we'll understand the functions $\psi_k = \psi_k(x,t)$, $k=1,2$, belonging to $B_1 \equiv C^0\left([0,T], \overset{\circ}{W}{}_2^2(0,l)\right) \cap C^1([0,T], L_2(0,l))$ and $B_2 \equiv C^0\left([0,T], W_2^2(0,l)\right) \cap C^1([0,T], L_2(0,l))$, respectively, and satisfying the conditions (2)-(5) for almost all $x \in (0,l)$ and $\forall t \in [0,T]$. The reduced problem consists of two boundary value problems, i.e. the first and second boundary value problems for Schtodinger equation. It should be noted that boundary value problems for equation (2) were earlier investigated in the papers [1-3] and others. However, these results are not sufficient for our goal. In the indicated papers a more wide class of admissi ble controls is a set from $W_\infty^1(0,l)$ but in our case a class of admissible controls is a set from the Hilbert space $W_2^1(0,l)$ that is wider than $W_\infty^1(0,l)$. Therefore, there again arises necessity to study the matter of correctness of the statement of boundary value problem (2)-(5) with a coeeficient from the set $V \subset W_2^1(0,l)$.

Using the Galerkin's method and the proof method of the papers [1,2,4,5] we can prove the validity of the statement:

Theorem 1. *Let $a(x)$, $\varphi_k(x)$, $f_k(x,t)$, $k=1,2$ satisfy the conditions (6)-(8). Then the reduced problem (2)-(5) has a unique solution for each $\nu \in V$, has a unique solution $\psi_1 \in B_1$ and $\psi_2 \in B_2$ and the estimations:*

$$\|\psi(\cdot,t)\|_{\overset{\circ}{W}{}_2^2(0,l)} + \left\|\frac{\partial\psi(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} \leq$$

$$\leq M_1\left(\|\varphi_1\|_{\overset{\circ}{W}{}_2^2(0,l)} + \|f_1\|_{\overset{\circ}{W}{}_2^{1,1}(\Omega)} + \|\varphi_1\|^3_{\overset{\circ}{W}{}_2^1(0,l)} + \|f_1\|^3_{\overset{\circ}{W}{}_2^{1,0}(\Omega)}\right) \tag{9}$$

$$\|\psi(\cdot,t)\|_{W_2^2(0,l)} + \left\|\frac{\partial\psi(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} \leq$$

$$\leq M_2\left(\|\varphi_2\|_{W_2^2(0,l)} + \|f_2\|_{W_2^{1,1}(\Omega)} + \|\varphi_2\|^3_{W_2^1(0,l)} + \|f_2\|^3_{W_2^{1,0}(\Omega)}\right) \tag{10}$$

are true for $\forall t \in [0,T]$, where M_1, M_2 are some positive constants.

Now, let's study differentiability of the functional $J_\alpha(\nu)$ on the set V. To this end we introduce the following adjoint problem on definition of the functions $\eta_k = \eta_k(x,t)$, $k=1,2$, from the conditions:

$$i\frac{\partial\eta_k}{\partial t} + a_0\frac{\partial^2\eta_k}{\partial x^2} - a(x)\eta_k - \nu(x)\eta_k - ia_1\left(2|\psi_k|^2\eta_k - \psi_k^2\overline{\eta}_k\right) =$$

$$= 2(-1)^k(\psi_1(x,t) - \psi_2(x,t)), \quad (x,t) \in \Omega, \tag{11}$$

$$\eta_k(x,T) = 0, \quad x \in (0,l), \quad k=1,2, \tag{12}$$

$$\eta_1(0,T) = \eta_1(l,t) = 0, \quad t \in (0,T), \tag{13}$$

$$\frac{\partial\eta_2(0,t)}{\partial x} = \frac{\partial\eta_2(l,t)}{\partial x} = 0, \quad t \in (0,T), \tag{14}$$

where $\psi_k = \psi_k(x,t)$, $k=1,2$ is the solution of the reduced problem (2)-(5) for $\nu \in V$.

Under the solution of this adjoint problem we'll understand the functions $\eta_k = \eta_k(x,t)$, $k=1,2$ from the space $C^0([0,T], L_2(0,l))$ satisfying the identity integrals:

$$\int_\Omega \left\{ \eta_k \left[-i\frac{\partial \overline{\Phi}_k}{\partial t} + a_0 \frac{\partial^2 \overline{\Phi}_k}{\partial x^2} - a(x)\overline{\Phi}_k - \nu(x)\overline{\Phi}_k - i2a_1 |\psi_k|^2 \overline{\Phi}_k \right] + \right.$$

$$\left. + ia_1 \overline{\eta}_k \psi_k^2 \overline{\Phi}_k \right\} dxdt = 2(-1)^k \int_\Omega (\psi_1(x,t) - \psi_2(x,t)) \overline{\Phi}_k(x,t)\, dxdt, \quad k=1,2 \quad (15)$$

for any functions $\Phi_1 \in \overset{\circ}{W}{}_2^{2,1}(\Omega)$, $\Phi_2 \in W_2^{2,1}(\Omega)$ satisfying the conditions:

$$\frac{\partial \Phi_2(0,t)}{\partial x} = \frac{\partial \Phi_2(l,t)}{\partial x} = 0, \quad t \in (0,T), \; \Phi_k(x,0) = 0, \; k=1,2.$$

Applying the method used for solving the reduced problem (2)-(5) we can prove the existence and uniqueness of the solution of the adjoint problem (11)-(14) and estimates for these solutions:

$$\|\eta_k(\cdot,t)\|_{L_2(0,l)} \le M_3 \|\psi_1 - \psi_2\|_{L_2(\Omega)}, \quad k=1,2, \qquad \forall t \in [0,T] \qquad (16)$$

Now, let's consider the increment of the functional $J_\alpha(\nu)$ on any element $\nu \in V$. Let $\delta\nu \in W_2^1(0,l)$ be an increment of the element $\nu \in V$ such that $\nu + \delta\nu \in V$. Then, we use the formulae for $J_\alpha(\nu)$ and have:

$$\delta J_\alpha(\nu) = J_\alpha(\nu + \delta\nu) - J_\alpha(\nu) = 2\int_\Omega \mathrm{Re}\,[(\psi_1(x,t) -$$

$$-\psi_2(x,t))\left(\delta\overline{\psi}_1(x,t) - \delta\overline{\psi}_2(x,t)\right)]\, dxdt +$$

$$+2\alpha \int_0^l (\nu(x) - \omega(x))\,\delta\nu(x)\, dx + 2\alpha \int_0^l \left(\frac{d\nu(x)}{dx} - \frac{d\omega(x)}{dx} \right) \frac{d\delta\nu(x)}{dx} dx +$$

$$+\alpha \|\delta\nu\|^2_{W_2^1(0,l)} + \|\delta\psi_1\|^2_{L_2(\Omega)} + \|\delta\psi_2\|^2_{L_2(\Omega)} - 2\int_\Omega \mathrm{Re}\left(\delta\psi_1(x,t)\,\delta\overline{\psi}_2(x,t)\right) dxdt, \quad (17)$$

where $\delta\psi_k = \delta\psi_k(x,t)$, $k=1,2$ are solutions of the boundary value problem

$$i\frac{\partial \delta\psi_k}{\partial t} + a_0 \frac{\partial^2 \delta\psi_k}{\partial x^2} - a(x)\,\delta\psi_k - (\nu + \delta\nu)\,\delta\psi_k + ia_1\left(|\psi_{k\delta}|^2 - |\psi_k|^2\right)\delta\psi_k +$$

$$+ia_1 \psi_{k\delta}\psi_k \delta\overline{\psi}_k = \delta\nu\,\psi_k(x,t;\nu), \quad (x,t) \in \Omega, \qquad (18)$$

$$\delta\psi_k(x,0) = 0, \quad x \in (0,l), \; k=1,2, \qquad (19)$$

$$\delta\psi_1(0,t) = \delta\psi_1(l,t) = 0, \quad t \in (0,T), \qquad (20)$$

$$\frac{\partial \delta\psi_2(0,t)}{\partial x} = \frac{\partial \delta\psi_2(l,t)}{\partial x} = 0, \quad t \in (0,T), \qquad (21)$$

where $\psi_k = \psi_k(x,t) \equiv \psi_k(x,t;\nu)$, $k=1,2$ is a solution of the reduced problem (2)-(5) for $\nu \in V$. For the solution of this problem we can establish the estimation.

$$\|\delta\psi_k(\cdot,t)\|^2_{L_2(0,l)} \le M_4 \|\delta\nu\|^2_{L_2(0,l)}, \quad \forall t \in [0,T] \qquad (22)$$

Theorem 2. *Let the conditions of theorem 1 be fulfilled and* $\omega \in W_2^1(0,l)$ *be a given element. Then for any function* $w = w(x)$ *from* $W_2^1(0,l)$ *it is valid the following expression for the first variation of the functional* $J_\alpha(\nu)$*:*

$$\delta J_\alpha(v,w) = \int_0^l \left[-\int_0^T \mathrm{Re}\left(\psi_1(x,t)\overline{\eta}_1(x,t) + \psi_2(x,t)\overline{\eta}_2(x,t)\right) dt \cdot w(x) + \right.$$

$$\left. +2\alpha(\nu(x)-\omega(x))w(x) + 2\alpha\left(\frac{d\nu(x)}{dx} - \frac{d\omega(x)}{dx}\right)\frac{dw(x)}{dx}\right] dx, \qquad (23)$$

where $\psi_k = \psi_k(x,t) \equiv \psi_k(x,t;\nu)$, $\eta_k = \eta_k(x,t) \equiv \eta(x,t;\nu)$, $k = 1,2$, *respectively are the solutions of the reduced problem (2)-(5) and adjoint problem (11)-(14) for* $\nu \in V$.

Proof. Clearly, the functions $\delta\psi_k(x,t)$, $k = 1,2$ - solution of the boundary value problem (18)-(21) will satisfy the following identity integrals:

$$\int_\Omega \left[i\frac{\partial\delta\psi_k}{\partial t} + a_0\frac{\partial^2\delta\psi_k}{\partial x^2} - a(x)\delta\psi_k - (\nu+\delta\nu)\delta\psi_k + ia_1\left(|\psi_{k\delta}|^2 - |\psi_k|^2\right)\delta\psi_k + \right.$$

$$\left. +ia_1\psi_{k\delta}\psi_k\delta\overline{\psi}_k\right]\overline{\Phi}_{1k}(x,t)\,dxdt = \int_\Omega \delta\nu(x)\psi_k(x,t)\overline{\Phi}_{1k}(x,t)\,dxdt, \quad k = 1,2,$$

for any functions $\Phi_{1k} = \Phi_{1k}(x,t)$, $k = 1,2$ from $L_2(\Omega)$. In view of $\eta_k \in L_2(\Omega)$, $k = 1,2$, in this equality instead of the functions $\overline{\Phi}_k = \overline{\Phi}_k(x,t)$, $k = 1,2$, we take $\overline{\eta}_k = \overline{\eta}_k(x,t)$, $k = 1,2$. Then we get the validity of the equality:

$$\int_\Omega \left[i\frac{\partial\delta\psi_k}{\partial t} + a_0\frac{\partial^2\delta\psi_k}{\partial x^2} - a(x)\delta\psi_k - (\nu+\delta\nu)\delta\psi_k + ia_1\left(|\psi_{k\delta}|^2 + |\psi_k|^2\right)\delta\psi_k + \right.$$

$$\left. +ia_1\psi_{k\delta}\psi_k\delta\overline{\psi}_k\right]\overline{\eta}_k(x,t)\,dxdt = \int_\Omega \delta\nu(x)\psi_k(x,t)\overline{\eta}_k(x,t)\,dxdt, \quad k = 1,2. \qquad (24)$$

As the functions $\delta\psi_1 \in \overset{\circ}{W}{}_2^{2,1}(\Omega)$, $\delta\psi_1(x,0) = 0$, $\delta\psi_2 \in W_2^{2,1}(\Omega)$, $\delta\psi_2(x,0) = 0$, $\dfrac{\partial\delta\psi_2(0,t)}{\partial x} = \dfrac{\partial\delta\psi_2(l,t)}{\partial x} = 0$, in identity integral (15) instead of $\overline{\Phi}_k = \overline{\Phi}_k(x,t)$, $k = 1,2$, we take the functions $\delta\psi_k = \delta\psi_k(x,t)$, $k = 1,2$. Then, taking the complex conjugation of the obtained equality we have:

$$\int_\Omega \left\{ \left[i\frac{\partial\delta\psi_k}{\partial t} + a_0\frac{\partial^2\delta\psi_k}{\partial x^2} - a(x)\delta\psi_k - \nu(x)\delta\psi_k + 2ia_1|\psi_k|^2\delta\psi_k\right]\overline{\eta}_k - \right.$$

$$\left. -ia_1\overline{\psi}_k^2\delta\psi_k\eta_k\right\} dxdt = 2(-1)^k\int_\Omega \left(\overline{\psi}_1(x,t) - \overline{\psi}_2(x,t)\right)\delta\psi_k(x,t)\,dxd\tau, k = 1,2. \quad (25)$$

If from the equality (24) we substract equality (25), we get:

$$\int_\Omega \left[-\delta\nu(x)\delta\psi_k\overline{\eta}_k + ia_1|\psi_{k\delta}|^2\overline{\eta}_k - ia_1|\psi_k|^2\delta\psi_k\overline{\eta}_k + ia_1\psi_{k\delta}\psi_k\delta\overline{\psi}_k\overline{\eta}_k + \right.$$

$$\left.+ia_1\overline{\psi}_k^2\delta\psi_k\eta_k\right]dxdt=\int\limits_{\Omega}\left(\overline{\psi}_1(x,t)-\overline{\psi}_2(x,t)\right)\delta\psi_k(x,t)\,dxdt,\quad k=1,2. \tag{26}$$

Using these equalities it is easy to get validity of the equality:

$$2\int\limits_{\Omega}\mathrm{Re}\left[\left(\psi_1(x,t)-\psi_2(x,t)\right)\left(\delta\overline{\psi}_1(x,t)-\delta\overline{\psi}_2(x,t)\right)\right]dxdt=$$

$$=-\int\limits_{\Omega}\mathrm{Re}\left(\psi_1(x,t)\,\overline{\eta}_1(x,t)+\psi_2(x,t)\,\overline{\eta}_2(x,t)\right)dxdt-$$

$$-\int\limits_{\Omega}\mathrm{Re}\left(\delta\psi_1(x,t)\,\overline{\eta}_1(x,t)+\delta\psi_2(x,t)\,\overline{\eta}_2(x,t)\right)dxdt+$$

$$+\int\limits_{\Omega}\mathrm{Im}\left[a_1\left(\psi_{1\delta}\left|\delta\psi_1\right|^2\overline{\eta}_1+\overline{\psi}_1\left(\delta\psi_1\right)^2\overline{\eta}_1\right)\right]dxdt+$$

$$+\int\limits_{\Omega}\mathrm{Im}\left[a_1\left(\psi_{2\delta}\left|\delta\psi_2\right|^2\overline{\eta}_2+\overline{\psi}_2\left(\delta\psi_2\right)^2\overline{\eta}_2\right)\right]dxdt+$$

$$+\int\limits_{\Omega}\mathrm{Im}\left[a_1\left|\delta\psi_1\right|^2\psi_1\overline{\eta}_1+a_1\left|\delta\psi_2\right|^2\psi_2\overline{\eta}_2\right]dxdt. \tag{27}$$

Now, we take this equality into account in the right hand side of (17). Hence we get the following formula for the increment of the functional $J_\alpha(\nu)$:

$$\delta J_\alpha(\nu)=\int\limits_0^l\left\{\left[-\int\limits_0^T\mathrm{Re}\left(\psi_1(x,t)\,\overline{\eta}_1(x,t)+\psi_2(x,t)\,\overline{\eta}_2(x,t)\right)dt+2\alpha\left(\nu(x)-\omega(x)\right)\right]\times\right.$$

$$\left.\times\,\delta\nu(x)+2\alpha\left(\frac{d\nu(x)}{dx}-\frac{d\omega(x)}{dx}\right)\frac{d\nu(x)}{dx}\right\}dx+R, \tag{28}$$

where R is determined by the formula:

$$R=\left\|\delta\psi_1\right\|^2_{L_2(\Omega)}+\left\|\delta\psi_2\right\|^2_{L_2(\Omega)}+\alpha\left\|\delta\nu\right\|^2_{W_2^1(0,l)}-$$

$$-2\int\limits_{\Omega}\mathrm{Re}\left(\delta\psi_1\delta\overline{\psi}_2\right)\delta\nu(x)\,dxdt-\int\limits_{\Omega}\mathrm{Re}\left(\delta\psi_1\overline{\eta}_1+\delta\psi_2\overline{\eta}_2\right)\delta\nu(x)\,dxdt+$$

$$+\int\limits_{\Omega}\mathrm{Im}\left[a_1\left(\psi_{1\delta}\left|\delta\psi_1\right|^2\overline{\eta}_1+\overline{\psi}_1\left(\delta\psi_1\right)^2\overline{\eta}_1\right)\right]dxdt+$$

$$+\int\limits_{\Omega}\mathrm{Im}\left[a_1\left(\psi_{2\delta}\left|\delta\psi_2\right|^2\overline{\eta}_2+\overline{\psi}_2\left(\delta\psi_2\right)^2\overline{\eta}_2\right)\right]dxdt+$$

$$+\int\limits_{\Omega}\mathrm{Im}\left[a_1\left(\left|\delta\psi_1\right|^2\psi_1\overline{\eta}_1+\left|\delta\psi_2\right|^2\psi_2\overline{\eta}_2\right)\right]dxdt. \tag{29}$$

By virtue of Cauchy-Bunyakovskii inequality, from this equality we get validity of the inequality:

$$|R| \leq 2\|\delta\psi_1\|^2_{L_2(\Omega)} + 2\|\delta\psi_2\|^2_{L_2(\Omega)} +$$
$$+\alpha\|\delta\nu\|^2_{W_2^1(0,l)} + \sqrt{T}\|\delta\nu\|_{L_2(0,l)}\|\delta\psi_1\|_{L_\infty(\Omega)}\|\eta_1\|_{L_2(\Omega)} +$$
$$+\sqrt{T}\|\delta\nu\|_{L_2(0,l)}\|\delta\psi_2\|_{L_\infty(\Omega)}\|\eta_1\|_{L_2(\Omega)} + a_1\|\delta\psi_1\|^2_{L_\infty(\Omega)}\|\psi_{1\delta}\|_{L_2(\Omega)}\|\eta_1\|_{L_2(\Omega)} +$$
$$+a_1\|\delta\psi_2\|^2_{L_\infty(\Omega)}\|\psi_{2\delta}\|_{L_2(\Omega)}\|\eta_2\|_{L_2(\Omega)} + 2a_1\|\delta\psi_1\|^2_{L_\infty(\Omega)}\|\psi_1\|_{L_2(\Omega)}\|\eta_1\|_{L_2(\Omega)} +$$
$$+2a_1\|\delta\psi_2\|^2_{L_\infty(\Omega)}\|\psi_2\|_{L_2(\Omega)}\|\eta_2\|_{L_2(\Omega)}. \quad (30)$$

Now, let's estimate the quantities $\|\delta\psi_k\|_{L_\infty(\Omega)}$, $k = 1, 2$. To this end, we multiply the equation (18) by the function $\frac{\partial^2\delta\psi_k}{\partial x^2}$ and integrate the obtained equality with respect to the domain Ω_t. Then, using integration by parts , and also complex conjugation of the obtained equalities, we have:

$$\int\limits_{\Omega_t}\frac{\partial}{\partial\tau}\left|\frac{\partial\delta\psi_k}{\partial x}\right|^2 dxd\tau = 2\int\limits_{\Omega_t}\operatorname{Im}\left(\frac{da}{dx}\delta\psi_k\cdot\frac{\partial\delta\overline{\psi}_k}{\partial x}\right)dxd\tau +$$
$$+2\int\limits_{\Omega_t}\operatorname{Im}\left(\frac{d(\nu+\delta\nu)}{dx}\delta\psi_k\cdot\frac{\partial\delta\overline{\psi}_k}{\partial x}\right)dxd\tau -$$
$$-2\int\limits_{\Omega_t}\operatorname{Re}\left\{a_1\frac{\partial}{\partial x}\left[\left(|\psi_{k\delta}|^2+|\psi_k|^2\right)\delta\psi_k\right]\frac{\partial\delta\overline{\psi}_k}{\partial x}\right\}dxd\tau -$$
$$-2\int\limits_{\Omega_t}\operatorname{Re}\left\{a_1\frac{\partial}{\partial x}\left[\psi_{k\delta}\psi_k\delta\overline{\psi}_k\right]\frac{\partial\delta\overline{\psi}_k}{\partial x}\right\}dxd\tau + 2\int\limits_{\Omega_t}\operatorname{Im}\left[\frac{\partial}{\partial x}\left(\delta\nu(x)\psi_k\right)\frac{\partial\delta\overline{\psi}_k}{\partial x}\right]dxd\tau.$$

By the conditions $\delta\psi_k(x,0) = 0$, $k = 1, 2$, and Cauchy-Bunyakovskii inequality and also the estimations for the solution of the reduced problem (2)-(3) from the last equality it is easy to get validity of the inequality:

$$\left\|\frac{\partial\delta\psi_k(\cdot,t)}{\partial x}\right\|^2_{L_2(0,l)} \leq M_5\|\delta\nu\|^2_{W_2^1(0,l)} + M_6\|\delta\psi_k\|^2_{L_\infty(\Omega)} +$$
$$+M_{32}\int\limits_0^t\left\|\frac{\partial\delta\psi_k(\cdot,\tau)}{\partial x}\right\|_{L_2(0,l)}d\tau, \quad k = 1, 2, \quad \forall t\in[0,T]. \quad (31)$$

For the function $\delta\psi_1(x,t)$ by virtue of multiplicative inequalities we have:

$$\|\delta\psi_1(\cdot,t)\|_{L_\infty(0,l)} \leq M_7\left\|\frac{\partial\delta\psi_1(\cdot,t)}{\partial x}\right\|^{\frac{1}{2}}_{L_2(0,l)}\|\delta\psi_1(\cdot,t)\|^{\frac{1}{2}}_{L_2(0,l)}, \forall t\in[0,T]. \quad (32)$$

For the function $\delta\psi_2(x,t)$ we acn establish the analog of the inequality of type (32) in the form:

$$\|\delta\psi_2(\cdot,t)\|_{L_\infty(0,l)} \leq M_8\left\|\frac{\partial\delta\psi_2(\cdot,t)}{\partial x}\right\|^{\frac{1}{2}}_{L_2(0,l)}\|\delta\psi_2(\cdot,t)\|^{\frac{1}{2}}_{L_2(0,l)} +$$

$$+M_9 \left\| \delta\psi_2 (\cdot, t) \right\|_{L_2(0,l)}, \qquad \forall t \in [0,T]. \tag{33}$$

Using the inequalities (32) and (33) in the inequality (31) and applying the Cauchy inequality with ε and the Gronwall lemma we get validity of the estimation

$$\left\| \frac{\partial \delta\psi_k (\cdot, t)}{\partial x} \right\|^2_{L_2(0,l)} \leq M_{10} \left\| \delta\nu \right\|^2_{W_2^1(0,l)}, \qquad k = 1,2. \tag{34}$$

for $\forall t \in [0,T]$. Thus, using the estimations (22), (34) we have [5]:

$$\left\| \delta\psi_k (\cdot, t) \right\|_{W_2^1(0,l)} \leq M_{11} \left\| \delta\nu \right\|_{W_2^1(0,l)}, \qquad k = 1,2, \qquad \forall t \in [0,T]. \tag{35}$$

By the estiamtions (16) and (35) from (30) we get validity of the inequality

$$|R| \leq M_{12} \left\| \delta\nu \right\|^2_{W_2^1(0,l)}. \tag{36}$$

This means that

$$R = o\left(\left\| \delta\nu \right\|_{W_2^1(0,l)} \right). \tag{37}$$

Then, allowing for this relation we can represent the increment of the functional $J_\alpha(\nu)$ in the form:

$$\delta J_\alpha(\nu) = \int_0^l \left\{ \left[-\int_0^T \mathrm{Re}\left(\psi_1(x,t)\,\overline{\eta}_1(x,t) + \psi_2(x,t)\,\overline{\eta}_2(x,t) \right) dt + 2\alpha\left(\nu(x) - \omega(x) \right) \right] \times \right.$$

$$\left. \times \delta\nu(x) + \left(\frac{d\nu(x)}{dx} - \frac{d\omega(x)}{dx} \right) \frac{d\nu(x)}{dx} \right\} dx + o\left(\left\| \delta\nu \right\|_{W_2^1(0,l)} \right). \tag{38}$$

Here, instead of $\delta\nu \in W_2^1(0,l)$ we take $\theta w \in W_2^1(0,l)$ such that $\nu + \theta w \in V$, where $0 < \theta < 1$. Taking this into account in (38) and calculating the first variation of the functional $J_\alpha(\nu)$, we have:

$$\delta J_\alpha(\nu, w) = \lim_{\theta \longrightarrow 0} \frac{J_\alpha(\nu + \theta w) - J_\alpha(\nu)}{\theta} =$$

$$= \int_0^l \left\{ \left[-\int_0^T \mathrm{Re}\left(\psi_1(x,t)\,\overline{\eta}_1(x,t) + \psi_2(x,t)\,\overline{\eta}_2(x,t) \right) dt + 2\alpha\left(\nu(x) - \omega(x) \right) \right] \times \right.$$

$$\left. \times w(x) + \left(\frac{d\nu(x)}{dx} - \frac{d\omega(x)}{dx} \right) \frac{dw(x)}{dx} \right\} \tag{39}$$

for any function $w \in W_2^1(0,l)$, whence the statement of the theorem follows. Theorem 2 is proved.

Now, let's establish necessary optimality condition in the form of variational inequality in the problem (1)-(5).

Theorem 3. *Let the conditions of theorem 2 be satisfied, and $\nu^* = \nu^*(x)$ from V be an optimal control in the problem (1)-(5). Then for $\forall \nu \in V$ it is fulfilled the inequality*

$$\int_0^l \left\{ \left[\int_0^T \mathrm{Re}\left(\psi_1^*(x,t)\,\overline{\eta}_1^*(x,t) + \psi_2^*(x,t)\,\overline{\eta}_2^*(x,t) \right) dt - 2\alpha\left(\nu^*(x) - \omega(x) \right) \times \right] \right.$$

$$\times(\nu(x)-\omega^*(x))-2\alpha\left(\frac{d\nu^*(x)}{dx}-\frac{d\omega(x)}{dx}\right)\left(\frac{d\nu(x)}{dx}-\frac{d\nu^*(x)}{dx}\right)\Bigg\}dx\le 0,\quad (40)$$

where $\psi_k^*(x,t)=\psi_k(x,t;\nu^*)$ *and* $\eta_k^*(x,t)=\eta_k(x,t;\nu^*)$, $k=1,2$, *are the solutions of the reduced problem (2)-(5) and adjoint problem (11)-(14) for* $\nu^*\in V$.

Proof. Let $\nu\in V$ be an arbitrary control. From the structure of the set it is clear that it is a convex set in the space $W_2^1(0,l)$. Therefore for $\forall\nu^*\in V$ and $\forall\nu\in V$ we have;

$$\nu^*+\theta(\nu-\nu^*)\in V,\ \ \forall\theta\in(0,1).$$

Consequently, for $\nu^*\in V$ to be an optimal control in the problem (1)-(5), i.e. a point of minimum of the functional $J_\alpha(\nu)$ the inequality (see [6], p.403):

$$\frac{d}{d\theta}J_\alpha(\nu^*+\theta(\nu-\nu^*))\Bigg|_{\theta=0}=\delta J_\alpha(\nu^*,\nu-\nu^*)\ge 0$$

should be fulfilled for $\forall\nu\in V$.

Hence, by the formula (40) for $w=\nu-\nu^*$ we have:

$$\delta J_\alpha(\nu^*,\nu-\nu^*)=\int_0^l\Bigg\{\Bigg[-\int_0^T\operatorname{Re}\left(\psi_1^*(x,t)\,\overline{\eta}_1^*(x,t)+\psi_2^*(x,t)\,\overline{\eta}_2^*(x,t)\right)dt+$$

$$+2\alpha(\nu^*(x)-\omega(x))\Bigg](\nu(x)-\nu^*(x))+$$

$$+2\alpha\left(\frac{d\nu^*(x)}{dx}-\frac{d\omega(x)}{dx}\right)\left(\frac{d\nu(x)}{dx}-\frac{d\nu^*(x)}{dx}\right)\Bigg\}\le 0$$

for $\forall\nu\in V$. We multiply this inequality by -1 and get the statement of the theorem. Theorem 3 is proved.

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Nurali M. MAHMUDOV
Nakhchivan State University
University campus, AZ7000, Nakhchivan, Azerbaijan
Tel.: (99412)4394720 (off.)

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