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ON AN INVERSE BOUNDARY VALUE PROBLEM FOR A THIRD ORDER PSEUDOHIPERBOLIC EQUATION WITH NOT SELF-ADJOINT BOUNDARY CONDITIONS

Abstract

An inverse boundary value problem is investigated for a third order pseudohiperbolic equation with not self-adjoint boundary conditions. A first, an initial problem is reduced to an equivalent problem for which a theorem on existence and uniqueness is proved. Further, using these facts, the existence and uniqueness of the classic solution of the initial problem is proved.

Let's consider the following problem:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial t}\left(a(t)\frac{\partial^2}{\partial x^2}u(x,t)\right) + F(u,a_0,a_1;t),$$

$$(x,t)\in D_T = \{(x,t): 0\le x\le 1,\ 0\le t\le T\}, \tag{1}$$

$$u(x,0)=\varphi(x),\ \frac{\partial u(x,0)}{\partial t} = \psi(x),\ 0\le x\le 1, \tag{2}$$

$$u(0,t)=0,\ \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x},\ 0\le t\le T, \tag{3}$$

$$u(1,t)=h_0(t),(1/2,t)=h_1(t),\ 0\le t\le T, \tag{4}$$

where

$$F(u,a_0,a_1;t) = a_0(t)u(x,t)+a_1(t)g(x,t)+f(x,t),$$

$a(t)>0,\ g(x,t),\ f(x,t),\varphi(x),\ \psi(x),\ h_\nu(t)\quad(\nu=0,1)$ are given functions, $u(x,t),\ a_0(t),\ a_1(t)$ are desired functions.

Accept the following

Definition. *A three $\{u(x,t), a_0(t), a_1(t)\}$ of the functions $u(x,t)$, $a_0(t)$, $a_1(t)$ satisfying the conditions:*

1) the function $u(x,t)$ is continuous in D_T together with all its derivatives contained in equation (1);

2) the functions $a_\nu(t)\,(\nu=0,1)$ are continuous on $[0,T]$;

3) equation (1) and all the conditions of (2)-(4) are satisfied in the ordinary sense, are said to be a classic solution of the inverse boundary value problem (1)-(4).

The following lemma is valid.

Lemma 1. *Let* $\varphi(x)$, $\varphi'(x)$, $\psi(x)$, $\psi'(x) \in C[0,1]$, $h_\nu(t) \in C^2[0,T]$ $(\nu = 0,1)$, $f(x,t)$, $g(x,t) \in C(D_T)$ *and the agreement conditions*

$$\varphi(1) = h_0(0), \quad \varphi(1/2) = h_1(0), \tag{5}$$

$$\psi(1) = h_0'(0), \quad \psi(1/2) = h_1'(0) \tag{6}$$

be satisfied.

Then, the problem on finding of classic solution to problem (1)-(4) is equivalent to the problem on definition of functions $u(x,t)$, $a_0(t)$, $a_1(t)$ possessing properties 1) and 2) on definition of classic solution of problem (1)-(4) from (1)-(3) and

$$h_0''(t) = \frac{d}{dt}\left(a(t)\,u_{xx}(1,t)\right) + a_0(t)\,h_0(t) + a_1(t)\,g(1,t) + f(1,t)\ (0 \le t \le T), \tag{7}$$

$$h_1''(t) = \frac{d}{dt}\left(a(t)\,u_{xx}(1/2,t)\right) + a_0(t)\,h_1(t) +$$
$$+ a_1(t)\,g(1/2,t) + f(1/2,t)\ (0 \le t \le T). \tag{8}$$

It is known that [2] the sequences

$$X_0(x) = x, ..., X_{2k-1}(x) = x\cos\lambda_k x,\ X_{2k}(x) = \sin\lambda_k x, ..., \tag{9}$$

and

$$Y_0(x) = 2, ..., Y_{2k-1}(x) = 4\cos\lambda_k x,\ Y_{2k}(x) = 4(1-x)\sin\lambda_k x, ... \tag{10}$$

form in $L_2(0,1)$ a biorthogonal system, and system (9) forms a basis in $L_2(0,1)$ where $\lambda_k = 2\pi k$.

For any function $r(x) \in L_2(0,1)$ the following estimation is true:

$$\frac{3}{4}\|r(x)\|^2_{L_2(0,1)} \le \sum_{k=0}^{\infty} r_k^2 \le 16\|r(x)\|^2_{L_2(0,1)}, \tag{11}$$

where

$$r_k = \int_0^1 r(x)\,Y_k(x)\,dx \quad (k = 0,1,...).$$

Under assumptions

$$r(x) \in C^{2i-1}[0,1],\ r^{(2i)}(x) \in L_2(0,1)$$

and

$$r^{(2s)}(0) = 0,\ r^{(2s+1)}(0) = r^{(2s+1)}(1)\ \left(s = \overline{0, i-1};\ i \ge 1\right)$$

validity of the estimations [4]:

$$\sum_{k=1}^{\infty}\left(k^{2i} r_{2k-1}\right)^2 \le \frac{8}{(2\pi)^{4i}}\left\|r^{(2i)}(x)\right\|^2_{L_2(0,1)}, \tag{12}$$

$$\sum_{k=1}^{\infty}\left(k^{2i}r_{2k}\right)^2\leq\frac{8}{(2\pi)^{4i}}\left\|r^{(2i)}(x)(1-x)-2ir^{(2i-1)}(x)\right\|^2_{L_2(0,1)}\quad(i\geq1)\qquad(13)$$

is established.

And under assumptions

$$r(x)\in C^{2i}[0,1],\ r^{(2i+1)}(x)\in L_2(0,1)$$

and

$$r^{(2s)}(0)=0,\ r^{(2s-1)}(0)=r^{2s-1}(1)\quad\left(s=\overline{0,i};\ i\geq1\right)$$

validity of the estimations [4]:

$$\sum_{k=1}^{\infty}\left(k^{2i+1}r_{2k-1}\right)^2\leq\frac{8}{(2\pi)^{2(2i+1)}}\left\|r^{(2i+1)}(x)\right\|^2_{L_2(0,1)},\qquad(14)$$

$$\sum_{k=1}^{\infty}\left(k^{2i+1}r_{2k}\right)^2\leq\frac{8}{(2\pi)^{2(2i+1)}}\left\|r^{(2i+1)}(x)(1-x)-(2i+1)r^{(2i)}(x)\right\|^2_{L_2(0,1)}$$
$$(i\geq1)\qquad(15)$$

is established.

By $B^3_{2,T}$ [3] we denote totality of all functions $u(x,t)$ of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x),$$

considered on D_T, for which all the functions $u_k(t)\in C[0,T]$ and

$$J_T(u)\equiv\left\{\|u_0(t)\|^2_{C[0,T]}+\sum_{k=1}^{\infty}\left(k^3\|u_{2k-1}(t)\|_{C_{[0,T]}}\right)^2+\right.$$
$$\left.+\sum_{k=1}^{\infty}\left(k^3\|u_{2k}(t)\|_{C_{[0,T]}}\right)^2\right\}^{1/2}<+\infty,$$

where $X_k(x)$ $(k=0,1,2,...)$ are determined by relation (8) The norm in this set is determined as follows:

$$\|u\|_{B^3_{2,T}}=J_T(u).$$

It is known that [3] all these spaces are Banach spaces.

By E^3_T we denote a space of vector-functions $z=\{u(x,t),a_0(t),a_1(t)\}$ such that $u(x,t)\in B^3_{2,T}, a_0(t),a_1(t)\in C[0,T]$, supplied with the norm

$$\|z\|_{E^3_T}=\|u(x,t)\|_{B^3_{2,T}}+\|a_0(t)\|_{C[0,T]}+\|a_1(t)\|_{C[0,T]}.$$

Obviously, E^3_T is a Banach space.

Sience system (9) forms a basis in $L_2(0,1)$, and systems (9), (10) form a system of functions biorthogonal in $L_2(0,1)$, we'll look for the first component $u(x,t)$ of the solution $\{u(x,t),a_0(t),a_1(t)\}$ of problem (1)-(3), (7),(8) in the form:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x),\qquad(16)$$

where

$$u_k(t)=\int_0^1 u(x,t)\,Y_k(x)\,dx \quad (k=0,1,...) \tag{17}$$

is a solution of the following problem:

$$\frac{d^2}{dt^2}u_0(t)=F_0(u,a_0,a_1;t), \tag{18}$$

$$\frac{d^2}{dt^2}u_{2k-1}(t)+\lambda_k^2\frac{d}{dt}\left(a(t)\,u_{2k-1}(t)\right)=F_{2k-1}(u,a_0,a_1;t)\ \ (k=1,2,...), \tag{19}$$

$$\frac{d^2}{dt^2}u_{2k}(t)+\lambda_k^2\frac{d}{dt}\left(a(t)\,u_{2k}(t)\right)+2\lambda_k\frac{d}{dt}\left(a(t)\,u_{2k-1}(t)\right)=$$
$$=F_{2k}(u,a_0,a_1;t)\ \ (k=1,2,...), \tag{20}$$

$$u_k(0)=\varphi_k,\ \ u_k'(0)=\psi_k \quad (k=0,1,...), \tag{21}$$

moreover

$$F_k(u,a_0,a_1;t)=f_k(t)+a_0(t)\,u_k(t)+a_1(t)\,g_k(t),$$

$$\varphi_k=\int_0^1\varphi(x)\,Y_k(x)\,dx,\psi_k=\int_0^1\psi(x)\,Y_k(x)\,dx,$$

$$f_k(t)=\int_0^1 f(x,t)\,Y_k(x)\,dx,\ \ g_k(t)=\int_0^1 g(x,t)\,Y_k(x)\,dx \quad (k=0,1,...).$$

Similar [4], from (18)-(21) we find:

$$u_0(t)=\varphi_0+t\psi_0+\int_0^1(t-\tau)\,F_0(u,a_0,a_1;\tau)\,d\tau, \tag{22}$$

$$u_{2k-1}(t)=\left(e^{-\lambda_k^2\int_0^t a(s)ds}+a(0)\,\lambda_k^2\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)\varphi_{2k-1}+$$
$$+\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\cdot\psi_{2k-1}+\int_0^t F_{2k-1}(u,a_0,a_1;\eta)\left(\int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)d\eta, \tag{23}$$

$$u_{2k}(t)=\left(e^{-\lambda_k^2\int_0^t a(s)ds}+a(0)\,\lambda_k^2\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)\varphi_{2k}+$$
$$+\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\cdot\psi_{2k}+\int_0^t F_{2k}(u,a_0,a_1;\eta)\left(\int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)d\eta-$$

$$-2\lambda_k\left\{\int_0^t\left[\left(a'(\eta)-\lambda_k^2 a(\eta)\,e^{-\lambda_k^2\int_0^\eta a(s)ds}+\right.\right.\right.$$

$$\left.\left.+a(0)\,\lambda_k^2\left(a(\eta)+\left(a'(\eta)-\lambda_k^2a^2(\eta)\right)\int_0^\eta e^{-\lambda_k^2\int_\tau^\eta a(s)ds}d\tau\right)\right]\times\right.$$

$$\times\left(\int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)d\eta\cdot\varphi_{2k-1}+$$

$$+\int_0^t\left(a(\eta)+\left(a'(\eta)-\lambda_k^2a^2(\eta)\right)\int_0^\eta e^{-\lambda_k^2\int_\tau^\eta a(s)ds}d\tau\right)\times$$

$$\times\left(\int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)d\eta\cdot\psi_{2k-1}+$$

$$+\int_0^t\left(\int_0^\eta F_{2k-1}(u,a_0,a_1;\xi)\left[a(\eta)+\left(a'(\eta)-\lambda_k^2a^2(\eta)\right)\int_\xi^\eta e^{-\lambda_k^2\int_\tau^\eta a(s)ds}d\tau\right]d\xi\right)\times$$

$$\left.\times\left(\int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)d\eta\right\}. \tag{24}$$

After substitution of the expression $u_k(t)\,(k=0,1,...)$ in (16), for determining the component $u(x,t)$ of the solution of problem (1)-(3), (7),(8) we get

$$u(x,t)=\left\{\varphi_0+t\psi_0+\int_0^t(t-\tau)\,F_0(u,a_0,a_1;\tau)\,d\tau\right\}x+$$

$$+\sum_{k=1}^{\infty}\left\{\left(e^{-\lambda_k^2\int_0^t a(s)ds}+a(0)\,\lambda_k^2\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)\varphi_{2k-1}+\right.$$

$$\left.+\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\cdot\psi_{2k-1}+\int_0^t F_{2k-1}(u,a_0,a_1;\eta)\left(\int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)d\eta\right\}x\times$$

$$\times\cos\lambda_kx+\sum_{k=1}^{\infty}\left\{\left(e^{-\lambda_k^2\int_0^t a(s)ds}+a(0)\,\lambda_k^2\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)\varphi_{2k}+\right.$$

$$+\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\cdot\psi_{2k}+\int_0^t F_{2k}(u,a_0,a_1;\eta)\left(\int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)d\eta-$$

$$-2\lambda_k\left\{\int_0^t\left[\left(a'(\eta)-\lambda_k^2 a(\eta)\right)e^{-\lambda_k^2\int_0^\eta a(s)ds}+\right.\right.$$

$$\left.+a(0)\lambda_k^2\left(a(\eta)+\left(a'(\eta)-\lambda_k^2 a^2(\eta)\right)\int_0^\eta e^{-\lambda_k^2\int_\tau^\eta a(s)ds}d\tau\right)\right]\times$$

$$\times\left(\int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)d\eta\cdot\varphi_{2k-1}+\int_0^t\left(a(\eta)+\left(a'(\eta)-\lambda_k^2 a^2(\eta)\right)\int_0^\eta e^{-\lambda_k^2\int_\tau^\eta a(s)ds}d\tau\right)\times$$

$$\times\left(\int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)d\eta\cdot\psi_{k2-1}+$$

$$+\int_0^t\left(\int_0^\eta F_{2k-1}(u,a_0,a_1;\xi)\left[a(\eta)+\left(a'(\eta)-\lambda_k^2 a^2(\eta)\right)\int_\xi^\eta e^{-\lambda_k^2\int_\tau^\eta a(s)ds}d\tau\right]d\xi\right)\times$$

$$\left.\left.\times\left(\int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)d\eta\right\}\right\}\sin\lambda_k x. \tag{25}$$

Now, in order to get an equation for the component $a_i(t)\,(i=0;1)$ of the solution $\{u(x,t),a_0(t),a_1(t)\}$ of problem (1)-(3), (7), (8) we substitute expression (16) in condition (7) (8)

$$a_0(t)h_0(t)+a_1(t)g(1,t)=h_0''(t)-f(1,t)+\sum_{k=1}^\infty\lambda_k^2\frac{d}{dt}\left(a(t)u_{2k-1}(t)\right), \tag{26}$$

$$a_0(t)h_1(t)+a_1(t)g(1/2,t)=h_1''(t)-f(1/2,t)+$$
$$+\frac{1}{2}\sum_{k=1}^\infty(-1)^k\lambda_k^2\frac{d}{dt}\left(a(t)u_{2k-1}(t)\right). \tag{27}$$

Assume that

$$h(t)=h_0(t)g(1/2,t)-h_1(t)g(1,t)\neq 0\ \ (0\le t\le T). \tag{28}$$

Then, from (26) (27) we have

$$a_0(t)=h^{-1}(t)\left\{h_0''(t)-f(1,t)g(1/2,t)-\left(h_1''(t)-f(1/2,t)\right)g(1,t)+\right.$$
$$\left.+\sum_{k=1}^\infty\nu_{2k-1}(t)\left(g(1/2,t)-\frac{1}{2}(-1)^k g(1,t)\right)\right\}, \tag{29}$$

$$a_1(t)=h^{-1}(t)\left\{h_0(t)\left(h_1(t)-f(1/2,t)\right)-h_1(t)\left(h_0''(t)-f(1,t)\right)+\right.$$
$$\left.+\sum_{k=1}^\infty\nu_{2k-1}(t)\left(h_0(t)(-1)^k-h_1(t)\right)\right\}, \tag{30}$$

where

$$\nu_{2k-1}(t)=\left[-\lambda_k^2\left(a'(t)-\lambda_k^2a^2(t)\right)\left(e^{-\lambda_k^2\int_0^t a(s)ds}+a(0)\lambda_k^2\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)-\right.$$

$$\left.-\lambda_k^4a(t)a(0)\right]\varphi_{2k-1}+\left[-\lambda_k^2\left(a'(t)-\lambda_k^2a^2(t)\right)\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau-\lambda_k^2a(t)\right]\psi_{2k-1}+$$

$$+\int_0^t F_{2k-1}(u,a_0,a_1;\eta)\times$$

$$\times\left(-\lambda_k^2\left(a'(t)-\lambda_k^2a^2(t)\right)\int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau-\lambda_k^2a(t)\right)d\eta. \quad (31)$$

Proceeding from definition of solution of problem (1)-(3), (7),(8), we prove the following lemma.

Lemma 2. *If* $\{u(x,t), a_0(t), a_1(t)\}$ *is any solution of problem (1)-(3), (7), (8), the functions*

$$u_k(t)=\int_0^1 u(x,t)\,Y_k(x)\,dx \quad (k=0,1,...)$$

satisfy system (22)-(24).

From (22)-(24), (31) we have

$$\|u_0(t)\|_{C[0,T]}\le|\varphi_0|+T|\psi_0|+T\sqrt{T}\left(\int_0^T|f_0(\tau)|^2d\tau\right)^{1/2}+$$

$$+T^2\|a_0(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]}+T\sqrt{T}\|a_1(t)\|_{C[0,T]}\left(\int_0^T|g_0(\tau)|^2d\tau\right)^{1/2}, \quad (32)$$

$$\left(\sum_{k=1}^{\infty}\left(k^3\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{1/2}\le\varepsilon_1\left(\sum_{k=1}^{\infty}\left(k^3|\varphi_{2k-1}|\right)^2\right)^{1/2}+$$

$$+\varepsilon_2\left(\sum_{k=1}^{\infty}\left(k|\psi_{2k-1}|\right)^2\right)^{1/2}+\varepsilon_2\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(k|f_{2k-1}(\tau)|\right)^2d\tau\right)^{1/2}+\right.$$

$$+T\|a_0(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(k\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{1/2}+$$

$$\left.+\sqrt{T}\|a_1(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(k|g_{2k-1}(\tau)|\right)^2d\tau\right)^{1/2}\right], \quad (33)$$

$$\left(\sum_{k=1}^{\infty}\left(k^{3}\left\|u_{2k}\left(t\right)\right\|_{C[0,T]}\right)^{2}\right)^{1/2}\leq\varepsilon_{3}\left(\sum_{k=1}^{\infty}\left(k^{3}\left|\varphi_{2k}\right|\right)^{2}\right)^{1/2}+$$

$$+\varepsilon_{2}\left(\sum_{k=1}^{\infty}\left(k\left|\psi_{2k}\right|\right)^{2}\right)^{1/2}+\varepsilon_{2}\left[\sqrt{T}\left(\int_{0}^{T}\sum_{k=1}^{\infty}\left(k\left|f_{2k}\left(\tau\right)\right|\right)^{2}d\tau\right)^{1/2}+\right.$$

$$+T\left\|a_{0}\left(t\right)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(k\left\|u_{2k}\left(t\right)\right\|_{C[0,T]}\right)^{2}\right)^{1/2}+$$

$$+\sqrt{T}\left\|a_{1}\left(t\right)\right\|_{C[0,T]}\left(\int_{0}^{T}\sum_{k=1}^{\infty}\left(k\left|g_{2k}\left(\tau\right)\right|\right)^{2}d\tau\right)^{1/2}+\varepsilon_{4}\left(T\right)\left(\sum_{k=1}^{\infty}\left(k^{3}\left|\varphi_{2k}\right|\right)^{2}\right)^{1/2}+$$

$$+\varepsilon_{5}\left(T\right)\left(\sum_{k=1}^{\infty}\left(k\left|\psi_{2k}\right|\right)^{2}\right)^{1/2}+\varepsilon_{5}\left(T\right)\left\{\sqrt{T}\left(\int_{0}^{T}\sum_{k=1}^{\infty}\left(k\left|f_{2k}\left(\tau\right)\right|\right)^{2}d\tau\right)^{1/2}+\right.$$

$$+T\left\|a_{0}\left(t\right)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(k\left\|u_{2k}\left(t\right)\right\|_{C[0,T]}\right)^{2}\right)^{1/2}+$$

$$\left.+\sqrt{T}\left\|a_{1}\left(t\right)\right\|_{C[0,T]}\left(\int_{0}^{T}\sum_{k=1}^{\infty}\left(k\left|g_{2k}\left(\tau\right)\right|\right)^{2}d\tau\right)^{1/2}\right\},\qquad(34)$$

$$\left(\sum_{k=1}^{\infty}\left(k\left\|\nu_{2k-1}\left(t\right)\right\|_{C[0,T]}\right)^{2}\right)^{1/2}\leq\varepsilon_{6}\left(\sum_{k=1}^{\infty}\left(k^{5}\left|\varphi_{2k-1}\right|\right)^{2}\right)^{1/2}+$$

$$+\varepsilon_{7}\left(\sum_{k=1}^{\infty}\left(k^{3}\left|\psi_{2k-1}\right|\right)^{2}\right)^{1/2}+\varepsilon_{7}\left\{\sqrt{T}\left(\int_{0}^{T}\sum_{k=1}^{\infty}\left(k^{3}\left|f_{2k-1}\left(\tau\right)\right|\right)^{2}\right)^{1/2}+\right.$$

$$+T\left\|a_{0}\left(t\right)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(k^{3}\left\|u_{2k-1}\left(t\right)\right\|_{C[0,T]}\right)^{2}\right)^{1/2}+$$

$$\left.+\sqrt{T}\left\|a_{1}\left(t\right)\right\|_{C[0,T]}\left(\int_{0}^{T}\sum_{k=1}^{\infty}\left(k^{3}\left|g_{2k-1}\left(\tau\right)\right|\right)^{2}d\tau\right)^{1/2}\right\},\qquad(35)$$

where

$$\varepsilon_{1}=\sqrt{5}\left(1+\frac{a\left(0\right)}{m_{0}}\right),\ \varepsilon_{2}=\frac{\sqrt{5}}{4m_{0}\pi^{2}},\varepsilon_{3}=\sqrt{10}\left(1+\frac{a\left(0\right)}{m_{0}}\right),$$

$$\varepsilon_{4}\left(T\right)=\sqrt{10}T\left(M_{1}+M+M\left(M+TM_{1}+\frac{M^{2}}{m_{0}}\right)\right)\frac{1}{m_{0}},$$

$$\varepsilon_{5}\left(T\right)=\frac{\sqrt{10}T}{4\pi^{2}}\left(M+\left(M_{1}+M^{2}\right)\frac{1}{m_{0}}\right)\frac{1}{m_{0}},$$

$$\varepsilon_{6}=16\sqrt{5}\pi^{4}\left(M_{1}+M^{2}\left(1+\frac{M}{m}\right)+M^{2}\right),$$

$$\varepsilon_7 = 4\sqrt{5}\pi^2 \left(\frac{1}{m_0} \left(M_1 + M^2 \right) + M \right),$$

$$m_0 = \min_{0\le t\le T} a(t) \le a(t) \le \max_{0\le t\le T} a(t) \equiv M, \quad M_1 = \left\| a'(t) \right\|_{C[0,T]}.$$

Assume that the data of problem (1)-(3), (7), (8) satisfy the following conditions:

1. $a(t) > 0,\ a(t) \in C^1[0,T]$ for $t \in [0,T]$;
2. $h_i(t) \in C^2[0,T]$ and $h_i(t) \ne 0\ (i=0,1)$ for $t \in [0,T]$;
3. $\varphi(x) \in C^5[0,1],\ \varphi^{(6)}(x) \in L_2(0,1),\ \varphi^{(2s)}(0) = 0,$ $\varphi^{(2s+1)}(0) = \varphi^{(2s+1)}(1)\ (s=0,1,2)$;
4. $\psi(x) \in C^3[0,1],\ \psi^{(4)}(x) \in L_2(0,1),\ \psi^{(2s)}(0) = 0,$ $\psi^{(2s+1)}(0) = \psi^{(2s+1)}(1)\ (s=0,1,)$;
5. $\dfrac{\partial^i f(x,t)}{\partial x^i} \in C(D_T)\quad (i=0,1,2,3),\ \dfrac{\partial^4 f(x,t)}{\partial x^4} \in L_2(D_T),$ $\dfrac{\partial^{2s} f(0,t)}{\partial x^{2s}} = 0,\ \dfrac{\partial^{2s+1} f(0,t)}{\partial x^{2s+1}} = \dfrac{\partial^{2s+1} f(1,t)}{\partial x^{2s+1}}\ (s=0,1)$;
6. $\dfrac{\partial^i g(x,t)}{\partial x^i} \in C(D_T)\quad (i=0,1,2,3),\ \dfrac{\partial^4 g(x,t)}{\partial x^4} \in L_2(D_T),$ $\dfrac{\partial^{2s} g(0,t)}{\partial x^{2s}} = 0,\ \dfrac{\partial^{2s+1} g(0,t)}{\partial x^{2s+1}} = \dfrac{\partial^{2s+1} g(1,t)}{\partial x^{2s+1}}\ (s=0,1)$;
7. $h(t) = h_0(t)\, g(1/2,t) - h_1(t)\, g(1,t) \ne 0$ for $t \in [0,T]$.

Then, from (32)-(35) we have

$$\|u_0(t)\|_{C[0,T]} \le 2\|\varphi(x)\|_{L_2(0,1)} + 2T\|\psi(x)\|_{L_2(0,1)} + 2T\sqrt{T}\|f(x,t)\|_{L_2(D_T)} +$$
$$+T^2\|a_0(t)\|_{C[0,T]}\|u(x,t)\|_{B^2_{2,T}} + 2T\sqrt{T}\|a_1(t)\|_{C[0,T]}\|g(x,t)\|_{L_2(D_T)}, \qquad (36)$$

$$\left(\sum_{k=1}^{\infty} \left(k^3 \|u_{2k-1}(t)\|_{C[0,T]} \right)^2 \right)^{1/2} \le$$
$$\le \varepsilon_1 \frac{\sqrt{2}}{4\pi^3} \left\| \varphi'''(x) \right\|_{L_2(0,1)} + \varepsilon_2 \frac{\sqrt{2}}{\pi} \left\| \psi'(x) \right\|_{L_2(0,1)} +$$
$$+\varepsilon_2 \left[\frac{\sqrt{2T}}{\pi} \|f_x(x,t)\|_{L_2(D_T)} + T\|a_0(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + \right.$$
$$\left. + \frac{\sqrt{2T}}{\pi} \|a_1(t)\|_{C[0,T]}\|g_x(x,t)\|_{L_2(D_T)} \right], \qquad (37)$$

$$\left(\sum_{k=1}^{\infty} \left(k^3 \|u_{2k}(t)\|_{C[0,T]} \right)^2 \right)^{1/2} \le \varepsilon_3 \frac{\sqrt{2}}{4\pi^3} \left\| \varphi'''(x) \right\|_{L_2(0,1)} + \varepsilon_2 \frac{\sqrt{2}}{\pi} \left\| \psi'(x) \right\|_{L_2(0,1)} +$$
$$+\varepsilon_2 \left[\frac{\sqrt{2T}}{\pi} \|f_x(x,t)\|_{L_2(D_T)} + T\|a_0(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + \right.$$
$$\left. + \frac{\sqrt{2T}}{\pi} \|a_1(t)\|_{C[0,T]}\|g_x(x,t)\|_{L_2(D_T)} \right] +$$
$$+\varepsilon_4(T) \frac{\sqrt{2}}{4\pi^3} \left\| \varphi'''(x)(1-x) - 3\varphi''(x) \right\|_{L_2(0,1)} +$$

$$+\varepsilon_5(T)\frac{\sqrt{2}}{\pi}\left\|\psi'(x)(1-x)-\psi(x)\right\|_{L_2(0,1)}+$$

$$+\varepsilon_5(T)\left\{\frac{\sqrt{2T}}{\pi}\left\|f_x(x,t)(1-x)-f(x,t)\right\|_{L_2(D_T)}+\right.$$

$$+T\left\|a_0(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^3_{2,T}}+$$

$$\left.+\frac{\sqrt{2T}}{\pi}\left\|a_1(t)\right\|_{C[0,T]}\left\|g_x(x,t)(1-x)-g(x,t)\right\|_{L_2(D_T)}\right\}, \qquad (38)$$

$$\left(\sum_{k=1}^{\infty}\left(k\left\|\nu_{2k-1}(t)\right\|_{C[0,T]}\right)^2\right)^{1/2}\leq\varepsilon_6\frac{\sqrt{2}}{16\pi^5}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\varepsilon_7\frac{\sqrt{2}}{4\pi^3}\left\|\psi'''(x)\right\|_{L_2(0,1)}+$$

$$+\varepsilon_7\left\{\frac{\sqrt{2T}}{4\pi^3}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}+T\left\|a_0(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^3_{2,T}}+\right.$$

$$\left.+\frac{\sqrt{2T}}{4\pi^3}\left\|a_1(t)\right\|_{C[0,T]}\left\|g_{xxx}(x,t)\right\|_{L_2(D_T)}\right\}. \qquad (39)$$

Further, from (36)-(38) we find

$$\left\|u(x,t)\right\|_{B^3_{2,T}}\leq A_1(T)+B_1(T)\times$$

$$\times\left(\left\|a_0(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^2_{2,T}}+\left\|a_1(t)\right\|_{C[0,T]}\right), \qquad (40)$$

where

$$A_1(T)=2\left\|\varphi(x)\right\|_{L_2(0,1)}+2T\left\|\psi(x)\right\|_{L_2(0,1)}+2T\sqrt{T}\left\|f(x,t)\right\|_{L_2(D_T)}+$$

$$+\varepsilon_1\frac{\sqrt{2}}{4\pi^3}\left\|\varphi'''(x)\right\|_{L_2(0,1)}+\varepsilon_2\frac{\sqrt{2}}{\pi}\left\|\psi'(x)\right\|_{L_2(0,1)}+\varepsilon_2\frac{\sqrt{2T}}{\pi}\left\|f_x(x,t)\right\|_{L_2(D_T)}+$$

$$+\varepsilon_3\frac{\sqrt{2}}{4\pi^3}\left\|\varphi'''(x)\right\|_{L_2(0,1)}+\varepsilon_2\frac{\sqrt{2}}{\pi}\left\|\psi'(x)\right\|_{L_2(0,1)}+\varepsilon_2\frac{\sqrt{2T}}{\pi}\left\|f_x(x,t)\right\|_{L_2(D_T)}+$$

$$+\varepsilon_4(T)\frac{\sqrt{2}}{4\pi^3}\left\|\varphi'''(x)(1-x)-3\varphi''(x)\right\|_{L_2(0,1)}+$$

$$+\varepsilon_5(T)\frac{\sqrt{2}}{\pi}\left\|\psi'(x)(1-x)-\psi(x)\right\|_{L_2(0,1)}+$$

$$+\varepsilon_5(T)\frac{\sqrt{2T}}{\pi}\left\|f_x(x,t)(1-x)-f(x,t)\right\|_{L_2(D_T)},$$

$$B_1(T)=T^2+2T\sqrt{T}\left\|g(x,t)\right\|_{L_2(D_T)}+\varepsilon_2\left(T+\frac{\sqrt{2T}}{\pi}\left\|g_x(x,t)\right\|_{L_2(D_T)}\right)+$$

$$+\varepsilon_2\left(T+\frac{\sqrt{2T}}{\pi}\left\|g_x(x,t)\right\|_{L_2(D_T)}\right)+$$

$$+\varepsilon_5(T)\left(T+\frac{\sqrt{2T}}{\pi}\left\|g_x(x,t)(1-x)-g(x,t)\right\|_{L_2(D_T)}\right).$$

Further, from (29), (30) allowing for (39) we find

$$\|a_0(t)\|_{C[0,T]}\le\left\|h^{-1}(t)\right\|_{C[0,T]}\times$$

$$\times\left\{\left\|h_0''(t)-f(1,t)\,g(1/2,t)-\left(h_1''(1)-f(1/2,t)\right)g(1,t)\right\|_{C[0,T]}+\right.$$

$$+\left(\|g(1/2,t)\|_{C[0,T]}+\frac{1}{2}\|g(1,t)\|_{C[0,T]}\right)\frac{\pi}{\sqrt{6}}\times$$

$$\times\left\{\varepsilon_6\frac{\sqrt{2}}{16\pi^5}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\varepsilon_7\frac{\sqrt{2}}{4\pi^3}\left\|\psi'''(x)\right\|_{L_2(0,1)}+\right.$$

$$+\varepsilon_7\left\{\frac{\sqrt{2T}}{4\pi^3}\|f_{xxx}(x,t)\|_{L_2(D_T)}+\right.$$

$$\left.\left.\left.+T\|a_0(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}+\frac{\sqrt{2T}}{4\pi^3}\|a_1(t)\|_{C[0,T]}\|g_{xxx}(x,t)\|_{L_2(D_T)}\right\}\right\}\right\},$$

$$\|a_1(t)\|_{C[0,T]}\le\left\|h^{-1}(t)\right\|_{C[0,T]}\times$$

$$\times\left\{\left\|h_0(t)\left(h_1(t)-f(1/2,t)\right)-h_1(t)\left(h_0''(t)-f(1/t)\right)\right\|_{C[0,T]}+\right.$$

$$+\left(\|h_0(t)\|_{C[0,T]}+\|h_1(t)\|_{C[0,T]}\right)\frac{\pi}{\sqrt{6}}\times$$

$$\times\left\{\varepsilon_6\frac{\sqrt{2}}{16\pi^5}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\varepsilon_7\frac{\sqrt{2}}{4\pi^3}\left\|\psi'''(x)\right\|_{L_2(0,1)}+\right.$$

$$+\varepsilon_7\left\{\frac{\sqrt{2T}}{4\pi^3}\|f_{xxx}(x,t)\|_{L_2(D_T)}+\right.$$

$$\left.\left.\left.+T\|a_0(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}+\frac{\sqrt{2T}}{4\pi^3}\|a_1(t)\|_{C[0,T]}\|g_{xxx}(x,t)\|_{L_2(D_T)}\right\}\right\}\right\}.$$

Hence we have :

$$\|a_0(t)\|_{C[0,T]}\le A_2(T)+B_2(T)\left(\|a_0(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}+\|a_1(t)\|_{C[0,T]}\right),\quad(41)$$

$$\|a_1(t)\|_{C[0,T]}\le$$

$$\le A_3(T)+B_3(T)\left(\|a_0(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}+\|a_1(t)\|_{C[0,T]}\right),\quad(42)$$

where

$$A_2(T)=\left\|h^{-1}(t)\right\|_{C[0,T]}\times$$

$$\times\left\{\left\|h_0''(t)-f(1,t)\,g(1/2,t)-\left(h_1''(t)-f(1/2,t)\right)g(1,t)\right\|_{C[0,T]}+\right.$$

$$+\frac{\pi}{\sqrt{6}}\left(\|g(1/2,t)\|_{C[0,T]}+\frac{1}{2}\|g(1,t)\|_{C[0,T]}\right)\times$$

$$\times\left\{\varepsilon_6\frac{\sqrt{2}}{16\pi^5}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\varepsilon_7\frac{\sqrt{2}}{4\pi^3}\left\|\psi'''(x)\right\|_{L_2(0,1)}+\right.$$

$$\left.+\varepsilon_7\frac{\sqrt{2T}}{4\pi^3}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}\right\},$$

$$B_2(T)=\left\|h^{-1}(t)\right\|_{C[0,T]}\frac{\pi}{\sqrt{6}}\left(\|g(1/2,t)\|_{C[0,T]}+\frac{1}{2}\|g(1,t)\|_{C[0,T]}\right)\times$$

$$\times\left(T+\frac{\sqrt{2T}}{4\pi^3}\left\|g_{xxx}(x,t)\right\|_{L_2(D_T)}\right),$$

$$A_3(T)=\left\|h^{-1}(t)\right\|_{C[0,T]}\times$$

$$\times\left\{\left\|h_0''(t)-f(1,t)\,g(1/2,t)-\left(h_1''(t)-f(1/2,t)\right)g(1,t)\right\|_{C[0,T]}+\right.$$

$$+\frac{\pi}{\sqrt{6}}\left(\|h_0(t)\|_{C[0,T]}+\|h_1(t)\|_{C[0,T]}\right)\times$$

$$\times\left\{\varepsilon_6\frac{\sqrt{2}}{16\pi^5}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\varepsilon_7\frac{\sqrt{2}}{4\pi^3}\left\|\psi'''(x)\right\|_{L_2(0,1)}+\right.$$

$$\left.+\varepsilon_7\frac{\sqrt{2T}}{4\pi^3}\left\|f_{xxx}(,tx)\right\|_{L_2(D_T)}\right\},$$

$$B_3(T)=\left\|h^{-1}(t)\right\|_{C[0,T]}\frac{\pi}{\sqrt{6}}\left(\|h_0(t)\|_{C[0,T]}+\|h_1(t)\|_{C[0,T]}\right)\times$$

$$\times\left(T+\frac{\sqrt{2T}}{4\pi^3}\left\|g_{xxx}(x,t)\right\|_{L_2(D_T)}\right).$$

From inequality (40)-(42) we conclude:

$$\|u(x,t)\|_{B^3_{2,T}}+\|a_0(t)\|_{C[0,T]}+\|a_1(t)\|_{C[0,T]}\le$$

$$\le A(T)+B(T)\left(\|a_0(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}+\|a_1(t)\|_{C[0,T]}\right), \tag{43}$$

where

$$A(T)=A_1(T)+A_2(T)+A_3(T),$$

$$B(T)=B_1(T)+B_2(T)+B_3(T).$$

So, we proved the following theorem

Theorem 1. *Let conditions 1-7 be satisfied and*

$$B(T)(A(T)+2)(A(T)+3)<1. \tag{44}$$

Then problem (1)-(3), (7), (8) has a unique solution in the ball

$$K = K_R \left(\|z\|_{E_T^3} \le R = A(T) + 2 \right) \text{ from } E_T^3.$$

Proof. In the space E_T^3 consider the equation

$$z = \Phi z, \tag{45}$$

where $z = \{u, a_0, a_1\}$, the components $\Phi_i(u, a_0, a_1)$ $(i = 1, 2, 3)$ of the operator $\Phi(u, a)$ are determined by the right hand sides of equations (25), (29), (30), respectively.

Consider the operator $\Phi(u, a_0, a_1)$ in the ball $K = K_R \left(\|z\|_{E_T^3} \le R = A(T) + 2 \right)$ from E_T^3.

Similar to (43) we get that the following estimations are valid for any $z, z_1, z_2 \in K_R$.

$$\|\Phi z\|_{E_T^3} \le A(T) + B(T) \left(\|u(x,t)\|_{B_{2,T}^3} \|a_0(t)\|_{C[0,T]} + \|a_1(t)\|_{C[0,T]} \right), \tag{46}$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^3} \le B(T)(R+1) \|z_1 - z_2\|_{E_T^3}. \tag{47}$$

Then, allowing for (44), it follows from inequalities (46), (47) that the operator Φ acts on the ball $K = K_R$ and is contractive. Therefore, in the ball, the operator Φ has a unique fixed point $\{u, a_0, a_1\}$ and this point is a solution of equation (45).

The function $u(x,t)$ as an element of the space $B_{2,T}^3$ is continuous and has continuous derivatives $u_x(x,t), u_{xx}(x,t)$.

We can show that $u_t(x,t)$, $u_{tx}(x,t), u_{txx}(x,t)$, $u_{tt}(x,t)$, $u_{ttx}(x,t)$ are continuous on D_T, equation (1), conditions (2), (3), (7) are satisfied in the ordinary sense. So, $\{u(x,t),\ a_0(t),\ a_1(t)\}$ is a solution of problem (1)-(3), (7), (8) and by lemma 2 it is unique. The theorem is proved. By means of lemma 1 we prove the following.

Theorem 2 *Let all the conditions of theorem 1 and agreement conditions (5) and (6) be satisfied.*

Then problem (1)-(4) has a has a ubique classic solution in the ball $K = K_R \left(\|z\|_{E_T^3} \le R = A(T) + 2 \right)$ from E_T^3.

Reference

[1]. Namazov G.K. *Inverse problems of theory of mathematical physics equations.* GU, 1984, 128p. (Russian).

[2]. Ionkin N.I. *Solution of a boundary problem of the heat conductivity with non-classic boundary condition.* DU, 1977, vol.13, No 2, pp.294-304 (Russian).

[3]. Khudaverdiyev K.I., Ismailov A.I. *Investiqation of a classic solution of a not self-adjoint one-dimensional inverse boundary value problem for a class of semi- linear differential equations of third order.* Copywright. Dep.in Az NIINTI, 03.07.1998, 1998, No 2566-Az 98, 110 p. (Russian)

[4]. Guliyeva A.M. *On solvability of an inverse boundary value problem for a third order pseudohyperbolic equation with not self-adjoint boundary conditions.* Vestnik Bakinskogo Universiteta. Ser. Fiz. Math. Nauk 2008, No 1, pp. 75-83 (Russian).

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