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ON BASICITY OF CLOSE SYSTEMS

Abstract

In the paper, the notion of K_p-closeness is introduced and a theorem on basicity of K_p-close systems is proved. The obtained results generalize the classic results of N.K. Bari and T. Kato.

Paley-Wiener type theorems [1] play an important role in establishing basicity in this or other sense of systems in Banach spaces. Concerning history of this issue we can consider the monographs of I.Zinger [2,3]. In the Hilbert case, there are different variants of these theorems. Essential result in this direction belongs to N.K. Bari [4]. Namely, let H be some Hilbert space with the norm $\|\cdot\|$. The systems $\{\varphi_n\}_{n\in N}$; $\{\psi_n\}_{n\in N} \subset H$ are said to be quadratically close if

$$\sum_{n=1}^{\infty} \|\varphi_n - \psi_n\|^2 < +\infty.$$

The system $\{\varphi_n\}_{n\in N} \subset H$ is called ω-linearly in dependent in H, if it follows from $\sum\limits_{n=1}^{\infty} \lambda_n \varphi_n = 0$ in H that $\lambda_n = 0$, $\forall n \in N$.

The Bari theorem. *Let $\{e_n\}_{n\in N} \subset H$ be an orthormed (on-basis) in H and $\{\varphi_n\}_{n\in N} \subset H$ be a system quadratically close to it. If $\{\varphi_n\}_{n\in N}$ is ω-linearly independent in H, it forms a Riesz basis in H, i.e. a basis izomorphic to $\{e_n\}_{n\in N}$.*

Another kind of this theorem is in Kato's monograph [5].

Theorem K. *Let $\{\varphi_n\}_{n\in N}$ be an on-basis in H. Then the sequence $\{\psi_n\}_{n\in N} \subset H$ of non-zero vectors is a basis in H if*

$$\sum_{n=1}^{\infty} \left(\|\psi_n - \varphi_n\|^2 - \frac{|(\psi_n - \varphi_n, \psi_n)|^2}{\|\psi_n\|^2} \right) < 1.$$

In the present note, one generalization of this theorem is given.

1. K-quadratically closeness. Accept the following

Definition 1. *The systems $\{\varphi_n\}_{n\in N}$; $\{\psi_n\}_{n\in N} \subset B$ are said to be K-quadratically close if*

$$\sum_{n=1}^{\infty} \left(\|\psi_n - \varphi_n\|^2 - \frac{|(\psi_n - \varphi_n, \psi_n)|^2}{\|\psi_n\|^2} \right) < +\infty.$$

The following theorem is valid

Theorem 1. *Let $\{\varphi_n\}_{n\in N}$ be an on-basis in H and $\{\psi_n\}_{n\in N} \subset H$ be a system K-quadratically close to it. Then the following statements are equivalent:*

a) $\{\psi_n\}_{n\in N}$ is ω-linearly independent in H;

b) $\{\psi_n\}_{n\in N}$ is minimal in H;

c) $\{\psi_n\}_{n\in N}$ is complete in H;

d) $\{\psi_n\}_{n\in N}$ forms a basis in H;

Before to prove this theorem we cite an easily proved lemma.

Lemma 1. *Let $\{x_n\}_{n\in N}$ be a basis in the Banach space X and $F : X \to X$ be a Fredholm operator, moreover $y_n = Fx_n$, $\forall n \in N$. Then the following statements are equavalent.*

1) $\{y_n\}_{n\in N}$ is ω-linearly independent;

2) $\{y_n\}_{n\in N}$ is minimal;

3) $\{y_n\}_{n\in N}$ is complete;

4) $\{y_n\}_{n\in N}$ forms a basis isomorphic to $\{x_n\}_{n\in N}$ in X.

Proof. Let 1) hold. Take $\forall x \in KerF$ and expand it in basis: $x = \sum_{n=1}^{\infty} a_n x_n \Longrightarrow$ $Fx = \sum_{n=1}^{\infty} a_n y_n = 0$. It follows from 1) that $a_n = 0, \forall n \in N$ i.e. $x = 0$. Thus, $KerF = \{0\}$ and as a result F is invertible. By that $\{\psi_n\}_{n\in N}$ forms a basis isomorphic to $\{\varphi_n\}_{n\in N}$ in X. Earlier it was shown in [6] that 4) follows form assumptions 2), 3).

Implication 4) $\Longrightarrow$ 1),2),3) is obvious.

The lemma is proved.

This lemma directly yields

Lemma 2. *Let $\{x_n\}_{n\in N}$ be a basis in X and $card\{n : y_n \neq x_n\} < +\infty$. Then the statement of lemma 1 is valid for the system $\{y_n\}_{n\in N}$.*

Indeed, denote by $\{x_n^*\}_{n\in N} \subset X^*$ a system conjugated to $\{x_n\}_{n\in N}$ and consider an operator $T : X \to X$ determined by the expression:

$$Tx = \sum_{n:y_n\neq x_n} x_n^*(x)(y_n - x_n), \ \forall x \in X.$$

It is obvious that T is a finite-dimensional operator and so $\Phi = I + T$ is of Fredholm property, where I is an identity operator. The furhter one follows from the relation $\Phi x_n = y_n$, $\forall n \in N$, and from lemma 1.

Now prove the theorem. Let the sysem $\{\psi_n\}_{n\in N}$ be K-quadratically close to on-basis $\{\varphi_n\}_{n\in N}$, i.e. the series converges

$$\sum_{n=1}^{\infty}\left(\|\psi_n - \varphi_n\|^2 - \frac{|(\psi_n - \varphi_n, \psi_n)|^2}{\|\psi_n\|^2}\right).$$

It is clear that

$$\exists n_0 \in N : \sum_{n=n_0}^{\infty}\left(\|\psi_n - \varphi_n\|^2 - \frac{|(\psi_n - \varphi_n, \psi_n)|^2}{\|\psi_n\|^2}\right) < 1.$$

Make up a new system $\{\widetilde{\psi}_n\}_{n\in N}$:

$$\widetilde{\psi}_n = \begin{cases} \psi_n \ , & n \geq n_0, \\ \varphi_n \ , & n < n_0. \end{cases}$$

Obviously,

$$\sum_{n=1}^{\infty}\left(\left\|\widetilde{\psi}_n - \varphi_n\right\|^2 - \frac{\left|\left(\widetilde{\psi}_n - \varphi_n, \widetilde{\psi}_n\right)\right|^2}{\left\|\widetilde{\psi}_n\right\|^2}\right) < 1.$$

Then by the theorem K, the system $\{\widetilde{\psi}_n\}_{n\in N}$ forms a Riesz basis in H. The further one follows directly from lemma 2.

The theorem is proved.

Really, these results may be generalized to the case of Riesz basis, i.e. it is valid

Theorem 2. *Let $\{\varphi_n\}_{n\in N}$ be a Riesz basis in H and $\{\psi_n\}_{n\in N} \subset H$ be a system K-quadratically close to it. Then theorem 1 is valid for $\{\psi_n\}_{n\in N}$.*

This theorem is proved as theorem 1.

2. K_p-closeness. Basicity. At first we prove K_p-closeness.

Definition 2. *The systems $\{\varphi_n\}_{n\in N}; \{\psi_n\}_{n\in N} \subset H$ are called K_p-close if*

$$\sum_{n=1}^{\infty} r_n^{p/2} < +\infty, \quad \text{where} \quad r_n = \|\varphi_n - \psi_n\|^2 - \frac{|(\varphi_n - \psi_n, \psi_n)|^2}{\|\psi_n\|^2}.$$

We'll need the following notion from the paper [7].

Definition 3. *The basis $\{x_n\}_{n\in N}$ in H is called a q-basis, if for $\forall x \in H$: $\{x_n^*(x)\}_{n\in N} \in l_q$, where $\{x_n^*\}_{n\in N} \subset H$ is a system conjugated to $\{x_n\}_{n\in N}$.*

The proof scheme of previous theorems will be suitable in the following case.

Theorem 3. *Let $\{\varphi_n\}_{n\in N}$ be a q-basis in H and $\{\psi_n\}_{n\in N} \subset H$ be a system K_p-close to it. Then the following statements are equivalent.*

1) $\{\psi_n\}_{n\in N}$ is complete;

2) $\{\psi_n\}_{n\in N}$ is minimal;

3) $\{\psi_n\}_{n\in N}$ is ω-linearly independent;

4) $\{\psi_n\}_{n\in N}$ is a basis in H.

Indeed, following [5], we have: the system $\{\psi_n\}_{n\in N}$ forms a basis in H iff the system $\{\rho_n\psi_n : \rho_n \neq 0\}_{n\in N}$ forms a basis in H. We choose ρ_n from the relation: $\|\rho_n\psi_n - \varphi_n\| = \min\limits_{\rho} \|\rho\psi_n - \varphi_n\|$; i.e. $\|\rho_n\psi_n - \varphi_n\|^2 = \min\limits_{\rho} \|\rho\psi_n - \varphi_n\|^2 = I_n$.

Having denoted by P_n an orthogonal project or on a subspace generated by the element ψ_n, we have ($I : H \to H$ is an identity operator)

$$I_n = \|(I - P_n)\,\varphi_n\|^2 = \|(I - P_n)\,(\varphi_n - \psi_n)\|^2 =$$

$$= \|\varphi_n - \psi_n\|^2 - \|P_n\,(\varphi_n - \psi_n)\|^2 = \|\varphi_n - \psi_n\|^2 - \frac{|(\varphi_n - \psi_n, \psi_n)|^2}{\|\psi_n\|^2} = r_n \ .$$

Consequently,

$$\sum_n \|\rho_n\psi_n - \varphi_n\|^p < +\infty.$$

Now, let's take $\forall x \in H$ and consider the operators $\widetilde{T}_m : H \to l_q$:

$$\widetilde{T}_m x = \{\varphi_n^*(x)\}_{n=1}^m,$$

where $\{\varphi_n^*\}_{n\in N} \subset H$ is a system conjugated to the basis $\{\varphi_n\}_{n\in N}$, and $\|\{a_n\}_{n\in N}\|_{l_q} = \left(\sum\limits_{n=1}^{\infty} |a_n|^q\right)^{1/q}$.

It is obvious that for $\forall x \in H$ the sequence $\{T_m x\}_{m\in N}$ is bounded in l_q : $\|T_m x\| \leq M_x$, $\forall m \in N$, where M_x is a constant dependent only on x. Then by Banach-Steinhaus theorem [8] for operators, we get $\sup\limits_m \|T_m\| < +\infty$.

Now, consider the operators $T_m : H \to H$:

$$T_m x = \sum_{n=1}^{m} \left(\widetilde{\psi}_n - \varphi_n\right) \varphi_n^* (x)\,,\ m \in N, \quad \text{where } \widetilde{\psi}_n = \rho_n \psi_n,\ n \in N.$$

It directly follows from the relation

$$\left\| \sum_{k=n}^{m} \left(\widetilde{\psi}_k - \varphi_k\right) \varphi_k^* (x) \right\| \leq \sum_{k=n}^{m} |\varphi_n^* (x)| \left\| \widetilde{\psi}_k - \varphi_k \right\| \leq$$

$$\leq \left(\sum_{k=n}^{m} \left\| \widetilde{\psi}_k - \varphi_k \right\|^p \right)^{1/p} \cdot \left(\sum_{k=n}^{m} |\varphi_n^* (x)|^q \right)^{1/q},$$

that $\lim\limits_{m\to\infty} T_m x$ is some element from H : $Tx \overset{def}{=} \lim\limits_{m\to\infty} T_m x$ and moreover T is a completely continuous operator, so $\Phi = I + T$ is Fredholm. It is easy to see that $\Phi\varphi_n = \widetilde{\psi}_n, \forall n \in N$. Since $\rho_n \neq 0, \forall n \in N$ the system $\{\psi_n\}_{n\in N}$ is complete; minimal; ω-linearly independent forms a basis in H iff the system $\{\widetilde{\psi}_n\}_{n\in N}$ prossess appropriate properties. By lemma 1 we get that statements 1)-4) of theorem 4 are equivalent for the system $\{\widetilde{\psi}_n\}_{n\in N}$. Consequently, this is true for the system $\{\psi_n\}_{n\in N}$ as well.

The theorem is proved.

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Received May 25, 2009; Revised September 22, 2009.