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ON UNBOUNDED POLYNOMIAL OPERATOR AND DIFFERENTIAL PENCILS

Abstract

The present work is dedicated to the investigation in Banach space of unbounded polynomial operator pencils that are not of Keldysh form. There has been found sufficien condition for operator coefficients under which in Banach space coercive solvability in the spectral parameter of the considered pencils is proved. Futher, using the known abstract result obtained by S.Y.Yakubov, n-fold completeness of root vectors of a polynomial operator pencil is established. This abstract results are applied to proving coercive solvability and n-fold completeness of eigenfunctions and associated functions of principally boundary value problem for ordinary differential equations with a spectral parameter. Earlie the analogous results were established for the square pencils

In [5], published in 1951, M.V. Keldysh introduced the important notion of n-fold completeness of root vectors and proved a fundamental theorem on n-fold completeness for a polynomial operator pencil with principal part generaled by a single self adjoint operator. Later, these results were extended in [1] , [9] , [10] and [12] .

At present there are a lot of works dedicated to the fold completeness of root vectors for operator pencils. However, all these investigations were carried out for the special class of operator pencils, namely for the pencils that is of Keldysh form or self adjoint pencils in a Hilbert spece. A detailed bibliography may be found, e.g. in the monographs of F.G. Maksudov [7], A.S. Markus [8] and G.V. Radzievsky's article [10].

In contrast to the existed results for the first time the S.Y. Yakubov's and M.K. Balaev's work [17] has proved coercive solvability in the spectral parameter and two-fold completeness of root vectors for square operator pencils in Banach space that are not of Keldysh form.

In the present paper there has been found sufficient conditions for the operator coefficients under which in Banach space coercive solvability in the spectral parameter of polynomial operator, pencils that are not of Keldysh form is proved.

Further, applying the known abstract results obtained by S.Y. Yakubov [13] we establish n-fold competeness of root vectors of a polynomial operator pencil. The got abstract results are applied to the investigation of n-fold completeness of root functions of a principally boundary value problem for ordinary differential equation with a polynomial parameter.

Consider in the Banach spave E the operator pencil

$$L(\lambda) = \lambda^n I + \lambda^{n-1} A_1 + ... + A_n, \tag{0.1}$$

where the A_k, $k = 1 + n$, are given, generally unbounded, operator. Clearly, $D(L(\lambda)) = \bigcap_1^n D(A_k)$ for $\lambda \neq 0$.

The number λ_0 is called an eigenvalue of the pencil $L(\lambda)$ if the equation $L(\lambda_0) u = 0$ has a nontrivial solution $u \in D(L(\lambda_0))$. A vector $u_0 \neq 0$ satisfying the equation $L(\lambda_0)u = 0$ is called an eigenvector of the pencil $L(\lambda)$ corresponding to the eigenvalue λ_0.

The vector $u_1, ..., u_k$ of E that are connected by the relations

$$L(\lambda_0)u_p + \frac{1}{1!}L'(\lambda_0)u_{p-1} + ... + \frac{1}{p!}L^{(p)}(\lambda_0)u_0 = 0, \ p = 0 \div k$$

are called the associated vectors for eigenvector u_0 of the pencil $L(\lambda)$.

The associated vectors and eigenvectors of a pencil are combined under the general name of root vectoras of the pencil.

A point μ of the complex plane is called a regular point of the pencil $L(\lambda)$, if the operator $L(\mu)$ has a bounded inverse $L^{-1}(\mu)$ defined on the whole space.

The complement of the set of regular points in the complex plane is called the spectrum of the pencil $L(\lambda)$.

The spectrum of the pencil $L(\lambda)$ is said to be disctere if:

a) all the points λ that are not eigenvalues of the pencil $L(\lambda)$ are regular points of $L(\lambda)$;

b) the eigenvalues are isolated and have finite algebtaic multiplicity;

c) infinity is the uniqur limit point of the set of eigenvalues of $L(\lambda)$.

Below $L^{-1}(\lambda)$ will be written only at the regular points or the pencil $L(\lambda)$.

Consider in the Banach space E the Cauchy problem

$$L(D_t)u = u^{(n)}(t) + A_1u^{(n-1)}(t) + ... + A_nu(t) = 0, \tag{0.2}$$

$$u^{(k)}(t) = u_{k+1}, \quad k = 0 \div (n-1), \tag{0.3}$$

where u_{k+1} are given vectors in E.

A function $u(t)$ of the form

$$u(t) = \exp(\lambda_0 t)\left(\frac{t^k}{k!}u_0 + \frac{t^{k-1}}{(k-1)!}u_1 + ... + \frac{t}{1!}u_{k-1} + u_k\right) \tag{0.4}$$

is a solution of equation (0.2) if and only if $u_0, u_1, ..., u_k$ is a chain of root vectors corresponding to an eigenvalue λ_0 of the pencil (0.1). A solution of the form (0.4) is called an elementary solution of equation (0.2).

The natural desire to approximate a solution of the problem (0.2), (0.3) by linear combinations of elementary solution (0.4) leads to necessity of approximating the vector $(v_1, ..., v_n)$ by linear combinations of vectors of the form

$$\left(u(0), u'(0), ..., u^{(n-1)}(0)\right). \tag{0.5}$$

A system of vectors (0.5) is called a Keldysh system of the pencil (0.1).

Let E_k, $k = 1 + n$, be Banach space continuously imbedded in E. A system of root vectors of the pencil $L(\lambda)$ is said to be $n-$fold complete in $E_1 \times ... \times E_n$ if the Keldysh system of the pencil $L(\lambda)$ is complete in $E_1 \times ... \times E_n$.

The infimum of numbers η satisfying the setimate

$$\sum_{k=0}^{n} |\lambda|^{n-k-\eta} \left\|L^{-1}(\lambda)\right\|_{B(E,E_k)} \le C, \quad \lambda \in G(a,\varphi), \quad |\lambda| \to \infty$$

where the $G(a,\varphi)$ is an angle in the complex plane with the center in a and angle value φ, is called a defect of coerciveness of the pencil $L(\lambda)$ in the angle $G(a,\varphi)$.

1. Coersive Solvability in the Spectral Parameter of Polynomial Operator Pencils

Consider the polynomial operator pencil

$$L(\lambda)=\lambda^n I+\lambda^{n-1}(A_1+B_1)+...+(A_n+B_n) \tag{1.1}$$

with unbounded operator coefficients A_k, B_k acting in a Banach space E.

Theorem 1. *Assume that the following conditions hold:*

1) The operators $A_k, k=1+n-1$ are closed and have everywhere dense range of definition and bounded inverse.

2) The closed operators $N_k=-A_kA_{k-1}^{-1}$, $k=1\div n$, $A_0\equiv I$ have everywhere dense range of definition in E and at some $\sigma_k\in(0,1]$, $k=1\div n$

$$\|R(\lambda,-N_k)\|\leq C|\lambda|^{-\sigma_k},\ \lambda\in S,\ |\lambda|\to\infty,$$

where $j-p<\delta_{pj}=\sum_{k=p}^{j}\sigma_k\leq n$, $1\leq p\leq j\leq n$; $\sigma_k\leq\delta_{1n}-(n-1)$, S is a set in the complex plane which goes to infinity.

3) For any $\varepsilon>0$ and $u\in D(N_k)\subset D(N_{k+1})$, $k=1\div n-1$,

$$\|N_{k+1}u\|\leq\varepsilon\|N_ku\|^{\sigma_k+\sigma_{k+1}-1}\|u\|^{2-\sigma_k+\sigma_{k+1}}+C(\varepsilon)\|u\|.$$

4) For any $\varepsilon>0$ and $u\in D(A_k)\subset D(B_k)$, $k=1\div n$

$$\|B_ku\|\leq\varepsilon\|A_ku\|^{\sigma_k+\sigma_{k+1}-1}\|u\|^{2-\sigma_k+\sigma_{k+1}}+C(\varepsilon)\|u\|.$$

5) For $n\leq 3$ the operator $N_1=A_1$ commutes the operators N_k, $k=2\div n$.

Then for all $\lambda\in S$, $|\lambda|\to\infty$ the operator

$$L(\lambda)=\lambda^n I+\sum_{k=1}^{n}\lambda^{n-k}(A_k+B_k)$$

is invertible in E and the following estimate takes place:

$$|\lambda|^{\delta_{1n}}\left\|L^{-1}(\lambda)\right\|+|\lambda|^{\delta_{1n}-1}\left\|A_1L^{-1}(\lambda)\right\|+|\lambda|^{\delta_{1n}-2}\left\|A_2L^{-1}(\lambda)\right\|+...$$

$$...+|\lambda|^{\delta_{1n}-n}\left\|A_nL^{-1}(\lambda)\right\|\leq C,\quad\lambda\in S,\quad|\lambda|\to\infty.$$

Proof. We will show at first that for the pencil

$$L_0(\lambda)=(\lambda I+N_n)(\lambda I+N_{n-1})...(\lambda I+N_1)=$$

$$=\lambda^nI+\lambda^{n-1}\sum_{k=1}^{n}N_k+\lambda^{n-2}\sum_{1\leq i_1\leq i_2\leq n}N_{i_1}N_{i_2}+...+N_nN_{n-1}...N_1 \tag{1.2}$$

the following estimate is true:

$$|\lambda|^{\delta_{1n}}\left\|L_0^{-1}(\lambda)\right\|+|\lambda|^{\delta_{1n}-1}\left\|A_1L_0^{-1}(\lambda)\right\|+...$$

$$...+|\lambda|^{\delta_{1n}-n}\left\|A_nL_0^{-1}(\lambda)\right\|\le C,\quad \lambda\in S,\quad |\lambda|\to\infty. \tag{1.3}$$

By virtue of the condition 2) and the representation

$$L_0^{-1}(\lambda)=(\lambda I+N_1)^{-1}(\lambda I+N_2)^{-1}...(\lambda I+N_n)^{-1}=R_1(\lambda)R_2(\lambda)...R_n(\lambda)$$

the first two summand in (1.3) are easily estimated. Indeed, for sufficiently large $|\lambda|$ it follows the inequality:

$$\|A_1R_1(\lambda)R_2(\lambda)...R_n(\lambda)\|\le$$
$$\le\|[I-\lambda R_1(\lambda)]R_2(\lambda)...R_n(\lambda)\|\le C|\lambda|^{1-\delta_{1n}}. \tag{1.4}$$

Since, from that we have

$$A_2L_0^{-1}(\lambda)=A_2A_1^{-1}\left(R_2(\lambda)...R_n(\lambda)-\lambda L_0^{-1}(\lambda)\right)=R_3(\lambda)...R_n(\lambda)-$$
$$-\lambda R_2(\lambda)...R_n(\lambda)-\lambda A_2A_1^{-1}L_0^{-1}(\lambda),$$

then under all $\lambda\in S,\quad |\lambda|\to\infty$

$$\left\|A_2L_0^{-1}(\lambda)\right\|\le|\lambda|^{-\delta_{3n}}+|\lambda|^{\delta_{2n}-1}+|\lambda|\left\|A_2A_1^{-1}\right\|\left\|A_1L_0^{-1}(\lambda)\right\|\le C|\lambda|^{2-\delta_{1n}}.$$

By virtue of the condition 2) the rest summand of (1.3) are estimated analogically. Thus, the estimate (1.3) is proved.

As the perturbed operator pencil is of the form

$$L_1(\lambda)=L(\lambda)-L_0(\lambda)=\sum_{k=1}^{n}\lambda^{n-k}B_k-\sum_{m=1}^{n}\lambda^{n-m}\sum_{i_1>m}N_{i_1}N_{i_2}...N_{i_m}, \tag{1.5}$$

where the latter sum applies to the indices satisfying the inequality $i_1>i_2>...>i_m$, so by virtue of the conditions 2)-4) and estimate (1.3) for all $\lambda\in S,\quad |\lambda|\to\infty$ and any $\varepsilon>0$ we have

$$\left\|\lambda^{n-k}B_kL_0^{-1}(\lambda)\right\|\le\varepsilon|\lambda|^{n-k}\left\|A_kL_0^{-1}(\lambda)\right\|^{\sigma_k+\sigma_{k+1}-1}\left\|L_0^{-1}(\lambda)\right\|^{2-\sigma_k+\sigma_{k+1}}+$$
$$+C(\varepsilon)|\lambda|^{n-k}\left\|L_0^{-1}(\lambda)\right\|\le\varepsilon+C(\varepsilon)|\lambda|^{-\alpha_k},$$

where $\alpha_k=\alpha_k(\sigma_1,...,\sigma_k)>0$.

From (1.4) for any $i_1>1$ we get the following estimate

$$\left\|\lambda^{n-k}N_{i_1}L_0^{-1}(\lambda)\right\|\le\varepsilon|\lambda|^{n-1}\left\|N_1L_0^{-1}(\lambda)\right\|^{\beta_{i_1}}\left\|L_0^{-1}(\lambda)\right\|^{\beta_{i_1}}+$$
$$+C(\varepsilon)|\lambda|^{n-1}\left\|L_0^{-1}(\lambda)\right\|\le\varepsilon+|\lambda|^{-\gamma_i}, \tag{1.6}$$

where β_{i_1},β'_{i_1} and γ_i are some functions of $\sigma_1,...,\sigma_i$.

On the analogy the operator $N_{i_1}N_{i_2}L_0^{-1}(\lambda)$ $(i_1>2)$ is estimated through the operator

$$N_2N_1R_1(\lambda)R_2(\lambda)...R_n(\lambda)=R_3(\lambda)...R_n(\lambda)-\lambda R_2(\lambda)R_3(\lambda)...R_n(\lambda)-$$
$$\lambda N_2R_1(\lambda)R_2(\lambda)...R_n(\lambda).$$

With the help of the estimate (1.6) we come to the inequality

$$|\lambda|^{n-2} \left\| N_{i_1} N_{i_2} L_0^{-1}(\lambda) \right\| \le \varepsilon + C(\varepsilon) |\lambda|^{-\delta_{1n}} .$$

All summands condidted in the operator

$$K(\lambda) = \sum_{m=1}^{n} \lambda^{n-m} \sum_{i_1 > m} N_{i_1} N_{i_2} ... N_{i_m} L_0^{-1}(\lambda)$$

are estimated analogically. So, for $\lambda \in S$, $|\lambda| \to \infty$ we have

$$\left\| L_1(\lambda) L_0^{-1}(\lambda) \right\| \le \left\| \sum_{k=1}^{n} \lambda^{n-k} B_k L_0^{-1}(\lambda) \right\| + \| K(\lambda) \| \le q < 1. \tag{1.7}$$

Then, from (1.4) it follows that under all $\lambda \in S$, $|\lambda| \to \infty$ the operator $L(\lambda)$ is invertible in E and

$$L^{-1}(\lambda) = L_0^{-1}(\lambda) \left[I + L_1(\lambda) L_0^{-1}(\lambda) \right] .$$

Hence, by virtue of (1.3) and (1.7) we have

$$\sum_{k=0}^{n} |\lambda|^{\delta_{1n} - k} \left\| A_k L^{-1}(\lambda) \right\| \le C, \quad A_0 \equiv I.$$

Theorem is proved.

Remark. The assertion of theorem 1 remains to be true if we substitute the condition 2) by the following condition: at some $c > 0$ and $\beta < \beta_0 = \delta_{1n} - (n-1)$ and under any $u \in D(N_k) \cap D(N_{k+1})$, $k = 1 \div n-1$, the following inequalities are true:

$$\| N_{k+1} u \| \le C \| N_k u \|^{\beta} \| u \|^{1-\beta} .$$

2. Multiple Completeness of Root Vectors of Unbounded Polynomial Operator Pencil

Let operator A acts in a Hilbert space with the range of definition $D(A)$. Convert $D(A)$ into Hilbert space $H(A)$ by the norm

$$\| u \|_{N(A)} = \left(\| u \|^2 + \| Au \|^2 \right)^{1/2} ,$$

where $\| \cdot \|$ is a norm in H. Denote by J the embedding operator from one Hilbert space to another.

Now let us give one theorem from [13].

Theorem 2. [13, p.430]. *Let the following conditions be satisfied:*

1) There exist Hilbert spaces H_k, $k = 0 \div 0$, for which the compact imbeddings $H_n \subset H_{n-1} \subset ... \subset H_0 = H$ hold.

2) $\overline{H}_k \big|_{H_{k-1}} = H_{k-1}$, $k = 1 \div n$.

3) $J \quad \sigma_p(H_k, H_{k-1})$, $k = 1 \div n$, for $p > 0$.

4) The operators $A_k, k = 1 \div n$, act boundedly from H_k to H.

5) There exist rays i_k with angles between neighboring rays of at most π/p and an integer m such that

$$\left\|L^{-1}(\lambda)\right\|_{B(H,H_{n-1})} \le c\,|\lambda|^m\,,\ \lambda\quad i_k,\ |\lambda|\to\infty.$$

Then the spectrum of the pencil (0.1) is discrete, and the system root vectors of the pencil (0.1) is $n-$fold compete in $H_n \times ... \times H_1$.

Theorem 3. *Let the following condition be satisfied:*

1) The operators $A_k, k = 1\div n$, are closed in H and have everywhere dense range of definition and bounded inverse; the operator A_n is closed in H.

2) $D(A_n) \subset ... \subset D(A_1)\,,\ \overline{H(A_{k+1})}\Big|_{H(A_k)} = H(A_k)\,,\ k = 1\div n-1$, and at some $p > 0$ $J \in \sigma_p(H(A_1), H)\,, ..., J \in \sigma_p(H(A_{n-1}), H(A_n))$.

The closed operators $-A_kA_{k-1}^{-1}$, $k = 1\div n-1$, $A_0 \equiv I$ have everywhere dense range of definition in H and there exist rays i_k with angles between adjacent rays at most π/p and numbers $\sigma_k \in (0,1]$ such that

$$\left\|R\left(\lambda, -A_kA_{k-1}^{-1}\right)\right\| \le C\,|\lambda|^{-\sigma_k}\,,\ \lambda\in i_k(a)\,,\quad |\lambda|\to\infty,$$

$$j-k < \sigma_k < \sum_{k=p}^{j}\sigma_k \le n;\quad 1\le p\le j\le n;\ \sigma_k \le \delta_{1n} - (n-1)\,.$$

4) For any $\varepsilon > 0$ and $u \in D\left(A_kA_{k-1}^{-1}\right) \subset D\left(A_{k+1}A_k^{-1}\right),\ k = 1\div n-1,$

$$\left\|A_{k+1}A_k^{-1}u\right\| \le \varepsilon\left\|A_kA_{k-1}^{-1}\right\|^{\sigma_k+\sigma_{k+1}-1}\|u\|^{2-\sigma_k+\sigma_{k+1}} + C(\varepsilon)\|u\|\,.$$

5) For any $\varepsilon > 0$ and $u \in D(A_k) \subset D(B_k)\,,\ k = 1\div n-1,$

$$\|B_ku\| \le \varepsilon\,\|A_ku\|^{\sigma_k+\sigma_{k+1}-1}\|u\|^{2-\sigma_k+\sigma_{k+1}} + C(\varepsilon)\|u\|\,.$$

6) For $n \ge 3$ the operator A_1 commutes the operators $A_kA_{k-1}^{-1}$, $k = 1\div n-1$.

Then the spectrum of the pencil (1.1) is discrete and the system of root vectors of the pencil (1.1) is $n-$fold complete in

$$H(A_n)\times H(A_{n-1})\times ...\times H(A_1)\,.$$

Proof. Let us show that theorem 2 can be applied to the pencil (1.1). By virtue of the conditions 1) and 2) of the theorem under $H_k = H(A_k)\,,\ k = 1\div n-1$, $H_0 = H(A_n) \equiv H$ the conditions 1-3 of the theorem 2 are satisfied. by virtue of theorem 1 the condition 4 and 5 of theorem 2 are also satisfied. So all the conditions of theorem 2 are satisfied and consequently the system of root vectors of the pencil (1.1) is $n-$fold complete in

$$H(A_n)\times H(A_{n-1})\times ...\times H(A_1)\,.$$

3. Coercive Problem for Ordinary Differential Equations with a Parameter

3.1. Coerciveness of the problem. Consider the problem with a polynomial spectral parameter

$$L(\lambda) \equiv \lambda^n u(x) + \lambda^{n-1} a_1 u^{(2p_1)}(x) + ... + a_n u^{(2p_1)}(x) +$$

$$+\lambda^{n-1} \sum_{\alpha=0}^{2p_1-1} b_{1\alpha}(x) u^{(\alpha)}(x) + ... + \sum_{\alpha=0}^{2p_n-1} b_{n\alpha}(x) u^{(\alpha)}(x) = f(x) \quad (3.1)$$

and the functional conditions

$$L_\nu u = \alpha_\nu u^{(q_\nu)}(0) + \beta_\nu u^{(q_\nu)}(1) + T_\nu u = 0, \quad (3.2)$$

under $p_1 \geq ... \geq p_n$, where $|\alpha_\nu| + |\beta_\nu| \neq 0$, $\nu = 1 \div 2p_1$, $0 \leq q_\nu \leq q_{\nu+1}$, $q_\nu < q_{\nu+2}$, T_ν is a linear continuous functional in $W_q^{q_\nu}(0,1)$ and under $p_1 < p_2 < 2p_1$, $p_2 \geq p_3 \geq ... \geq p_n$ with the additional conditions

$$L_{2p_1+s} u = L_{n_s} u^{(2p_1)} = a_{n_s} u^{(a_{n_s}+2p_1)}(0) + \beta_{n_s} u^{(a_{n_s}+2p_1)}(1) +$$

$$+T_{n_s} u^{(2p_1)}(\cdot) = 0, \quad (3.3)$$

where $s = 1 \div 2(p_2 - p_1)$, $1 \leq n_s \leq 2p_1$, $k = 1 \div n$.

Theorem 4. *Suppose that the following conditions hold:*

1. $a_k \neq 0$, $k = 1 \div n$, $p_1 \geq 1$, $q_\nu \leq 2p_1 - 1$.

2. Under $p_1 \geq p_2 \geq ... \geq p_n$

$$\theta_1 = \begin{vmatrix} \alpha_1 \omega_1^{q_1} ... \alpha_1 \omega_1^{q_1} \beta_1 \omega_{k+1}^{q_1} ... \beta_1 \omega_{2k}^{q_1} \\ .... \\ \alpha_{2k} \omega_1^{q_{2k}} ... \alpha_{2k} \omega_k^{q_{2k}} \beta_{2k} \omega_{k+1}^{q_{2k}} ... \beta_{2k} \omega_{2k}^{q_{2k}} \end{vmatrix} \neq 0 \quad (3.4)$$

where $\omega_1, ..., \omega_{2k}$ *are roots of the equation* $a_1 \omega^{2k} + 1 = 0$ *number such that* $\operatorname{Re} \omega_j < 0$ *at* $j = 1, .., k$ *and* $\operatorname{Re} \omega_j > 0$ *at* $j = k+1, ..., 2k$.

3. Under $p_1 < p_2 < 2p_1$, $p_2 \geq ... \geq p_n$ *let* $\theta_1 \neq 0$ *and*

$$\theta_2 = \begin{vmatrix} \alpha_{n_1} S_1^{q_{n_1}} & ... \; . \; ... & \beta_{n_1} S_{2(p_2-p_1)}^{q_{n_1}} \\ . & ... \; . \; ... & . \\ \alpha_{n_{2(p_2-p_1)}} S_1^{q_{n_{2(p_2-p_1)}}} & ... \; . \; ... & \beta_{n_{2(p_2-p_1)}} S_{2(p_2-p_1)}^{q_{n_{2(p_2-p_1)}}} \end{vmatrix} \neq 0, \quad (3.5)$$

where $S_1, ..., S_{2(p_2-p_1)}$ *are roots of the equation* $\alpha_2 S^{2(p_2-p_1)} + a_1 = 0$ *such that* $\operatorname{Re} S_j < 0$ *at* $j = 1, ..., p_2 - p_1$, *and* $\operatorname{Re} S_j > 0$ *at* $j = p_2 - p_1 + 1, ..., 2(p_2 - p_1)$.

4. $b_{k\alpha}(x) \in L_q(0,1)$, $k = 1 \div n$, $f(\cdot) \in L_q(0,1)$.

5. At some $\eta \in [1, \infty)$ *functionals* T_ν *are continuous in* $W_\eta^{m_\nu}(0,1)$.

Then for any $\varepsilon > 0$ there is $R_\varepsilon > 0$ such that at all complex λ for which $|\lambda| > R_\varepsilon$ and

$$\lambda \in G(0; p_1\pi - \pi + \arg a_1 + \varepsilon, p_1\pi + \pi + \arg a_1 - \varepsilon)$$

under $p_1 \geq p_2 \geq ... \geq p_n$ and

$$\lambda \in G(0; p_1\pi - \pi + \arg a_1 + \varepsilon, p_1\pi + \pi + \arg a_1 - \varepsilon) \cap$$

$$\cap G\left(0;\left(p_2-p_1\right)\pi-\pi+\arg a_2+\arg a_1+\varepsilon,\left(p_2-p_1\right)\pi+\pi+\arg a_2-\arg a_1-\varepsilon\right)$$

under $p_1 < p_2 < 2p_1$ the problem (3.1) – (3.3) has a unique solution $u\in W_q^{\max(2p_1,2p_2)}(0,1)$ and at those λ for a solution of the problem (3.1) – (3.3) the following estimate is true:

$$|\lambda|^n\|u\|_{L_q(0,1)}+|\lambda|^{n-1}\|u\|_{W_q^{2p_1}(0,1)}+...+\|u\|_{W_q^{2p_n}(0,1)}\leq C\|f\|_{L_q(0,1)}.$$

Proof. To reduce the problem (3.1) – (3.3) to the operator pencil for which we will able to apply results of §1 let us introduce the operators $A_k, B_k,\ k=1\div n$ by the equals

$$D\left(A_k\right)=W_q^{2p_1}\left(\left(0,1\right);L_\nu u|_{\nu=1}^{2p_k}=0\right),\ \ A_k u=a_k u^{2p_k}\left(x\right)+\omega_k u\left(x\right),$$

$$k=1\div n,\ D\left(B_k\right)=D\left(A_k\right),\ \ B_k u=\sum_{\alpha=0}^{2p_k-1}b_{k\alpha}u^{(\alpha)}\left(x\right)-\omega_k u\left(x\right).$$

Then the problem (3.1) – (3.3) is reduced in $L_q(0,1)$ to the equation

$$L\left(\lambda\right)u=\lambda^n u+\lambda^{n-1}\left(A_1+B_1\right)u+...+\left(A_n+B_n\right)u=f. \tag{3.6}$$

From conditions 1,2 and 5 by virtue of theorem 2 [14] it follows that the operators $A_k,\ k=1\div n-1$ at some $\omega_k\in C$ are invertible, i.e. they satisfy the condition 1 of theorem 1. From theorem 2 [14] it also follows that at $\lambda\in S_1=G\left(0;p_1\pi-\pi+\arg a_1+\varepsilon,p_1\pi+\pi+\arg a_1-\varepsilon\right),\ |\lambda|\to\infty$

$$\left\|R\left(\lambda,-A_1\right)\right\|\leq C|\lambda|^{-1} \tag{3.7}$$

i.e. the operator A_1 satisfies in S_1 the condition 2 of the theorem 1. Under $p_1\geq p_2\geq...\geq p_n$ the operator $A_kA_{k-1}^{-1}$ is bounded in $L_q(0,1)$.

So, at all $|\lambda|\to\infty$

$$\left\|R\left(\lambda,-A_kA_{k-1}^{-1}\right)\right\|\leq C|\lambda|^{-1},$$

i.e. the operator $A_kA_{k-1}^{-1}$ in $S_2=C$ satisfies the condition 2 of theorem 1.

Now, consider the case of $p_1<p_2<2p_1$. Since under $2p_1\leq k$ and $u\in W_q^{2p_n}(0,1)$ the following correlation takes place

$$D^kA_1^{-1}u=a_1^{-1}D^{k-2p_1}\left[\left(a_1D^{2p_1}+\omega_1\right)-\omega_1\right]A_1^{-1}u=a_1^{-1}D^{k-2p_1}u-\omega_1D^{k-2p_1}A_1^{-1}u,$$

then for $u\in D\left(A_kA_1^{-1}\right)\subset W_q^{2p_2-2p_1}(0,1)$ we have

$$A_kA_1^{-1}u=a_2a_1^{-1}D^{2p_2-2p_1}u-a_2\omega_1D^{2p_2-2p_1}A_1^{-1}u \tag{3.8}$$

and for $s=1\div(2p_2-2p_1)$

$$L_{2p_1+s}A_1^{-1}u=a_1^{-1}\left(\alpha_{\nu_s}u^{(q_{\nu_s})}\left(0\right)+\beta_{\nu_s}u^{(q_{\nu_s})}\left(1\right)\right)-$$

$$-\omega_1\left(\alpha_{\nu_s}\left[D^{q_{\nu_s}}A_1^{-1}u\right]_{x=0}+\beta_{\nu_s}\left[D^{q_{\nu_s}}A_1^{-1}u\right]_{x=1}\right)+T_{\nu_s}A_1^{-1}u.$$

Obviously, the functional a $L_{2p_1+s}A_1^{-1}$ is of the form

$$\widetilde{L}_s u = a_1 L_{2p+s} A_1^{-1} u = \alpha_{\nu_s} u^{(q_{\nu_s})}(0) + \beta_{\nu_s} u^{(q_{\nu_s})}(1) + \widetilde{T}_s u,$$

where the functional

$$\widetilde{T}_s u = -a_1\omega_1 \left(\alpha_{\nu_s}\left[D^{q_{\nu_s}} A_1^{-1} u\right]_{x=0} + \beta_{\nu_s}\left[D^{q_{\nu_s}} A_1^{-1} u\right]_{x=1}\right) + a_1 T_{\nu_s} A_1^{-1} u,$$

is continuous in $L_q(0,1)$. Thus, the operator $A_2A_1^{-1}$ is defined by the equals

$$D\left(A_2A_1^{-1}\right) = W_q^{2p_2-2p_1}\left((0,1);\widetilde{L}_s u = 0,\ s = 1 \div (2p_2 - 2p_1)\right)$$

and (3.8).

By virtue of the conditions 1,3 and 5 the theorem 2 [4] can be applied to the operator $A_2A_1^{-1}$ from which it follows that at

$$\lambda \in S_2 = G\left(0; (p_2 - p_1)\pi - \pi + \arg a_2 - \arg a_1 + \varepsilon,\right.$$

$$\left.(p_2 - p_1)\pi + \pi + \arg a_2 - \arg a_1 - \varepsilon\right),\ |\lambda| \to \infty$$

$$\left\| R\left(\lambda, -A_2A_1^{-1}\right)\right\| \le C\,|\lambda|^{-1}, \tag{3.9}$$

i.e. the operator $A_2A_1^{-1}$ satisfies in S_2 the condition 2 of theorem 1.

Let us show that the condition 3 of theorem 1 takes place, i.e. $D(A_2) \supset D\left(A_1^2\right)$ and for any $\varepsilon > 0$

$$\|A_2u\| \le \varepsilon\left\|A_1^2u\right\| + C(\varepsilon)\|u\|,\ u \in D\left(A_1^2\right).$$

Indeed, from $u \in D\left(A_1^2\right)$ it follows that $u \in D(A_1)$ and $A_1u \in D(A_1)$. Therefore, $L_\nu u = 0$, $L_\nu A_1 u = a_1 L_\nu u^{(2p_1)} + \omega_1 L_\nu u = 0$, $\nu = 1 \div 2p_1$. Thus, the function $u(x)$ satisfies both the condition (3.2) and (3.3), i.e. $u \in D(A_2)$.

On the other hand, by virtue of the well-known estimate $[3, p.145]$

$$\left\|u^{(k)}\right\|_{L_\infty(0,1)} \le \varepsilon\left\|u^{(n)}\right\|_{L_q(0,1)} + C(\varepsilon)\|u\|_{L_q(0,1)},\ k < n$$

for $u \in D\left(A_1^2\right)$ we have

$$\|A_2u\|_{L_q(0,1)} \le C\left\|u^{(2p_1)}\right\|_{L_q(0,1)} \le \varepsilon\left\|u^{(4p_1)}\right\|_{L_q(0,1)} +$$

$$+C(\varepsilon)\|u\|_{L_q(0,1)} \le \varepsilon\left\|A_1^2u\right\|_{L_q(0,1)} + C(\varepsilon)\|u\|_{L_q(0,1)},$$

Q.E.D.

From condition 4 it follows that

$$\|B_ku\|_{L_q(0,1)} \le \varepsilon\|u\|_{W_q^{2p_k}(0,1)} + C(\varepsilon)\|u\|_{L_q(0,1)} \le$$

$$\le \varepsilon\|A_ku\|_{L_q(0,1)} + C(\varepsilon)\|u\|_{L_q(0,1)},$$

i.e. the operator B_k satisfy the condition 4 of theorem 1. Thus, for the problem (3.6) we have verified all the condtions of theorem 1 from which it follows the assertion of theorem 4.

3.2. Multiple Completeness of Root Functions of the Problem (3.1) – (3.3)
Consider under $p_1 < p_2$ the homogeneous equation

$$L(\lambda)u = \lambda^n u(x) + \lambda^{n-1} a_1 u^{(2p_1)}(x) + ... + a_n u^{(2p_n)}(x) +$$

$$+\lambda^{n-1} \sum_{\alpha=0}^{2p_1-1} b_{1\alpha} u^{(\alpha)} x + ... + \sum_{\alpha=0}^{2p_n-1} b_{n\alpha} u^{(\alpha)} x = 0 \qquad (3.10)$$

with the functional conditions

$$L_\nu u = 0, \quad \nu = 1 \div 2p_1, \qquad (3.11)$$

where L_ν are defined by the equals (3.2) and (3.3).

Theorem 5. *Let the following conditions be satisfied:*

1. $a_k \neq 0,\ p_1 \geq 1,\ p_1 < p_2 < 2p_1,\ p_2 = p_3 = ... = p_n;\ q_v \leq 2p_n - 1,\ q_{\nu_s} \leq 2(p_2 - p_1) - 1;$

2. The determinants (3.4) *and* (3.5) *are not equal to zero;*

3. $b_{k\alpha}(\cdot) \in L_2(0,1),\ l = 1 \div n.$

4. At some $\eta \in [1, \infty)$ *the functionals* T_ν *are continuous in* $W_2^{q_\nu}(0,1)$.

Then the spectrum of the problem (3.10) – (3.11) is discrete and the system of the root vectors of the problem (3.10) – (3.11) is n–fold complete in

$$W_q^{2p_2}((0,1); L_\nu u = 0, \nu = 1 \div 2p_2) \times ... \times W_q^{2p_1}(0,1); L_\nu u = 0,$$

$$\nu = 1 \div 2p_2) \times W_q^{2p_1}((0,1); L_\nu u = 0, \nu = 1 \div 2p_1).$$

Proof. Introducing in $L_2(0,1)$ the operators A_k and B_k by the equals

$$D(A_1) = W_q^{2p_1}((0,1); L_\nu u = 0, \nu = 1 \div 2p_1)$$

$$A_1 u = a_1 u^{(2p_1)}(x) + \omega_1 u(x),\ \omega_1 \in C,$$

$$D(A_k) = W_q^{2p_2}((0,1); L_\nu u = 0, \nu = 1 \div 2p_2)$$

$$A_k u = a_k u^{(2p_1)}(x) + \omega_k u(x),\ \omega_k \in C,\ k = 2 \div n,$$

$$D(B_k) = D(A_k),\ k = 1 \div n,$$

$$B_k u = \sum_{\alpha=0}^{2p_n-1} b_{k\alpha}(x) u^{(\alpha)} x - \omega_k u(x).$$

On the other hand,

$$H(A_1) = W_q^{2p_1}\left((0,1); L_\nu u|_{\nu=1}^{2p_1} = 0\right)$$

and

$$H(A_k) = W_q^{2p_2}\left((0,1); L_\nu u|_{\nu=1}^{2p_2} = 0\right)$$

are subspecies of $W_q^{2p_1}(0,1)$ and $W_q^{2p_2}(0,1)$ correspondingly.

Then at

$$\rho > \max\left\{\frac{1}{2p_1}, \frac{1}{2(p_2 - p_1)}\right\} \quad J \in \sigma_\rho(H(A_1), H)$$

and $J \in \sigma_\rho (H (A_2), H (A_1))$. Thus the condition 2 of theorem 3 has been verified.

Since the estimates (3.7) and (3.9) gold everywhere in the complex plane except of two arbitrary small angles

$$G (0; p_1\pi + \pi + \arg a_1 - \varepsilon,\ p_1\pi + \pi + \arg a_1 - \varepsilon,)$$

and

$$G (0; (p_2 - p_1) \pi + \pi + \arg a_2 - \arg a_1 - \varepsilon,$$
$$(p_2 - p_1) \pi + \pi + \arg a_2 - \arg a_1 - \varepsilon)$$

then the condition 3 of theorem 3 hold completely. Thus the theorem 3 can be applied to the problem (3.12) from which it follows the assertion of theorem 5.

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