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SPECTRAL CHARACTERISTICS OF STARK'S FINITE-PERTURBED OPERATOR

Abstract

Consider a boundary value problem of the form

$$-y'' - [x - p(x)]\, y' = \lambda y, \quad 0 \le x < \infty, \tag{1}$$

$$y(0) = 0, \tag{2}$$

where $p(x)$ *is a real continuous function and* $p(x) \equiv 0$ *for* $x \ge a > 0$.

Let the functions $\varphi_0(x,\lambda), \theta_0(x,\lambda)$ be the solutions of equation (1) with the following initial conditions:

$$\varphi_0(0,\lambda) = 0, \quad \varphi_0'(0,\lambda) = 1, \tag{3}$$

$$\theta_0(0,\lambda) = 1, \quad \theta_0'(0,\lambda) = 0, \tag{4}$$

By K^2 denote a set of finite functions from the space $L_2[0,+\infty)$, by $\widetilde{f}(\lambda) - \varphi(x,\lambda)$ a Fourier transformation $f(x) \in K^2$

$$\hat{f}(\lambda) = \int_x^{+\infty} f(x)\, \varphi(x,\lambda)\, dx.$$

It is known that for any λ with $\operatorname{Im}\lambda \neq 0$ equation (1) has a unique solution,

$$\psi_0(x,\lambda) = \theta_0(x,\lambda) + m_0(\lambda)\, \varphi_0(x,\lambda) \in L_2,$$

moreover, if $f(x) \in K^2$, then

$$\int_0^{+\infty} f(x)\, g(x)\, dx = \frac{1}{\pi} \int_x^{+\infty} \hat{f}(\lambda)\, \hat{g}(\lambda)\, dM_0(\lambda),$$

where the spectral function

$$M_0(\lambda) = \lim_{\sigma \to 0} \int_0^{+\infty} [-\operatorname{Im} m_0(\lambda)(S + i\sigma]\, d\sigma, [1].$$

Investigate some special solutions of the equation

$$-y'' - xy' = \lambda y. \tag{5}$$

It is known that [2] the solution of the equation of the form

$$y'' + \left[\frac{1}{2}\frac{g'''}{g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 + \left(\frac{1}{4} - \nu^2\right)\left(\frac{g'}{g}\right)^2 + g'^2\right] y = 0 \tag{6}$$

is the function

$$y=\sqrt{\frac{g}{g'}}H_\nu\left(g\right) \tag{7}$$

$H_\nu\left(g\right)$ are the cylindric functions.

Take $g\left(x\right)=\frac{2}{3}\left(x+\lambda\right)^{\frac{3}{2}}$, then

$$g'^2=\left(x+\lambda\right). \tag{8}$$

Consequently

$$\frac{1}{2}\frac{g'''}{g'}=\frac{1}{2}\left[\frac{-\frac{1}{4}\left(x+\lambda\right)^{-\frac{3}{2}}}{\left(x+\lambda\right)^{-\frac{1}{2}}}\right]=-\frac{1}{8}\left(x+\lambda\right)^{-2} \tag{9}$$

$$-\frac{3}{4}\left(\frac{g'''}{g'}\right)^2=-\frac{3}{4}\left[\frac{\frac{1}{2}\left(x+\lambda\right)^{-\frac{1}{2}}}{\left(x+\lambda\right)^{-\frac{1}{2}}}\right]^2=-\frac{3}{16}\left(x+\lambda\right)^{-2}. \tag{10}$$

Then,

$$\left(\frac{1}{4}-\nu^2\right)\left(\frac{g'}{g}\right)^2=\left(\frac{1}{4}-\nu^2\right)\left[\frac{\left(x+\lambda\right)^{\frac{1}{2}}}{\frac{2}{3}\left(x+\lambda\right)^{\frac{3}{2}}}\right]^2=\left(\frac{1}{4}-\nu^2\right)\cdot\frac{9}{4}\left(x+\lambda\right)^{-2}. \tag{11}$$

Substituting (8) (11) in (6), we get

$$-\frac{1}{8}\left(x+\lambda\right)^{-2}-\frac{3}{16}\left(x+\lambda\right)^{-2}\left(\frac{1}{4}-\nu^2\right)\frac{9}{4}\left(x+\lambda\right)^{-2}=0.$$

Hence

$$\left(-\frac{1}{8}-\frac{3}{16}+\frac{9}{16}-\frac{9}{4}\nu^2\right)\left(x+\lambda\right)^{-2}=0,$$

$$\frac{9}{4}\nu^2=\frac{1}{4}\Rightarrow\nu=\pm\frac{1}{3}.$$

Taking into account these reasomings, from formula (7) we get

$$\psi_0\left(x,\lambda\right)=y\left(x,\lambda\right)=\sqrt{\left[\frac{\frac{2}{3}\left(x+\lambda\right)^{\frac{3}{2}}}{\left(x+\lambda\right)^{\frac{1}{2}}}\right]}H_{\frac{1}{3}}\sqrt{\frac{2}{3}\left(x+\lambda\right)^{\frac{3}{2}}}=$$

$$=\sqrt{\frac{2}{3}\left(x+\lambda\right)}H_{\frac{1}{3}}\sqrt{\frac{2}{3}\left(x+\lambda\right)^{\frac{3}{2}}}. \tag{12}$$

Thus, (12) is a special solution of equation (15)

$$H_\nu\left(x\right)=J_\nu\left(x\right)+iY_\nu\left(x\right).$$

Theorem 1. *For any λ from the upper half-plane the function $\psi_0\left(x,\lambda\right)$ belongs to the space $L_2\left[0,\infty\right)$.*

Proof. Above we proved that the solution of equation (5) is the function

$$\psi_0\left(x,\lambda\right)=\sqrt{\frac{2}{3}\left(x+\lambda\right)}H_{\frac{1}{3}}\sqrt{\frac{2}{3}\left(x+\lambda\right)^{\frac{3}{2}}},$$

the function $H_{\nu}(x)$ is called Henkel's cylindrical function $\left(H_{\nu}^{(1)}, H_{\nu}^{(2)}\right)$, moreover,

$$H_{\nu}^{(1)}(x) = J_{\nu}(x) + iY_{\nu}(x), \quad H_{\nu}^{(2)} = J_{\nu}(x) - iY_{\nu}(x). \tag{13}$$

In the given case, the functions

$$\theta_1(x,\lambda) = \sqrt{\frac{2}{3}(x+\lambda)} J_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}, \tag{14}$$

$$\theta_2(x,\lambda) = \sqrt{\frac{2}{3}(x+\lambda)} Y_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\} \tag{15}$$

are linear independent solutions of equation (5).

($J_{\nu}(x)$ is Bessel's function of first kind, $Y_{\nu}(x)$ a second kind).

Asymptotics of these functions

$$J_{\nu}(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \tag{14'}$$

$$Y_{\nu}(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \tag{15'}$$

for fixed ν and $|z| \to \infty$ is known.

Notice that the asymptotics of the function

$$H_{\frac{1}{3}}^{(1)}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\} = J_{\frac{1}{3}}(x) + iY_{\frac{1}{3}}(x)$$

as $|x+\lambda| \to \infty$ contains a multiplier e^{iX} $\left(X = \frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right)$

$$e^{iX} = e^{i\left[\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right]} = e^{i\frac{2}{3}x^{\frac{3}{2}}\left(1+\frac{3}{2}\frac{\lambda}{x}+\ldots\right)} \sim e^{i\lambda\sqrt{x}},$$

that for $Jm\lambda > 0$ exponentially decreases as $x \to \infty$, so that the function

$$\psi_0(x,\lambda) = Y_{\frac{2}{3}}\sqrt{(x+\lambda)} H_{\frac{1}{3}}^{(1)}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\} \tag{16}$$

belongs to $L_2[0,\infty)$. The theorem is proved.

Remark 1. If by $\psi(x,\lambda)$ we denote the solution of problem (1)-(2) coinciding with $\psi_0(x,\lambda)$ for $x \geq a > 0$, theorem 1 remains valid for $\psi(x,\lambda)$ as well.Further, there exists a kernel $K(x,t)$ such that (see [3])

$$\psi(x,\lambda) = \psi_0(x,\lambda) + \int_x^{2a-x} K(x,t)\,\psi_0(t,\lambda)\,dt.$$

We proved that the solutions of the equations $y'' + (x+\lambda)y = 0$ are

$$\theta_1(x,\lambda) = \sqrt{(x+\lambda)} J_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}, \tag{17}$$

$$\theta_2(x,\lambda)=\sqrt{(x+\lambda)}Y_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}. \tag{18}$$

Calculate their wronskain introducing the denotation

$$X=\frac{2}{3}(x+\lambda)^{\frac{3}{2}},\quad T=\frac{2}{3}(t+\lambda)^{\frac{3}{2}}$$

$$W[\theta_1(x,\lambda);\theta_2(x,\lambda)]=\theta_1(x,\lambda)\theta_2'(x,\lambda)-\theta_1'(x,\lambda)\theta_2(x,\lambda)=$$

$$=\Big\{(x+\lambda)^{\frac{1}{2}}J_{\frac{1}{3}}(X_1)(x+\lambda)^{\frac{1}{2}}Y_{\frac{1}{3}}'(X_1)-$$

$$-\Big((x+\lambda)^{\frac{1}{2}}J_{\frac{1}{3}}'(X_1)\Big)\cdot\Big((x+\lambda)^{\frac{1}{2}}Y_{\frac{1}{3}}(X_1)\Big)\Big\}=$$

$$=\left\{(x+\lambda)^{\frac{1}{2}}J_{\frac{1}{3}}(X_1)\left[\frac{1}{2}(x+\lambda)^{-\frac{1}{2}}Y_{\frac{1}{3}}(X_1)+(x+\lambda)^{\frac{1}{2}}Y_{\frac{1}{3}}(X_1)\cdot X_1'\right]\right\}=$$

$$=\left[\frac{1}{2}(x+\lambda)^{-\frac{1}{2}}J_{\frac{1}{3}}(X_1)+(x+\lambda)^{\frac{1}{2}}J_{\frac{1}{3}}(X_1)\cdot X_1'\right]\frac{1}{2}(x+\lambda)^{\frac{1}{2}}Y_{\frac{1}{3}}(X_1)=$$

$$=\left\{(x+\lambda)X_1'\left[J_{\frac{1}{3}}(X_1)\cdot Y_{\frac{1}{3}}(X_1)-J_{\frac{1}{3}}'(X_1)\cdot Y_{\frac{1}{3}}(X_1)\right]\right\}=$$

$$=(x+\lambda)^{\frac{1}{2}}X_1'(x)W\left[J_{\frac{1}{3}}(X_1);Y_{\frac{1}{3}}(X_1)\right]_{x=0}=\left\{(x+\lambda)\frac{2}{3}\cdot\frac{3}{2}(x+\lambda)^{\frac{3}{2}-1}\right\}\times$$

$$\times W\left[J_{\frac{1}{3}}(X_1);Y_{\frac{1}{3}}(X_1)\right]_{x=0}=\left\{(x+\lambda)^{\frac{3}{2}}\cdot\frac{2}{\pi(x+\lambda)^{\frac{3}{2}}\cdot\frac{3}{2}}\right\}_{x=0}=\frac{3}{4\pi}$$

Consequently,

$$W[\theta_1;\theta_2]=\frac{3}{4\pi}. \tag{19}$$

Here we take into account that $W[J_\nu(z);Y_\nu(z)]=\frac{2}{\pi z}$.

By $\varphi_0(x,\lambda)$, $\theta_0(x,\lambda)$ we denote the solutions of equation (1) $(p(x)=0, x>a)$ with initial conditions

$$\varphi_0'(0,\lambda)=0,\quad \varphi_0'(0,\lambda)=1, \tag{20}$$

$$\theta_0(0,\lambda)=1,\quad \theta_0'(0,\lambda)=0. \tag{21}$$

Obviously,

$$\varphi_0(x,\lambda)=a(\lambda)\theta_1(x,\lambda)+b(\lambda)\theta_2(x,\lambda). \tag{22}$$

Taking into account (20), we have

$$\varphi_0'(0,\lambda)=a(\lambda)\theta_0'(0,\lambda)+b(\lambda)\theta_2'(0,\lambda)=1,$$

$$\varphi_0(0,\lambda)=a(\lambda)\theta_1(0,\lambda)+b(\lambda)\theta_2(0,\lambda)=0.$$

Hence

$$a(\lambda)=\frac{\begin{vmatrix}0 & \theta_2(0,\lambda)\\ 1 & \theta_2'(0,\lambda)\end{vmatrix}}{W[\theta_1,\theta_2]}=-\frac{4\pi}{3}\theta_2(0,\lambda),$$

$$b(\lambda)=\frac{\begin{vmatrix}\theta_1(0,\lambda) & 0\\ \theta_1'(0,\lambda) & 1\end{vmatrix}}{W(\lambda)}=\frac{4\pi}{3}\theta_1(0,\lambda).$$

Substituting these values in (22), we get

$$\varphi_0(x,\lambda)=\frac{4\pi}{3}\left[-\theta_1(x,\lambda)\,\theta_2(0,\lambda)+\theta_2(x,\lambda)\,\theta_1(0,\lambda)\right]. \tag{23}$$

Taking into attention formulae (17) and (18), we get

$$\varphi_0(x,\lambda)=\frac{4\pi}{3}\left[\sqrt{(x+\lambda)}Y_{\frac{1}{3}}\left(\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right)\cdot\sqrt{\lambda}J_{\frac{1}{3}}\left(\frac{2}{3}\lambda^{\frac{3}{2}}\right)-\right.$$
$$\left.-\sqrt{(x+\lambda)}J_{\frac{1}{3}}\left(\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right)\cdot\sqrt{\lambda}Y_{\frac{1}{3}}\left(\frac{2}{3}\lambda^{\frac{3}{2}}\right)\right]=$$
$$=\frac{4\pi}{3}\sqrt{\lambda(x+\lambda)}\left[Y_{\frac{1}{3}}\left(\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right)\cdot J_{\frac{1}{3}}\left(\frac{2}{3}\lambda^{\frac{3}{2}}\right)-\right.$$
$$\left.-Y_{\frac{1}{3}}\left(\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right)\cdot\sqrt{\lambda}Y_{\frac{1}{3}}\left(\frac{2}{3}\lambda^{\frac{3}{2}}\right)\right]. \tag{23'}$$

Introducing the denotation $Z=\frac{2}{3}\lambda^{\frac{3}{2}}$, we can write the last one in the form

$$\varphi_0(x,\lambda)=\frac{4\pi}{3}\sqrt{\lambda}\sqrt{(x+\lambda)}\left[Y_{\frac{1}{3}}(X)\cdot J_{\frac{1}{3}}(Z)-J_{\frac{1}{3}}(X)\cdot Y_{\frac{1}{3}}(z)\right]. \tag{23''}$$

We can calculate the asymptotics $\varphi_0(x,\lambda)$. To this end we use the asymptotics of the functions $\theta_1(x,\lambda)$, $\theta_2(x,\lambda)$.

$$\theta_1(x,\lambda)\underset{|X|\to\infty}{\sim_{(x+\lambda)^{\frac{1}{2}}}}\sqrt{\frac{2}{\pi\cdot\frac{2}{3}(x+\lambda)^{\frac{3}{2}}}}\cdot\cos\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}-\frac{\pi}{6}-\frac{\pi}{4}\right\}=$$
$$=(x+\lambda)^{\frac{1}{2}}\sqrt{\frac{3}{\pi\cdot(x+\lambda)^{\frac{3}{2}}}}\cdot\cos\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}-\frac{5\pi}{12}\right\}=$$
$$=\sqrt{\frac{3}{\pi\cdot(x+\lambda)^{\frac{1}{2}}}}\cdot\cos\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}-\frac{5\pi}{12}\right\} \tag{24}$$

$$\theta_1(0,\lambda)\sim\sqrt{\frac{3}{\pi\cdot\lambda^{\frac{1}{2}}}}\cdot\cos\left\{\frac{2}{3}\lambda^{\frac{3}{2}}-\frac{5\pi}{12}\right\} \tag{25}$$

$$\underset{|X|\to\infty}{\theta_2(x,\lambda)}\sim_{(x+\lambda)^{\frac{1}{2}}}\sqrt{\frac{2}{\pi\cdot\frac{2}{3}(x+\lambda)^{\frac{3}{2}}}}\cdot\sin\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}-\frac{5\pi}{12}\right\}=$$
$$=(x+\lambda)^{\frac{1}{2}}\sqrt{\frac{3}{\pi\cdot(x+\lambda)^{\frac{3}{2}}}}\cdot\sin\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}-\frac{5\pi}{12}\right\}=$$
$$=\sqrt{\frac{3}{\pi\cdot(x+\lambda)^{\frac{1}{2}}}}\cdot\sin\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}-\frac{5\pi}{12}\right\} \tag{26}$$

$$\theta_2(0,\lambda)\sim\sqrt{\frac{3}{\pi\cdot\lambda^{\frac{1}{2}}}}\cdot\sin\left\{\frac{2}{3}\lambda^{\frac{3}{2}}-\frac{5\pi}{12}\right\} \tag{27}$$

Substituting (24)-(27) in (23), we get

$$\varphi_0(x,\lambda)=\frac{4\pi}{3}\left[-\sqrt{\frac{3}{\pi(x+\lambda)^{\frac{3}{2}}}}\cdot\cos\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}-\frac{5\pi}{12}\right\}\times\right.$$

$$\times\sqrt{\frac{3}{\pi\cdot\lambda^{\frac{1}{2}}}}\cdot\sin\left\{\frac{2}{3}\lambda^{\frac{3}{2}}-\frac{5\pi}{12}\right\}+\sqrt{\frac{3}{\pi\cdot(x+\lambda)^{\frac{3}{2}}}}\cdot\sin\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}-\frac{5\pi}{12}\right\}\times$$

$$\times\sqrt{\frac{3}{\pi\cdot\lambda^{\frac{1}{2}}}}\cdot\cos\left\{\frac{2}{3}\lambda^{\frac{3}{2}}-\frac{5\pi}{12}\right\}=$$

$$=\frac{4\pi}{3}\sqrt{\frac{3}{\pi(x+\lambda)^{\frac{1}{2}}\lambda^{\frac{3}{2}}}}\cdot\sin\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}-\frac{5\pi}{12}\right\}\cdot\cos\left\{\frac{2}{3}\lambda^{\frac{3}{2}}-\frac{5\pi}{12}\right\}-$$

$$-\sin\left\{\frac{2}{3}\lambda^{\frac{3}{2}}-\frac{5\pi}{12}\right\}\cos\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}-\frac{5\pi}{12}\right\}=$$

$$=\frac{4\pi^{\frac{1}{2}}}{\sqrt{3}\sqrt{(x+\lambda)^{\frac{1}{2}}\lambda^{\frac{3}{2}}}}\left[\sin\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}-\frac{2}{3}\lambda^{\frac{3}{2}}\right\}\right],$$

i.e.

$$\underset{|X|\to\infty}{\varphi_0(x,\lambda)}\sim\frac{4\pi^{\frac{1}{2}}}{\sqrt{3(x+\lambda)^{\frac{1}{2}}}}\sin\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}-\frac{2}{3}\lambda^{\frac{3}{2}}\right\}. \tag{28}$$

Similarly we find

$$\theta_0(x,\lambda)=\widetilde{a}(\lambda)\,\theta_1(x,\lambda)+\widetilde{b}(\lambda)\,\theta_2(x,\lambda),$$

$$\theta_0(0,\lambda)=\widetilde{a}(\lambda)\,\theta_0'(0,\lambda)+\widetilde{b}(\lambda)\,\theta_2'(x,\lambda)=1,$$

$$\theta_0'(0,\lambda)=\widetilde{a}(\lambda)\,\theta_1'(0,\lambda)+\widetilde{b}(\lambda)\,\theta_2'(x,\lambda)=0,$$

$$\widetilde{a}(\lambda)=\frac{\begin{vmatrix}1 & \theta_2(0,\lambda)\\ 0 & \theta_2'(0,\lambda)\end{vmatrix}}{W[\lambda]}=\frac{\pi}{2}\theta_2'(0,\lambda),\qquad b(\lambda)=-\frac{\pi}{2}\theta_1'(0,\lambda),$$

$$\theta_0(x,\lambda)=\frac{\pi}{2}\left[\theta_2'(0,\lambda)\,\theta_1(x,\lambda)-\theta_1'(0,\lambda)\,\theta_2(x,\lambda)\right]. \tag{29}$$

Calculate $\theta_1'(x,\lambda),\theta_2(x,\lambda)$

$$\theta_1'(x,\lambda)=\left[\sqrt{x+\lambda}J_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}\right]'=\left(\sqrt{x+\lambda}\right)'J_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}+$$

$$+\sqrt{x+\lambda}J_{\frac{1}{3}}'\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}\cdot\left(\frac{3}{2}(x+\lambda)^{\frac{3}{2}}\right)'=$$

$$=\frac{1}{2}(x+\lambda)^{\frac{1}{2}-1}J_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}+\sqrt{x+\lambda}J'_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}\frac{\cancel{2}}{\cancel{3}}\frac{\cancel{3}}{\cancel{2}}(x+\lambda)^{\frac{3}{2}-1}=$$

$$=\frac{1}{2}(x+\lambda)^{-\frac{1}{2}}J_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}+\sqrt{x+\lambda}J'_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}(x+\lambda)^{\frac{1}{2}}=$$

$$=\frac{1}{2}(x+\lambda)^{-\frac{1}{2}}J_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}+(x+\lambda)J'_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}.$$

Thus,

$$\theta_1'(x,\lambda)=\frac{1}{2}(x+\lambda)^{-\frac{1}{2}}J_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}+(x+\lambda)J'_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}. \tag{30}$$

Hence we have

$$\theta_1'(0,\lambda)=\frac{1}{2}\lambda^{-\frac{1}{2}}J_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}+\lambda J'_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}. \tag{31}$$

Similarly we get

$$\theta_2'(x,\lambda)=\frac{1}{2}(x+\lambda)^{-\frac{1}{2}}Y_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}+(x+\lambda)Y'_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}, \tag{32}$$

$$\theta_2'(0,\lambda)=\frac{1}{2}\lambda^{-\frac{1}{2}}Y_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}+\lambda Y'_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}. \tag{33}$$

Substituting (30)-(33) in formula (29), we get

$$\theta_0(x,\lambda)=\frac{\pi}{2}\left[\sqrt{x+\lambda}J_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}\left[\frac{1}{2}\lambda^{-\frac{1}{2}}Y_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}+\lambda Y'_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}\right]-\right.$$

$$\left.-\sqrt{x+\lambda}Y_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}\left[\frac{1}{2}\lambda^{-\frac{1}{2}}J_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}+\lambda J'_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}\right]\right]=$$

$$=\frac{\pi}{2}\left[\frac{1}{2}\sqrt{x+\lambda}\cdot\lambda^{-\frac{1}{2}}J_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}Y_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}+\right.$$

$$+\lambda\sqrt{x+\lambda}J_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}Y'_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}-$$

$$-\sqrt{x+\lambda}\cdot\lambda^{-\frac{1}{2}}Y_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}J_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}-$$

$$\left.-\sqrt{x+\lambda}\cdot\lambda Y_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}J'_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}\right]=$$

$$=\frac{\pi}{2}\left\{\frac{1}{2}\sqrt{x+\lambda}\left(J_{\frac{1}{3}}\left(\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right)Y_{\frac{1}{3}}\left(\frac{2}{3}\lambda^{\frac{3}{2}}\right)-\right.\right.$$

$$\left.-Y_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}J_{\frac{1}{3}}\left(\frac{2}{3}\lambda^{\frac{3}{2}}\right)\right)+\lambda\sqrt{x+\lambda}\left[J_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}Y'_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}-\right.$$

$$\left.\left.-Y_{\frac{1}{3}}\left\{\frac{2}{3}(x+\lambda)^{\frac{3}{2}}\right\}J'_{\frac{1}{3}}\left\{\frac{2}{3}\lambda^{\frac{3}{2}}\right\}\right]\right\}.$$

Taking into account the denotation reduced above, we can rewrite:

$$\theta_0(x,\lambda)=\frac{\pi}{2}\left\{\frac{\lambda}{2}\sqrt{x+\lambda}\left(J_{\frac{1}{3}}(X)\cdot Y'_{\frac{1}{3}}(Z)-Y_{\frac{1}{3}}(X)\,J_{\frac{1}{3}}(Z)\right)+\right.$$

$$\left.+\lambda\sqrt{x+\lambda}\left(J_{\frac{1}{3}}(X)\cdot Y'_{\frac{1}{3}}(Z)-Y_{\frac{1}{3}}(X)\cdot J_{\frac{1}{3}}(X)\right)\right\}. \tag{34}$$

Then, as is known [1], the derivative from the spectral function of the boundary value problem (1)-(2) ($p(x)\equiv 0$ for $x\geq a$) is determined as:

$$K_0(\lambda)=-\operatorname{Im}\left[\frac{\psi'_0(0,\lambda)}{\psi_0(0,\lambda)}\right]=\frac{1}{2i}\left[\frac{\psi'_0(0,\lambda)}{\psi_0(0,\lambda)}-\frac{\overline{\psi}'_0(0,\lambda)}{\overline{\psi}_0(0,\lambda)}\right]=$$

$$=-\frac{1}{2i}\frac{W\left[\psi_0(0,\lambda)\,\overline{\psi}_0(0,\lambda)\right]_{x=0}}{\left|\psi_0(0,\lambda)\right|^2}.$$

Taking into account [5]

$$H_\nu^{(1)}(z)=J_\nu(z)+iY_\nu(z)\,,\quad H_\nu^{(2)}(z)=J_\nu(z)-iY_\nu(z)\,,$$

and the fact that

$$W_0\left[H_{\frac{1}{3}}^{(1)}(z)\,,H_{\frac{1}{3}}^{(2)}(z)\right]=\frac{4i}{\pi z},$$

from the last formula we get

$$K_0(\lambda)=\frac{2}{\pi\lambda\left|H_{\frac{1}{3}}^{(1)}\frac{2}{3}\lambda^{\frac{3}{2}}\right|^2}, \tag{35}$$

therefore, the expansion formula looks like

$$\int_{-\infty}^{+\infty}\varphi_0(x,\lambda)\,\varphi_0(t,\lambda)\,K_0(\lambda)\,d\lambda=\delta(x,t)\,.$$

So, we proved the theorem.

Theorem 2. *The derivative $K_0(\lambda)$ of the spectral function of the operator generated by boundary value problem (1)-(2) is determined by formula (35).*

By remark (1), the solution $\psi(x,\lambda)$ belongs to the space $L_2[0,\infty)$ for $\operatorname{Im}>0$. Therefore, the Weyl function $m(\lambda)$ is of the form: $m(\lambda)=\dfrac{\psi'(0,\lambda)}{\psi(0,\lambda)}$. The derivative $K(\lambda)$ from the spectral function of the operator generated by problem (1)-(2) is determined as is known, in the following way

$$K(\lambda)=\operatorname{Im}m(\lambda+i0)$$

for real λ [1].

Therefore,

$$K(\lambda) = -\frac{1}{2}\left[\frac{\psi'(0,\lambda)}{\psi(0,\lambda)} - \frac{\overline{\psi'(0,\lambda)}}{\overline{\psi(0,\lambda)}}\right] = -\frac{1}{2i}\frac{W\left[\psi(x,\lambda);\overline{\psi(x,\lambda)}\right]_{x=a}}{\psi(0,\lambda)\overline{\psi(0,\lambda)}} =$$

$$= -\frac{1}{2i}\frac{W_0\left[\psi_0(x,\lambda);\overline{\psi_0(x,\lambda)}\right]_{x=0}}{\psi_0(0,\lambda)\overline{\psi_0(x,\lambda)}\eta_1(\lambda)\eta_2(\lambda)} = \frac{K_0(\lambda)}{\eta_1(\lambda)\eta_2(\lambda)},$$

where $W\left[\psi(x,\lambda);\overline{\psi(x,\lambda)}\right]$; $W_0\left[\psi_0(x,\lambda);\overline{\psi_0(x,\lambda)}\right]$ is a Wronskian of the solutions $\psi(x,\lambda)$, $\overline{\psi}(x,\lambda)$, $\psi_0(x,\lambda)$, $\overline{\psi_0(x,\lambda)}$, respectively.

Thus,

$$K(\lambda) = \frac{K_0(\lambda)}{\eta_1(\lambda)\eta_2(\lambda)},$$

$$K_0(\lambda) = -\frac{1}{2}\frac{W_0\left[\psi_0(x,\lambda);\overline{\psi_0(x,\lambda)}\right]_{x=0}}{\psi_0(0,\lambda)\overline{\psi_0(0,\lambda)}},$$

$$\eta_1(\lambda) = 1 + \int_0^{2a} K(0,t)F_0(t,\lambda)\,dt, \quad F_0(t,\lambda) = \frac{\psi_0(t,\lambda)}{\psi_0(0,\lambda)}$$

for real λ, $\eta_2(\lambda) = \overline{\eta_1(\lambda)}$.

Theorem 3. *If for $x \to a$, $p(x) \sim C_0(a-x)^l$, (A) where $l \geq 0$*[1], *then*

$$K(0,s) = \frac{C_0}{2}\left(a-\frac{s}{2}\right)^{l+1} + O\left[\left(a-\frac{s}{2}\right)^{l+1}\right] \tag{36}$$

for $S \to 2a$ [3].

Proof. It is known that $K(x,t)$ for $t>0$, satisfies the integral equation of the form

$$K(x,t) = \frac{1}{2}\int_{\frac{x+t}{2}}^{\infty} p(s)\,ds + \frac{1}{2}\int_x^{a-\frac{t-x}{2}}\int_{t+x-\xi}^{t-x+\xi} p(\xi,\eta)K(\xi,\eta)\,d\xi d\eta +$$

$$+\frac{1}{2}\int_{a-\frac{t-x}{2}}^{\frac{x+t}{2}}\int_{t+x-\xi}^{2a-\xi} p(\xi,\eta)K(\xi,\eta)\,d\eta d\xi + \frac{1}{2}\int_{\frac{x+t}{2}}^{a}\int_{\xi}^{2a-\xi} p(\xi,\eta)K(\xi,\eta)\,d\eta d\xi, \tag{37}$$

where $p(\xi,\eta) = -\xi+\eta+p(\xi)$, It is seen from (37) that as $t \to 2a$, $K(0,t)$ tends to zero more rapidly than $\int_{\frac{t}{2}}^{a} p(s)\,ds$.

Now, assume

$$K(0,t) \sim A\left(a-\frac{t}{2}\right)\left(1+o\left(a-\frac{t}{2}\right)\right) +$$

[1]In sequel, everywhere we'll assume that condition A is fulfilled.

$$+\frac{1}{2}\left(a-\frac{t}{2}\right)^2 p(0,t)\,K(0,t)\left(1+o\left(a-\frac{t}{2}\right)\right)+$$

$$+\frac{1}{2}\int\limits_{a-\frac{t}{2}}^{\frac{t}{2}}(2a-t)\,p(\xi,2a-\xi+\triangle)\,K(\xi,2a-\xi+\triangle)\,d\xi, \tag{38}$$

where $|\triangle|\le a-\dfrac{t}{2}$.

For $K(x,t)$ [4] the following estimates of the form

$$|K(x,t)|\le\frac{1}{4}\sigma\left(\frac{x+t}{2}\right)\sum_{n=0}^{\infty}\frac{C^n(a-x)^{2n}}{(2n-1)!},$$

are known, where $C=\max|p(\xi,\eta)|$, max is taken from the domain of definition $K(x,t)$, $b(x)=\int\limits_x^\infty|p(s)|\,ds$. It is seen from this estimate that $|K(\xi,2a-\xi)|\le(\triangle)\,C(a-\xi)$ i.e. $K(\xi,2a-\xi+\triangle)$ tends to zero as $t\to 2a$ more rapidly than $\left(a-\frac{t}{2}\right)^{l+1}$. Dividing the both hand sides of equality (38) by $\frac{C_0}{2(l+1)}\left(a-\frac{t}{2}\right)$ and passing to limit as $t\to 2a$, we get

$$\lim_{t\to 2a}K(0,t)\frac{2(l+1)}{C_0\left(a-\frac{t}{2}\right)^{(l+1)}}=1,$$

i.e. $K(0,t)\sim\frac{C_0}{2(l+1)}\left(a-\frac{t}{2}\right)^{(l+1)}$.

Consider the solution $\varphi_0(x,\lambda)$ of the equation $y''+(x+\lambda)\,y=0$ satisfying the conditions

$$\varphi_0(0,\lambda)=0,\quad \varphi_0'(0,\lambda)=1. \tag{39}$$

Noticing that for real λ, the $\overline{\psi}_0(x,\lambda)$ is a linear independent solution with $\psi_0(x,\lambda)$, we have

$$\varphi_0(x,\lambda)=a(\lambda)\,\psi_0(x,\lambda)+b(\lambda)\,\overline{\psi}_0(x,\lambda),$$

where $a(\lambda)$, $b(\lambda)$ are the unknown constants.

Taking into account condition (), by the method of variation of constants we get

$$\varphi_0(x,\lambda)=-\frac{1}{W_0(\lambda)}\left[\psi_0(x,\lambda)\,\overline{\psi}_0(0,\lambda)-\psi_0(0,\lambda)\,\overline{\psi}_0(x,\lambda)\right], \tag{40}$$

where

$$W_0(\lambda)=\psi_0(0,\lambda)\,\overline{\psi}_0'(0,\lambda)-\psi_0'(0,\lambda)\,\overline{\psi}_0(0,\lambda).$$

Let $\varphi(x,\lambda)$ be a solution of the equation

$$y''+[x+p(x)]\,y=\lambda y$$

$(p(x)\equiv 0$, for $x\ge a)$ with conditions (39).

Noticing that for real λ the $\overline{\psi}(x,\lambda)$, is a linear independent solution with $\psi(x,\lambda)$, we have $\varphi(x,\lambda)=a(\lambda)\psi(x,\lambda)+b(\lambda)\overline{\psi}(x,\lambda)$, where $a(\lambda), b(\lambda)$ are the unknown constants.

From condition (40) we have:

$$a(\lambda)\psi(0,\lambda)+b(\lambda)\overline{\psi}(0,\lambda)=0,\quad a(\lambda)\psi'(0,\lambda)+b(\lambda)\overline{\psi}'(0,\lambda)=1,$$

$$a(\lambda)=\frac{\psi(0,\lambda)}{W\left[\psi(x,\lambda),\overline{\psi}(x,\lambda\right]_{x=0}}=\frac{\overline{\psi}(0,\lambda)}{W\left[\psi(x,\lambda),\overline{\psi}(x,\lambda\right]_{x=a}}=$$

$$=\frac{\overline{\psi}(0,\lambda)}{W_0\left[\psi_0(x,\lambda),\overline{\psi}_0(x,\lambda\right]_{x=0}},\ b(\lambda)=\frac{\psi(0,\lambda)}{W_0(\lambda)},$$

where

$$W\left[\psi(x,\lambda),\overline{\psi}(x,\lambda\right]=\psi(x,\lambda),\overline{\psi}'(x,\lambda)-\psi'(x,\lambda),\overline{\psi}(x,\lambda).$$

Then,

$$\varphi(x,\lambda)=-\frac{1}{W_0(\lambda)}\left[\psi(x,\lambda),\overline{\psi}(x,\lambda)-\psi(0,\lambda)\overline{\psi}(x,\lambda)\right].\tag{41}$$

From the last one we have

$$U(x,\lambda)=-\frac{1}{\eta_1(\lambda)W_0(\lambda)}\left[\psi(x,\lambda)\overline{\psi}(0,\lambda)-\overline{\psi}(x,\lambda)\psi(0,\lambda)\right]=$$

$$=-\frac{1}{W_0(\lambda)}\left[\psi(x,\lambda)\overline{\psi}_0(0,\lambda)S(\lambda)-\overline{\psi}(x,\lambda)\psi_0(0,\lambda)\right],\tag{42}$$

Taking into account that $\varphi(0,\lambda)=\psi_0(0,\lambda)\eta_1(\lambda)$, where

$$U(x,\lambda)=\frac{\varphi(x,\lambda)}{\eta_1(\lambda)},\quad S(\lambda)=\frac{\eta_2(\lambda)}{\eta_1(\lambda)}=\frac{\overline{\eta_1(\lambda)}}{\eta_1(\lambda)}.$$

$S(\lambda)$ is called the S - function of problem (1)-(2) (the sceattering function) It is abvious from (42) that the asymptotics of the normed eigen functions of boundary value problem (1)-(2) is determined by the function $S(\lambda)$ as $x\to\infty$.

Notice that the S-function will play an important part in deriving the basic integral equation.

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Received December 07, 2009; Revised March 03, 2010.