Bilal T. BILALOV, Zafar G. GUSEYNOV

BASIS PROPERTIES OF UNITARY SYSTEMS OF POWERS

Abstract

In the paper we consider unitary systems of powers with complex valued coefficients that are natural generalizations of the systems of sines and cosines. Basis properties of such systems in Lebesgue spaces are investigated.

Application of the Fourier method to the solution of many problems of mathematical physics and mechanics requires to investigate spectral properties of operator pencils $A(\lambda) = A_0 + \lambda A_1 + ... + \lambda^n A_n$ in specific Banach spaces. Usually, if a domain associated with spatial variables is bounded, the resolvent of these pencils (of course, under definite boundary conditions) have isolated singularities on a complex domain. In its turn, the question on studying spectral singularities of a pencil reduces to investigation of basis properties of a part of appropriate root vectors.

In majority of cases, the principal parts of asymptotic expansions of root functions have the forms

$$\{a(t)\varphi^n(t) + b(t)\psi^n(t)\} \tag{1}$$

$$\{a(t)\varphi^n(t) + b(t)\overline{\varphi}^n(t)\}, \tag{2}$$

where $a(t)$, $b(t)$, $\varphi(t)$ and $\psi(t)$, generally speaking, are complex-valued functions.

In order to illustrate the reasoning mentioned above, we consider the quadratic pencil

$$A(\lambda) \equiv D^2 + 2b\lambda D + c\lambda^2 I,$$

where $D = \dfrac{d}{dx}$ is a differentiation operator, I is an identity operator. If in the place of domain of definition of the pencil $A(\lambda)$: $L_p(0,\pi) \to L_p(0,\pi)$ we take the manifold $(1 < p < +\infty)$

$$D_{A(\lambda)} \equiv \left\{ y \in W_p^2(0,\pi) \colon y'(0) + a\lambda y(0) = 0,\ y'(\pi) + a\lambda y(\pi) = 0 \right\},$$

where $a, b, c \in R$ are real parameters, we get that the eigen values of this pencil are $\lambda_k = \dfrac{k}{\sqrt{c - b^2}}$, $k = 0, \pm 1, ...$; $(b^2 - c > 0)$ and the appropriate eigen functions look like

$$y_k = Ae^{\alpha k x} + \overline{e^{\alpha k x}},\ k = 0, \pm 1, ...;$$

where

$$A = \frac{b - a + i\sqrt{c - b^2}}{a - b + i\sqrt{c - b^2}}, \quad \alpha = -\frac{b}{\sqrt{c - b^2}} + i.$$

Evidently, M.G. Javadov [1] first paid attention to the fact that a half of eigen functions $y_k^-(x) = y_k(x)$, $k = 0, -1, ...$; of the pencil $A(\lambda)$ is complete in $L_2(0,\pi)$.

The so-called A.G. Kostyuchenko system $\left\{e^{i\alpha n z} \sin\ nz\right\}$ should also be noted. As early as 1969 A.G. Kostynchenko stated a problem on investigation of basis properties of this system by purely functional methods [2]. The first theorem on completeness of this system $L_2(0,\pi)$ for $\alpha \in R$ was reduced by B.Ya. Levin [2].

A.A. Shkalikov [14] proved the Riesz basicity in $L_2(0,\pi)$ for $\alpha \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The final result for $\alpha \in R$ was obtained by the author [3].

It is notable that very special cases of the system of the form (1), when the coefficients are piece-wise constant, arise in the course of solution of specific problems of mechanics. On this occasion one can see the papers [4-8].

In connection with spectral theory of differential operators, special cases of system (1) were studied in the papers [9-14]. Therewith, the systems of sines with linear phase, i.e. the systems of the form

$$\sin\left[(n+\alpha)t+\beta\right], \quad n \geq 1, \tag{3}$$

where $\alpha, \beta \in R$ were obtained. Basicity criterion of this system in $L_p(0,\pi)$ was obtained in the papers [15, 16] of E.I. Moiseev. By the other method, the same result for $\beta = 0$ was obtained in the paper [17] of A.M. Sedletskiy. A complex case of the parameter $\alpha \left(\beta = \left\{0, \frac{\pi}{4}\right\}\right)$ was considered in the paper [18] of G.G. Devdariani.

In connection with reasonings formulated above, there is an increasing interest to studying basis properties of the systems of the form (1). On the other hand, it is easy to note that the systems of the form (1) are natural generalizations of classic systems of sines (cosines).

From this point of view, it is suitable to treat the Runge theorem [19] on generalization of Weierstrass classic theorem on approximation of a continuous function by polynomials on the segment of a real axis. So, let λ be Jordan's open arch on a complex plane, $z = \varphi(t)$ be its parametric notation, i.e. $\lambda \equiv \varphi\left\{[a,b]\right\}$. The Runge theorem affirms that any function continuous on λ (generally speaking, a complex valued function) may be approximated by the polynomials from $z \in \gamma$, otherwise, the system $\{\varphi^n(t)\}_{n\geq 0}$ is complete in $C[a,b]$. Considering the real space $C_R[a,b]$, we directly see that the systems $\{\mathrm{Re}\,[\varphi^n(t)]\}_{n\geq 0}$ and $\{\mathrm{Im}\,[\varphi^n(t)]\}_{n\geq 1}$ are also complete in $C_R[a,b]$. Apparently, having not observed this treat, in this papers [20,21] Yu.A. Kuzmin proves this result in another way and refinely under more rigid restrictions on the function $\varphi(t)$. Afterwards, the questions on completeness and minimality of the systems of the form (1) were considerered by many mathematicians (see f.i. [12, 23-38]) under different restrictions (mainly, the Holder property of coefficients and continuous differentiability (or piecewise) of the functions $\varphi(t)$, $\psi(t)$) on the functions contained in (1).

It should be noted that, mainly, the question on completeness and minimality of the systems of the form (1) in specific Banach spaces of functions is reduced to investigation of some shear boundary values problems of the theory of analytic functions dictated by the systems (1) itself. Therewith, the obtained problems have many difficulties in comparison with the ones known in references (see f.i. [29-31]). Therefore, different authors investigated these problems in their own way and got different results (even criteria) with respect to basis properties of the systems of the form (1). In the case (2), a more simple problem (namely, the Carleman problem) is obtained. Therefore, in this case, one can consider a more general case with respect to the functions a,b, and a, b and φ.

In the paper we consider completeness and minimality of system (2) in the spaces $L_p(0,\pi)$, $1 < p < +\infty$. Therewith another approach to the investigation of such properties of system (2) is suggested, namely, this question is reduced to similar property for some "binary" system connected with system (2).

A part of results were earlier announced in the papers [32, 33].

1. Main assumption and auxiliary facts

We'll assume that the following conditions are fulfilled for the data of systems (2).

1) $|a(t)|$, $|b(t)|$ and $|\varphi'(t)|$ are measurable on (a,b) and

$$\sup_{(a,b)} vrai \left\{ |a(t)|^{\pm 1};\ \ |b(t)|^{\pm 1};\ \ \left|\varphi'(t)\right|^{\pm 1} \right\} < +\infty.$$

is valid.

2) $\Gamma_\varphi = \varphi\left\{[a,b]\right\}$is Jordan's open, rectifiable, prime curve that doesn't intersect the real axis except the points a and b: $\operatorname{Im}\varphi(a) = \operatorname{Im}\varphi(b) = 0$, and this axis is not tangential to Γ_φ. Γ_φ is either a Radon's curve (i.e. the angle $\theta(\varphi(t))$ between the tangent at the point $\varphi = \varphi(t)$ to the curve Γ_φ and real axis is either a bounded variation function on $[a,b]$ or a piecewise Liapunov curve. By $\{\varphi_k\}$ we denote the discontinuity point of the function $\arg\varphi'(t)$ on (a,b). Γ_φ has no spinodes. Let $\Gamma = \Gamma_\varphi \cup \overline{\Gamma_\varphi}$, where $\overline{\Gamma_\varphi} = \overline{\varphi}\left\{[a,b]\right\}$, $\overline{(\cdot)}$ is a complex conjugation. For definiteness, we'll assume that the point $\varphi = \varphi(t)$ $(\overline{\varphi} = \overline{\varphi}(t)))$ by increase of t runs the curve $\Gamma_\varphi(\overline{\Gamma_\varphi})$, the internal domain int $int\Gamma$ remains in the left.

3) $\alpha(t) \equiv \arg\ a(t)$, $\beta(t) \equiv \arg\ b(t)$ are piecewise continuous functions on $[a,b]$ and may have infinitely many discontinuity points of first kind. Let $\{\alpha_k\}$ and $\{\beta_k\}$ be discontinuity points of these functions on (a,b).

4) The set $\{\widetilde{s}_k\} \equiv \{\alpha_k\} \cup \{\varphi_k\}$ may have a unique limit point $\widetilde{s}_0 \in (a,b)$. The function $\theta(t) \equiv \beta(t) - \alpha(t) + \dfrac{2}{p}\arg\alpha'(t)$ at the point $\widetilde{s}_0$ has from the left and right finite limits, where $p \in (1,+\infty)$ is some number.

Without loss of generality, we'll assume that the functions $\alpha(t)$, $\beta(t)$ and $\arg\varphi'(t)$ are continues from the left on (a,b).

Under the function $\arg\varphi'(t)$ we understand the following: at each initial discontinuity point $\varphi_k((\varphi_k,\varphi_{k+1})$ is discontinuity integral of the function $\arg\varphi'(t))$ we'll determine any branch $\arg\varphi'(\varphi_k+0)$; at the final point φ_{k+1}, we'll get the value $\arg\varphi'(\varphi_{k+1}-0)$ from the chosen branch $\arg\varphi'(\varphi_k+0)$ by means of continuous change. We'll assume that the conditions

$$0 \le \arg\varphi'(a+0) < 2\pi, \qquad \left|\arg\varphi'(\varphi_k+0) - \arg\varphi'(\varphi_k-0)\right| < \pi.$$

are observed.

This is possible, as Γ has no spinodes.

5) $\sum\limits_{i\ge1}\left|\widetilde{h}_i\right| < +\infty$, where $\widetilde{h}_i = \theta(\widetilde{s}_i+0) - \theta(\widetilde{s}_i-0)$.

Everywhere in the sequel, not loosing generality, we'll assume that the system

$$\upsilon_n^{\pm}(t) \equiv a(t)\varphi^n(t) \pm b(t)\overline{\varphi}^n(t),\ n \geq 1,$$

is determined on the segment $[0, a]$. Consider the function

$$\begin{gathered} A(t) \equiv \begin{cases} a(t),\ t \in [0, a], \\ \\ b(-t),\ t \in [-a, 0], \end{cases}, \\ W(t) \equiv \begin{cases} \varphi(t),\ t \in [0, a], \\ \\ \overline{\varphi}(-t),\ t \in [-a, 0], \end{cases}, \\ B(t) \equiv A(-t), \quad t \in [-a, a]. \end{gathered} \tag{4}$$

Let

$$\{V_n(t)\} \equiv \left\{A(t)W^n(t);\ B(t)\overline{W}^n(t)\right\}_{n\geq 1}. \tag{5}$$

It is easy to prove the lemma:

Lemma 1. *The system $\{V_n(t)\}$ is complete (minimal) in $L_p(-a, a)$ iff the systems $\{\upsilon_n^+\}_{n\geq 1}$ and $\{\upsilon_n^-\}_{n\geq 1}$ are simultaneously complete (minimal) in $L_p(0, a)$.*

We'll need the following system

$$\left\{A(t)W^n(t);\ B(t)\overline{W}^k(t)\right\}_{n\geq 0; k\geq 1}. \tag{6}$$

Before we state the main results, define the necessary quantities. Let r be a number after which it holds

$$-\frac{2\pi}{q} < \widetilde{h}_k < \frac{2\pi}{p},\ \ \forall k \geq r;\ \ \frac{1}{p} + \frac{1}{q} = 1.$$

Enumerate the elements of the set $\{\widetilde{s}_i\}_0^r$ by increase and denote by $\{s_i\}_0^r$. Enumerable their appropriate jumps by $\left\{\widetilde{h}_i\right\}_0^r$ and denote $\{h_i\}_0^r$. Determine the integers $n_i,\ i = \overline{0, r}$ from the conditions:

$$\left.\begin{aligned} \frac{2\pi}{p} + 2(n_0 - 1)\pi < \beta(+0) - \alpha(+0) + \frac{2}{p}\arg\varphi'(+0) \leq \frac{2\pi}{p} + 2n_0\pi \\ -\frac{1}{q} < \frac{h_{i-1}}{2\pi} + n_{i-1} - n_i \leq \frac{1}{p},\ i = \overline{1, r+1} \end{aligned}\right\} \tag{7}$$

Denote

$$\omega = \frac{1}{\pi}\left[\alpha(a-0) - \beta(a-0) - \frac{2}{p}\arg\varphi'(a-0)\right] + \frac{3}{p} + n_{r+1} - 1. \tag{7'}$$

2. Completeness of a unitary system

At first we consider completeness of the unitary system $\{\upsilon_n^-(t)\}_{n\geq 1}$ in $L_p(0,a)$. Obviously, the completeness of this system is equivalent to equality almost everywhere to zero of any function f from $L_q(0,a)$, $\frac{1}{p}+\frac{1}{q}=1$, that satisfies the relation

$$\int_0^a \upsilon_n^-(t)\overline{f(t)}dt=0, \quad \forall n\geq 1.$$

Thus, we have

$$\int_0^a a(t)\varphi^n(t)\overline{f}(t)dt-\int_0^a b(t)\overline{\varphi}^n(t)\overline{f}(t)dt=\int_{-a}^a A(t)\overline{F(t)}W^n(t)dt=0, \quad \forall n\geq 1,$$

where

$$A(t)\equiv\begin{cases} a(t), \ t\in[0,a], \\ b(-t), \ t\in[-a,0], \end{cases},$$

$$F(t)\equiv\begin{cases} f(t), \ t\in[0,a], \\ -f(-t), \ t\in[-a,0], \end{cases}.$$

Consequently

$$\int_\Gamma \Phi(\tau)\tau^n d\tau=0, \quad \forall n\geq 0, \tag{7_1}$$

where $\Phi(\tau)\equiv A(\psi(\tau))\,\overline{F}(\psi(\tau))\,\tau[\varphi'(\psi(\tau))]^{-1}$ and $\psi(\tau)$ is inverse to φ on $\Gamma\backslash\{\varphi(0)\}$ function. It is easy to see that the function $\Phi(\tau)$ on Γ satisfies the relation

$$\Phi(\tau)-\frac{A(\psi(\tau))\ \tau[\varphi'(\psi(\tau))]^{-1}}{A(\psi(\overline{\tau}))\ \tau[\varphi'(\psi(\overline{\tau}))]^{-1}}\Phi(\overline{\tau})=0, \quad \tau\in\Gamma. \tag{7_2}$$

Further, conformably mapping *int* Γ into a unit circle in such a way that (ω: *int* $\Gamma\to|z|<1$) $\omega(\varphi(0))=1$, $\omega(\varphi(a))=-1$, from the symmetry principle we get $\omega(\overline{\tau})=\overline{\omega(\tau)}$, $\tau\in\Gamma$. Then acting in the same way in proof of theorem 1 of the paper [34] we have that the completeness of the system $\{\upsilon_n^-(t)\}_{n\geq 1}$ in $L_p(0,a)$ is equivalent to the completeness of the system of exponents $\left\{\widetilde{a}(t)e^{int}-\widetilde{b}(t)e^{-int}\right\}_{n\geq 1}$ in $L_p(0,\pi)$ with appropriate coefficients $\widetilde{a}(t)$ and $\widetilde{b}(t)$.

As a result, using the results of the paper [35] we get the following theorem.

Theorem 1. *Let the functions $a(t)$ and $\varphi(t)$ satisfy conditions (1)-(5) and the quantity ω be determined as above. Then the system $\{\upsilon_n^-(t)\}_{n\geq 1}$ is complete in $L_p(0,a)$ only for $\omega\leq-\frac{1}{q}$.*

Comparing this theorem with theorem 1 of the paper [34], we get the following

Corollary 1. *Let the functions $a(t)$, $b(t)$ and $\varphi(t)$ satisfy conditions 1)-5). Then system (6) is complete in $L_p(-a,a)$ only in the case if the system $\{\upsilon_n^-\}_{n\geq 1}$ is* *complete in* $L_p(0,a)$.

3. Completeness and miniality criteria

The following theorem is valid.

Theorem 2. *Let* $a(t)$, $b(t)$ *and* $\varphi(t)$ *be the functions introduced above, and* $\varphi(a) < 0 < \varphi(0)$. *The quantity* ω *is defined from formulae (7), (7'). The system* $\{v_n^-\}_{n\geq N}$ *is complete in* $L_p(0,a)$ *only for* $\omega \leq -\frac{1}{q}$; *is minimal in* $L_p(0,a)$ *iff* $\omega < -\frac{1}{q} - 1$.

Proof. Let $A(t)$, $B(t)$ and $W(t)$ be the functions introduced above, and $\theta_W(t) \equiv \arg B(t) - \arg A(t) + \frac{2}{p}\arg W'(t)$. Obviously, the discontinuity points of this function on $[-a,a]$ are $\{0\} \cup \{\pm\widetilde{s}_k\}$, and appropriate jumps will be:

$$h_0(W) = \theta_W(+0) - \theta_W(-0) = 2\theta(+0);\ \widetilde{h}_k(W;-\widetilde{s}_k) = \widetilde{h}_k(W;+\widetilde{s}_k),$$

where $\widetilde{h_k}(W;\pm\widetilde{s}_k)$ is a jump of the function $\theta_W(t)$ at the point $\pm\widetilde{s}_k$. At first we assume that the conditions (7) are satisfied for $n_i = 0$, $i = \overline{1,r+1}$ and denote

$$\omega_\varphi = \frac{1}{\pi}\left[\alpha(a-0) - \beta(a-0) - \frac{2}{p}\arg\varphi'(a-0)\right] + \frac{3}{2} - 1.$$

Let the system determined on $[-a,a]$ have the form

$$\left\{c(t)\psi^n(t);\ d(t)\overline{\psi}^n(t)\right\}_{n\geq 0}. \tag{k}$$

By $\omega_{(k)}$ we'll denote the quantity

$$\omega_{(k)} = \frac{1}{2\pi}\left[\arg c(-a+0) - \arg c(a-0) + \arg d(a-0) - \right.$$

$$\left. - \arg d(-a+0) + \frac{2}{p}(\arg\psi'(-a+0) - \arg\psi'(a-0))\right] + \frac{2}{p} - 1,$$

that corresponds to the system (k).

We separately consider the following cases:

a) $-\frac{1}{q} < \frac{h_0(W)}{2\pi} \leq \frac{1}{p}$; $\frac{1}{p} - 3 < \omega_\varphi \leq \frac{1}{p} - 2$.

Consider system (5) and reduce it to the form (k):

$$\left\{\widetilde{A}(t)W^n(t);\ \ \widetilde{B}(t)\overline{W}^n(t)\right\}_{n\geq 0},$$

where $\widetilde{A}(t) \equiv A(t)W(t)$; $\arg\widetilde{A}(t) \equiv \arg A(t) + \arg W(t)$; $\widetilde{B}(t) = B(t)\overline{W}(t)$; $\arg\widetilde{B}(t) = \arg B(t) - \arg W(t)$. Following the paper [34], we calculate the quantity $\omega_{(5)}$ corresponding to this system. Obviously, the discontinuity points on $[-a,a]$ and jumps at these points of the functions $\theta_W(t)$ and

$$\widetilde{\theta}_W(t) \equiv \arg\widetilde{B}(t) - \arg\widetilde{A}(t) + \frac{2}{p}W'(t)$$

coincide. Therefore, it follows from conditions (7) that the quantities
$i=\overline{1,r+1}$; corresponding to system (5) equal zero. Consequently, we have

$$\omega_{(5)}=\frac{1}{2\pi}\Big[\arg\widetilde{B}(-a+0)-\arg\widetilde{B}(a-0)+\arg\widetilde{A}(a-0)-$$

$$-\arg\widetilde{A}(-a+0)+\frac{2}{p}(\arg W'(-a+0)-\arg W'(a-0))\Big]+\frac{2}{p}-1=\omega_\varphi+2,$$

since $\arg W(a-0)-\arg W(-a+0)=2\pi$. Thus, $-\frac{1}{q}<\omega_\varphi\le\frac{1}{p}$. As a result, it follows from theorems 1,2 of the paper [34] that the system (5) is complete and minimal in $L_p(-a,a)$. By lemma 1, the system $\{\upsilon_n^-\}_{n\ge1}$ is complete and minimal in $L_p(0,a)$, as well.

b) $-\frac{1}{q}-1<\frac{h_0(W)}{2\pi}\le\frac{1}{p}-1;\quad -\frac{1}{q}-2<\omega_\varphi\le\frac{1}{p}-2.$

We again consider system (5). Applying the results of the paper [34] to this system, we get that the quantities n_i determined by theorem 2 of this paper are: $n_i=-1,\ i=\overline{1,r}$. Thus, the appropriate quantity $\omega_{(5)}$ equals:

$$\omega_{(5)}=\frac{1}{2\pi}\Big[\arg\widetilde{B}(-a+0)-\arg\widetilde{B}(a-0)+\arg\widetilde{A}(a-0)-\arg\widetilde{A}(-a+0)+$$

$$+\frac{2}{p}(\arg W'(-a+0)-\arg W'(a-0))\Big]+\frac{2}{p}+n_r-1=\omega_\varphi+1,$$

i.e. $-\frac{1}{q}<\omega_{(5)}\le\frac{1}{p}$.

Again, by the results of the paper [34], system (5) and as a result the system $\{\upsilon_n^-\}_{n\ge1}$ is complete and minimal.

c) $-\frac{1}{q}<\frac{h_0(W)}{2\pi}\le\frac{1}{p};\quad -\frac{1}{p}-2<\omega_\varphi\le\frac{1}{p}-1.$

In this case we consider system (6). Acting as above, we easily see that the quantities $n_i,\ i=\overline{1,r}$ that correspond to this system, equal zero and moreover, $\omega_{(6)}=\omega_\varphi+1$, i.e. $-\frac{1}{q}<\omega_{(6)}\le\frac{1}{p}$. By the results of the paper [34], in this case, system (6) is complete and minimal in $L_p(-a,a)$, consequently, the system $\{\upsilon_n^-\}_{n\ge1}$ is minimal in $L_p(0,a)$.

On the other hand, it follows from theorem 1 that it is complete in $L_p(0,a)$, as well. Thus, in the present case, the system, $\{\upsilon_n^-\}_{n\ge1}$ is complete and minimal.

d) Finally, consider the case when it holds $\frac{1}{p}-1<\frac{h_0(W)}{2\pi}\le\frac{1}{p};\quad -\frac{1}{p}-2<$ $<\omega_\varphi\le-\frac{1}{q}-1$. Represent the system $\{\upsilon_n^+\}_{n\ge2}$ in the form:

$$\widetilde{\upsilon}_n^+(t)\equiv a_1(t)\cdot\varphi^n(t)-b_1(t)\cdot\overline{\varphi}^n(t),\quad n\ge1,$$

where

$$a_1(t)\equiv a(t)\cdot\varphi(t),\ b(t)\equiv-b(t)\cdot\overline{\varphi}(t).$$

Assume that $\arg a_1(t)\equiv\alpha(t)+\arg\varphi(t),\quad \arg b_1(t)\equiv\beta(t)-\arg\varphi(t)+\pi$. By the conditions of the theorem we have: $\arg\varphi(+0)=0,\ \arg\varphi(a-0)=\pi$. If in the place of initial functions $a(t)$ and $b(t)$ we take the functions $a_1(t)$, $b_1(t)$, respectively, we find the quantities $h_0(W)$, ω_φ for this case and denote them by $h_0'(W)$, ω_φ'. So,

$$h_0'(W) = 2\left[\arg b_1(+0) - \arg a_1(+0) + \frac{2}{p}\arg\varphi'(+0)\right] - \frac{2\pi}{p} = h_0(W) + 2\pi;$$

$$\omega_\varphi' = \frac{1}{\pi}\left[\arg a_1(a-0) - \arg b_1(a-0) - \frac{2}{p}\arg\varphi'(a-0)\right] + \frac{3}{p} - 1 = \omega_\varphi + 1.$$

Consequently, in this case, the following inequalities:

$$-\frac{1}{q} < \frac{h_0'(W)}{2\pi} \le \frac{1}{p}; \quad -\frac{1}{q} - 1 < \omega_W' \le -\frac{1}{q}.$$

are fulfilled.

Then by the case c) the system $\left\{\widetilde{\upsilon}_n^+(t)\right\}_{n\ge1}$ and as a result, the system $\{\upsilon_n^+\}_{n\ge2}$ is complete and minimal in $L_p(0,a)$.

Now, let's consider the system

$$\left\{A(t)W^{n+1}(t);\ B(t)\overline{W}^n(t)\right\}_{n\ge1}, \tag{8}$$

where $A(t)$, $B(t)$ are the functions determined by (4). Represent it in the form:

$$\left\{A_0(t)W^n(t);\ B_0(t)\overline{W}^n(t)\right\}_{n\ge0} \tag{9}$$

where $A_0(t) \equiv A(t)\cdot W^2(t)$, $B_0(t) \equiv B(t)\cdot\overline{W}(t)$.

Following the paper [34], we find the quantities n_r and $\omega_{(9)}$ that correspond to system (9). So, denoting $\theta_0(t) \equiv \arg B_0(t) - \arg A_0(t) + \frac{2}{p}\arg W'(t)$ we have that the jump at the point $t=0$ is

$$h_0 = \theta_0(+0) - \theta_0(-0) = 2\left[\beta(+0) - \alpha(+0) + \frac{2}{p}\arg\varphi'(+0)\right] - \frac{2\pi}{p},$$

since $\arg W(+0) = 0$. Since $-\frac{1}{q} < \frac{h_0}{2\pi} + 1 \le \frac{1}{p}$ it follows from the conditions (5) of the paper [34] that $n_r = -1$. Finding the quantity $\omega_{(9)}$ in accordance with formula (6) of the paper [34], we have:

$$\omega_{(9)} = \frac{1}{2\pi}\left[\beta_0(-a+0) - \beta_0(a-0) + \alpha_0(a-0) - \alpha_0(-a+0) + \right.$$

$$\left. + \frac{2}{p}(\arg W'(-a+0) - \arg W'(a-0))\right] + \frac{2}{p} + n_r - 1 =$$

$$= \frac{1}{\pi}\left[\alpha(a-0) - \beta(a-0) - \frac{2}{\pi}\arg\varphi'(a-0)\right] + \frac{3}{2} + 1.$$

Here, we took into account that $\arg\varphi(a) = \pi$. Consequently, $\omega_{(9)} = \omega + 2$ and as a result $-\frac{1}{q} < \omega_{(9)} \le \frac{1}{p}$. Then by the results of the paper [34], system (9) and at the same time system (8) is complete and minimal in $L_p(-a,a)$. Then, it is clear the system $\{\upsilon_n(t)\}_{n\geq 1}$ will be complete, but not minimal in $L_p(-a,a)$. Hence, the completeness of $\{\upsilon_n^-\}_{n\geq 1}$ in $L_p(-0,a)$ follows from lemma 1. Minimality of the system $\left\{A(t)W^n(t);\ B(t)\overline{W}^n(t)\right\}_{n\geq 2}$ in $L_p(-a,a)$ yields the minimality of $\{\upsilon_n^-\}_{n\geq 2}$ in $L_p(0,a)$. Show the minimality of $\{\upsilon_n(t)\}_{n\geq 1}$. Let it be not so. Since it is complete in $L_p(0,a)$, then by lemma 1 of the paper [34], we have that the function $\upsilon_1^-(t)$ belongs to the closure of a linear span of the system $\{\upsilon_n^-\}_{n\geq 2}$ and consequently, $\{\upsilon_n^-\}_{n\geq 2}$ is complete in $L_p(0,a)$. As we have shown, the system is complete and minimal. Then by lemma 1, the system $\{\upsilon_n^+\}_{n\geq 2}$ will be complete and minimal. This contradicts to what has been proved above. Thus, the system $\left\{AW^n;\ B\overline{W}^n\right\}_{n\geq 2}$ is complete and minimal. This contradicts to what has been proved above. Thus, the system $\{\upsilon_n^-\}_{n\geq 1}$ is complete and minimal.

As a result, we get that if the conditions $n_i=0,\ i=\overline{1,r};\ \frac{1}{p}-2<\frac{h_0(W)}{2\pi}\leq \leq\frac{1}{p}-1$ and $-\frac{1}{q}-2<\omega_\varphi\leq-\frac{1}{q}-1$ are fulfilled, then the system $\{\upsilon_n^-\}_{n\geq 1}$ is complete and minimal. General case, i.e. the case $\sum\limits_{i=1}^{r}|n_i|>0$ is reduced to the considered case by multiplication of each term of the system $\{\upsilon_n^-\}_{n\geq 1}$ by the function

$$g(t)\equiv\begin{cases} e^{i2n_0\pi}, & 0\leq t<s_1,\\ e^{i2n_1\pi}, & s_1\leq t<s_2,\\ \quad\dots\dots\dots\dots & \\ e^{i2n_r\pi}, & s_r\leq t<\pi, \end{cases}$$

The cases when the system $\{\upsilon_n^-\}_{n\geq 1}$ is minimal, but not complete for $\omega>-\frac{1}{q}$, is complete but not minimal for $\omega\leq-\frac{1}{q}-2$, similarly proved in the paper [34].

The theorem is proved.

As applications of the obtained results we consider the system

$$\left\{(t+c)^n\sin\left[nt+\gamma(t)\right]\right\}_{n\geq 1}, \tag{10}$$

where $c\in R$ is a real parameter, $\gamma(t)$ is a function continuous on $[0,\pi]$. It is clear that, depending on the parameter c, the curve $\Gamma=\varphi\{[0,\pi]\}$ has different configurations.

1.1) $c>0$. Γ wholly lies on a upper half-plane. Not loosing generality, we'll assume that it holds $\frac{1}{p}-1<\frac{1}{p\pi}arc\ ctg\frac{1}{c}-\frac{\gamma(0)}{\pi}\leq\frac{1}{p}$. Obviously, in this case $n_0=n_r=0$ and $\omega=\frac{2}{\pi}\left[\gamma(\pi)-\frac{1}{p}arc\ ctg\frac{1}{\pi+c}\right]+\frac{1}{p}-1$. By the proved theorem, the system (10) is complete in $L_p(0,\pi)$ only for $\gamma(\pi)\leq\frac{1}{p}arc\ ctg\frac{1}{\pi+c}$, is minimal in $L_p(0,\pi)$ iff $\gamma(\pi)>\frac{1}{p}arc\ ctg\frac{1}{\pi+c}-\pi$. For $\gamma(t)\equiv 0$, we get that the system

$\{(t+c)^n \sin\ nt\}_{n\geq 1}$ is complete and minimal in $L_p(0,\pi)$ for $\forall c>0$.

4. Non-minimality of unitary systems

As we have noted, distinct from theorem 1, there is a condition $0\in(\varphi(a),\ \varphi(0))$ for validity of the statement of theorem 2. It turns that this condition for minimality of the system $\{\upsilon_n^-(t)\}_{n\geq 0}$ in L_p is necessary, namely the following theorem is valid.

Theorem 3. *Let the functions $a(t)$, $b(t)$ and $\varphi(t)$ satisfy conditions (1)-(5).If $0\notin(\varphi(a),\ \varphi(0))$, the system $\{\upsilon_n^-(t)\}_{n\geq 0}$ is not minimal in L_p.*

Proof. The case $0\in ext\ \Gamma$ is proved similar to theorem 2 of the paper [34]. Let $0\in\Gamma$. Without loss of generality, we assume $\varphi(a)=0$. Suppose $\{\upsilon_n^-(t)\}_{n\geq 0}$ is minimal in L_p and $\{h_n\}_{n\geq 0}\subset L_q(0,a)$ is an appropriate biorthogonal system. So,

$$\int\limits_0^a \upsilon_n^-(t)\overline{h_1(t)}dt=\begin{cases}1, & n=0\\ 0, & n>0.\end{cases}$$

Let $h(t)\equiv\begin{cases}\overline{h_1(t)}, & t\in(0,a)\\ \overline{h_1(-t)}, & t\in(-a,0).\end{cases}$

As a result, we have

$$\int\limits_{-a}^a A(t)h(t)W^n(t)dt=\begin{cases}1, & n=0\\ 0, & n>0.\end{cases}$$

By $t=\psi(\varphi)$ we denote a function inverse to $W=W(t)$ on $(-a,a)$. Let

$$\omega(\varphi)=\int\limits_{\varphi_0^-}^{\varphi} A(\psi(\xi))\frac{h(\psi(\xi))}{W'(\psi(\xi))}d\xi,$$

where $\varphi_0^-=W(-a)$, $\varphi_0^+=W(a)$ are two different bound together end points $\Gamma_0=\Gamma\backslash\{\varphi(a)\}$, the integration is conducted along the contour Γ_0 from the point φ_0^- to the point $\varphi\in\Gamma$.

Consequently

$$\int\limits_{\Gamma_0}\omega'(\varphi)\varphi^n d\varphi=\begin{cases}1, & n=0\\ 0, & n>0.\end{cases}\tag{11}$$

Integrating for $n\geq 1$ by parts, we have

$$0=\varphi^n\omega(\varphi)\Big|_{\varphi_0^-}^{\varphi_0^+}-n\int\limits_{\Gamma_0}\omega(\varphi)\varphi^{n-1}d\varphi=0,\ n\geq 1.$$

Considering that $\left[\varphi_0^-\right]^n=\left[\varphi_0^+\right]^n=0,\ \ \forall n\geq 1$, we get

$$\int\limits_{\Gamma_0}\omega(\varphi)\varphi^n d\varphi=0,\ \ \forall n\geq 0\tag{12}$$

Obviously, $\omega(\varphi)$ has bounded variation with respect to the arch Γ_0.

Then by F. and M.Riesz theorem [36] (p. 209), it follows from (12) that the function $\omega(\varphi)$ is absolutely continuous on Γ_0 and $\omega(\varphi_0^-) = \omega(\varphi_0^+)$. On the other hand, for $n = 0$ it follows from (11) that:

$$\omega(\varphi_0^+) - \omega(\varphi_0^-) = 1.$$

The obtained contradiction proves the theorem.

Now, let's consider the case when Γ_φ either has common points with a real axis, or has a unique common point. It is valid

Theorem 4. *Let the functions $a(t)$, $b(t)$ and $\varphi(t)$ satisfy conditions (1)-(5) and $|\operatorname{Im}\varphi(a)| + |\operatorname{Im}\varphi(0)| > 0$. Then the system $\{\upsilon_n^-\}_{n\geq 1}$ is not minimal in L_p, $\forall p \in [1,+\infty)$.*

The proof of this theorem is obvious. So, continue the functions $a(t)$, $b(t)$ and $\varphi(t)$ on a wider segment $[c,d] \supset [0,a]$ so that the extended functions $\widetilde{a}(t)$, $\widetilde{b}(t)$ and $\widetilde{\varphi}(t)$ satisfy conditions (1)-(5), moreover $0 \in \Gamma_{\widetilde{\varphi}}$. Then, by theorem 3, the system $\left\{\widetilde{\upsilon}_n^-(t) \equiv \widetilde{a}(t)\widetilde{\varphi}^n(t) - \widetilde{b}(t)\overline{\varphi}^n(t)\right\}_{n\geq 1}$ is not minimal in $L_p(c,d)$. As a result, we get non-minimality of the system $\{\upsilon_n^-\}_{n\geq 1}$ in $L_p(0,a)$. Since minimality of the system $\{\upsilon_n^-\}_{n\geq 1}$ in $L_p(0,a)$ yields the minimlaity of the system $\left\{\widetilde{\upsilon}_n^-\right\}_{n\geq 1}$ in $L_p(c,d)$.

Remark. When the conditions of theorem 4 are fulfilled, the system $\{\upsilon_n^-\}_{n\geq 0}$ is complete in $L_p(0,a)$.

Indeed, let $\operatorname{Im}\varphi(a) > 0$, $\operatorname{Im}\varphi(0) = 0$. It is easy to note that the completeness of the system $\{A(t)W^n(t)\}_{n\geq 0}$ in $L_p(-a,a)$ yields the completeness of the system $\{\upsilon_n^-\}_{n\geq 0}$ in $L_p(0,a)$ where $A(t)$, $W(t)$ are the functions introduced earlier. It is clear that in this case $\Gamma = W\{[-a,a]\}$ is a Jordan arch, then by the Walsh theorem [19] (p. 56), the system $\{W^n(t)\}_{n\geq 0}$, consequently the system $\{A(t)W^n(t)\}_{n\geq 0}$ is complete in $L_p(-a,a)$. General case is proved in the same way.

Coming back to system (10) and applying theorems 3,4 to it, we get the following possibilities.

1.2. $c \in (-\pi, 0)$. Calculating the quantities according to theorem 1, we have $n_0 =$
$= n_r = 0$ and so $\omega = \dfrac{2}{\pi}\left[\gamma(\pi) - \dfrac{1}{p} arc\ ctg \dfrac{1}{\pi + c}\right] + \dfrac{1}{p} - 1$, where we assume that the inequalities $-\dfrac{1}{p} \leq \dfrac{c}{p\pi} + \dfrac{\gamma(-c)}{\pi} < 1 - \dfrac{1}{p}$ are fulfilled.

As a result we get that in this case system (10) is not minimal in $L_p(0,\pi)$ and complete in $L_p(-c,\pi)$, if $\omega \leq -\dfrac{1}{q}$.

1.3. $c \in (-\infty, -\pi)$ Let it hold $\dfrac{\pi}{p} - \pi < \gamma(\pi) + \dfrac{1}{p} arc\ ctg \dfrac{1}{c+\pi} \leq \dfrac{\pi}{p}$. In this case we find $n_0 = n_r = 0$ and $\omega = -\dfrac{2}{\pi}\left[\dfrac{1}{p} arc\ ctg \dfrac{1}{c} - \gamma(0)\right] + \dfrac{5}{p} - 1$. By theorem 2, system (10) is complete in $L_p(0,\pi)$ only for $\gamma(0) \leq \dfrac{1}{p} arc\ ctg \dfrac{1}{c} - \dfrac{2\pi}{p}$ and minimal iff $\gamma(0) > \dfrac{1}{p} arc\ ctg \dfrac{1}{c} - \dfrac{2\pi}{p} - \pi$.

For the A.G. Kostynchenko system $\left\{e^{ant}\sin\ nt\right\}_{n\geq 1}$ we get the earlier known result, i.e. it is complete and minimal in $L_p(0,\pi)$, $p\in(1,+\infty)$ for any $a\in R$.

Remark 2. Similar to the paper [4] we can get the results with respect to $L_1(0,a)$.

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Bilal T. Bilalov
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 47 20 (off.).

Zafar G. Guseynov
Sumgayit State University.
1, Baku str., 43 ward., AZ5008, Sumgayit, Azerbaijan.
Tel.: (99018) 642 12 63 (off.)

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