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A BOUNDARY VALUE PROBLEM FOR AN ORDINARY GENERAL INTEGRO-DIFFERENTIAL, LOADED EQUATION OF SECOND ORDER WITH NON-LOCAL AND GLOBAL TERMS IN THE BOUNDARY CONDITIONS

Abstract

It is known that in mathematical models, of different natural phenomena and optimal control there appear problems for loaded equations. If in integro-differential equations, the liger order derivative under is given by a conditions differentiable kernel, then decreasing the order of the derivative in these integrals by means of integration by parts, there appear loaded terms in this equation.

1. Introduction. In the paper we consider a problem for an ordinary, integro-differential, loaded equation of second order under general, linear (containing non-local and global terms) boundary conditions. We construct necessary conditions by means of the fundamental solutions of the principal part of the equation. By means of these necessary and given boundary conditions we determine the values of the unknown function and its derivative at the ends of the considered interval using the integral of the unknown function and its derivative.

Having written these values in the first relations of the basic expressions, we get a system of second kind with weak singularity in the kernel Fredholm integral equations of for the unknown function and its derivative. The three-point problem for a second order nonlinear differential equation was considered in [4]. For this equation, the problem containing an integral in the boundary condition was researched in [5]. The problem under point Dirichlet type boundary condition for second order ordinary differential equation was considered in [6] and at last the existence of the solution of three point and -point for such an equation in [7] and [8].

2. Problem statement. Let's consider the following boundary value problem:

$$\begin{gathered} ly = y''(x) + a_1(x)y'(x) + a_0(x)y(x) + \int\limits_0^1 \left[K_1(x,\xi)y'(\xi)+\right. \\ \left.+K_0(x,\xi)y(\xi)\right]d\xi + \alpha_1(x)y'(0) + \alpha_0(x)y(0)+ \\ +\beta_1(x)y'(1) + \beta_0(x)y(1) = f(x), \quad x \in (0,1), \end{gathered} \tag{2.1}$$

$$\begin{gathered} l_j y \equiv \alpha_{j1}y'(0) + \alpha_{j0}y(0) + \beta_{j1}y'(1) + \beta_{j0}y(1)+ \\ + \int\limits_0^1 \left[K_{j1}(\xi)\, y'(\xi) + K_{j0}(\xi)\, y(\xi)\right] d\xi = \gamma_j, \quad j = 1,2; \end{gathered} \tag{2.2}$$

here $a_i(x)$, $K_i(x,\xi)$, $\alpha_i(x)$, $\beta_i(x)$, $K_{ji}(\xi)$, $j=1,2$; $i=0,1$ and $f(x)$ are the known continuous real functions, α_{ji}, β_{ji} and γ_j $j=1,2$; $i=0,1$; are the given material (real) constants and $y(x)$ is an unknown (desired) function. If the kernels $K_1(x,\xi)$ and $K_{j1}(\xi)$ are differentiable functions with respect to the variable ξ, then in problem (2.1), (2.2) one can release integrand functions from $y'(\xi)$ (by means of integration by parts). In the problem we accept that the boundary conditions (2.2) are linear independent.

3. Fundamental solution. Considering that the principal part of the equation (2.1), (2.2) is

$$L_0 y \equiv y''(x),$$

for the fundamental solution of the equation

$$y''(x) = f(x),$$

we get [1]:

$$Y(x-t) = \frac{x-t}{2}. \tag{3.1}$$

4. Basic expressions. We multiply the both hand sides of the equation (2.1) by the fundamental solution (3.1), integrate on with respect to the variable x. Using the integration by parts, we construct the Lagrange formula [2]:

$$-\int_0^1 \left\{ a_1(x)y'(x) + a_0(x)y(x) + \int_0^1 \left[K_1(x,\xi)y'(\xi) + K_0(x,\xi)\,y(\xi)\right] d\xi \right\} \times$$

$$\times Y(x-t)\,dx - y'(1)\left[Y(1-t) + \int_0^1 \beta_1(x)Y(x-t)dx\right] +$$

$$+y'(0)\left[Y(-t) - \int_0^1 \alpha_1(x)Y(x-t)dx\right] +$$

$$+y(1)\left[Y'(1-t) - \int_0^1 \beta_1(x)Y(x-t)dx\right] - y(0)\left[Y'(-t) + \int_0^1 \alpha_0(x)Y(x-t)dx\right] +$$

$$+\int_0^1 f(x)Y(x-t)dx = \begin{cases} y(t), & \text{if } t\in(0,1), \\ \frac{1}{2}y(t), & \text{if } t=0 \text{ or } t=1. \end{cases} \tag{4.1}$$

In the same way, multiplying equation (2.1) by the function

$$Y'(x-\xi) = e(x-\xi)$$

being a derivative of the fundamental solution (3.1) and integrating on with respect to x, then similar to (4.1) we get [3], [5];

$$\int_0^1 \left\{ a_1(x)y'(x) + a_0(x)y(x) + \int_0^1 \left[K_1(x,\xi)y'(\xi) + K_0(x,\xi)y(\xi) \right] d\xi \right\} e(x-t)dx +$$

$$+y'(1)\left[e(1-t) + \int_0^1 \beta_1(x)e(x-t)dx \right] + y'(0)\left[-e(-t) + \int_0^1 \alpha_1(x)e(x-t)dx \right] +$$

$$+y(1)\int_0^1 \beta_0(x)e(x-t)dx + y(0)\int_0^1 \alpha_0(x)e(x-t)dx -$$

$$-\int_0^1 f(x)e(x-t)dx = \begin{cases} y'(t), & \text{if } \ t \in (0,1), \\ \frac{1}{2}y'(t), & \text{if } \ t = 0 \ \text{ or } \ t = 1. \end{cases} \tag{4.2}$$

Thus, we prove the following statement [9], [10]:

Theorem 1. *If* $a_i(x)$, $K_i(x,\xi)$, $\alpha_i(x)$, $\beta_i(x)$, $i = 0,1$ *and* $f(x)$ *are continuous real-valued functions, then each solution of the equation (2.1) determined on [0,1] satisfies the basic relations (4.1), (4.2).*

5. Necessary conditions. The second expressions of the basic relations (4.1) and (4.2) (obtained for $t = 0$ and $t = 1$) are necessary conditions.

The two of these four necessary conditions are linearly independent. We can easily see that the expressions obtained from (4.2) for $t = 0$ and $t = 1$ are the same and the remaining two expressions obtained from (4.1) for $t = 0$ and $t = 1$ are also linearly dependent. Really, the second expression (obtained for $t = 1$) is easily obtained from the first one (obtained for $t = 0$) if we add to it the integral of the equation (2.1) from 0 to 1. Let's write these two necessary conditions (that are linearly independent) in the following form:

$$y'(0)\int_0^1 \alpha_1(x)xdx + y(0)\left[1 + \int_0^1 \alpha_0(x)xdx \right] + y'(1)\left[1 + \int_0^1 \beta_1(x)xdx \right] -$$

$$-y(1)\left[1 - \int_0^1 \beta_0(x)xdx \right] + \int_0^1 \left\{ a_1(x)y'(x) + a_0(x)y(x) + \int_0^1 \left[K_1(x,\xi)y'(\xi) + \right. \right.$$

$$\left. \left. +K_0(x,\xi)y(\xi) \right] d\xi \right\} xdx = \int_0^1 f(x)xdx, \tag{5.1}$$

$$y'(0)\left[\int_0^1 \alpha_1(x)dx-1\right]+y(0)\int_0^1 \alpha_0(x)dx+$$

$$+y'(1)\left[1+\int_0^1 \beta_1(x)dx\right]+y(1)\int_0^1 \beta_0(x)dx+\int_0^1 \left\{a_1(x)y'(x)+a_0(x)y(x)+\right.$$

$$\left.+\int_0^1 \left[K_1(x,\xi)y'(\xi)+K_0(x,\xi)y(\xi)\right]d\xi\right\}dx=\int_0^1 f(x)dx, \qquad (5.2)$$

6. Definition of boundary values. Combining the necessary conditions (5.1) and (5.2) obtained above with the boundary conditions (2.2) we write the obtained expressions in the form of a system of linear algebraic equations in the following way:

$$\sum_{n=0}^{1}\left[\alpha_{jn}y^{(n)}(0)+\beta_{jn}y^{(n)}(1)\right]=$$
$$=\gamma_j-\int_0^1 \sum_{n=0}^{1} K_{jn}(\xi)\, y^{(n)}(\xi)\, y^{(n)}(\xi)\, d\xi, \quad j=\overline{1,4}, \qquad (6.1)$$

here

$$\alpha_{31}=\int_0^1 \alpha_1(x)xdx, \quad \alpha_{30}=1+\int_0^1 \alpha_0(x)xdx, \quad \beta_{31}=1+\int_0^1 \beta_1(x)xdx,$$

$$\beta_{30}=1-\int_0^1 \beta_0(x)xdx, \quad K_{31}(\xi)=-a_1(\xi)\xi-\int_0^1 K_1(x,\xi)xdx,$$

$$K_{30}(\xi)=-a_0(\xi)\xi-\int_0^1 K_0(x,\xi)xdx, \quad \gamma_3=\int_0^1 f(x)xdx,$$

$$\alpha_{41}=\int_0^1 \alpha_1(x)dx-1, \quad \alpha_{40}=\int_0^1 \alpha_0(x)dx, \quad \beta_{41}=1+\int_0^1 \beta_1(x)dx, \qquad (6.2)$$

$$\beta_{40}=\int_0^1 \beta_0(x)dx, \quad K_{41}(\xi)=-a_1(\xi)-\int_0^1 K_1(x,\xi)dx,$$

$$K_{40}(\xi)=-a_0(\xi)-\int_0^1 K_0(x,\xi)dx, \quad \gamma_4=\int_0^1 f(x)dx.$$

If the condition

$$\triangle = \begin{vmatrix} \alpha_{11} & \alpha_{10} & \beta_{11} & \beta_{10} \\ \alpha_{21} & \alpha_{20} & \beta_{21} & \beta_{20} \\ \alpha_{31} & \alpha_{30} & \beta_{31} & \beta_{30} \\ \alpha_{41} & \alpha_{40} & \beta_{41} & \beta_{40} \end{vmatrix} \neq 0, \tag{6.3}$$

is satisfied, then the following statement is true for (6.1).

Theorem 2. *If the coefficients $K_{jn}(\xi)$ $j = \overline{1,4}$; $n = 0,1$ contained in the system (6.1) are continuous and for the coefficients of this system given in (2.2) and (6.2) the condition (6.3) is satisfied, the system (6.1) has a unique solution representable in the form:*

$$y'(0) = -\int_0^1 \sum_{j=1}^4 \sum_{n=0}^1 K_{jn}(\xi) \frac{\triangle_{j1}}{\triangle} y^{(n)}(\xi)\, d\xi + \sum_{j=1}^4 \gamma_j \frac{\triangle_{j1}}{\triangle},$$

$$y(0) = -\int_0^1 \sum_{j=1}^4 \sum_{n=0}^1 K_{jn}(\xi) \frac{\triangle_{j2}}{\triangle} y^{(n)}(\xi)\, d\xi + \sum_{j=1}^4 \gamma_j \frac{\triangle_{j2}}{\triangle},$$

$$y'(1) = -\int_0^1 \sum_{j=1}^4 \sum_{n=0}^1 K_{jn}(\xi) \frac{\triangle_{j3}}{\triangle} y^{(n)}(\xi)\, d\xi + \sum_{j=1}^4 \gamma_j \frac{\triangle_{j3}}{\triangle},$$

$$y(1) = -\int_0^1 \sum_{j=1}^4 \sum_{n=0}^1 K_{jn}(\xi) \frac{\triangle_{j4}}{\triangle} y^{(n)}(\xi)\, d\xi + \sum_{j=1}^4 \gamma_j \frac{\triangle_{j4}}{\triangle}, \tag{6.4}$$

here $\triangle_{js}$ $j,s = \overline{1,4}$ is a cofactor of the $j-$ th row and the $s-$th column element of the determinant (6.3).

7. Fredholm property. Finally, having written the boundary values determined in the form (6.4) in the left hand side of the basic expressions (4.1) and (4.2), from the first relations of these expressions we get a system of Fredholm integral equations of second kind for $y(t)$ and $y'(t)$ in the following form

$$y(t) = -\int_0^1 \sum_{n=0}^1 \left[a_n(\xi)\, Y(\xi - t) + \int_0^1 K_n(x,\xi) Y(x-t)dx \right] y^{(n)}(\xi)d\xi +$$

$$+\int_0^1 \sum_{n=0}^1 \left\{ \left[Y(1-t) + \int_0^1 \beta_1(x) Y(x-t)dx \right] \sum_{j=1}^4 K_{jn}(\xi) \frac{\triangle_{j3}}{\triangle} \right\} y^{(n)}(\xi)d\xi -$$

$$-\int_0^1 \sum_{n=0}^1 \left\{ \left[Y(-t) - \int_0^1 \alpha_1(x) Y(x-t)dx \right] \sum_{j=1}^4 K_{jn}(\xi) \frac{\triangle_{j1}}{\triangle} \right\} y^{(n)}(\xi)d\xi -$$

$$-\int_0^1 \sum_{n=0}^1 \left\{ \left[Y'(1-t) - \int_0^1 \beta_0(x) Y(x-t)dx \right] \sum_{j=1}^4 K_{jn}(\xi) \frac{\triangle_{j4}}{\triangle} \right\} y^{(n)}(\xi)d\xi +$$

$$+\int_0^1 \sum_{n=0}^1 \left\{ \left[Y'(-t) + \int_0^1 \alpha_0(x) Y(x-t)dx \right] \sum_{j=1}^4 K_{jn}(\xi) \frac{\triangle_{j2}}{\triangle} \right\} y^{(n)}(\xi)d\xi +$$

$$+\left\{ \int_0^1 f(x) Y(x-t)dx - \sum_{j=1}^4 \gamma_j \frac{\triangle_{j3}}{\triangle} \left[Y(1-t) + \int_0^1 \beta_1(x) Y(x-t)dx \right] \right\} +$$

$$+\sum_{j=1}^4 \gamma_j \frac{\triangle_{j1}}{\triangle} \left[Y(-t) - \int_0^1 \alpha_1(x) Y(x-t)dx \right] +$$

$$+\sum_{j=1}^4 \gamma_j \frac{\triangle_{j4}}{\triangle} \left[Y'(1-t) - \int_0^1 \beta_0(x) Y(x-t)dx \right] -$$

$$-\sum_{j=1}^4 \gamma_j \frac{\triangle_{j2}}{\triangle} \left[Y'(-t) + \int_0^1 \alpha_0(x) Y(x-t)dx \right], \quad t \in (0,1)\,, \; (7.1)$$

$$y'(t) = \int_0^1 \sum_{n=0}^1 \left[a_n(\xi)\, e(\xi - t) + \int_0^1 K_n(x,\xi)\, e(x-t)dx \right] y^{(n)}(\xi)d\xi -$$

$$-\int_0^1 \sum_{n=0}^1 \left\{ \left[e(1-t) + \int_0^1 \beta_1(x)\, e(x-t)dx \right] \sum_{j=1}^4 K_{jn}(\xi) \frac{\triangle_{j3}}{\triangle} \right\} y^{(n)}(\xi)d\xi -$$

$$-\int_0^1 \sum_{n=0}^1 \left\{ \left[-e(-t) + \int_0^1 \alpha_1(x)\, e(x-t)dx \right] \sum_{j=1}^4 K_{jn}(\xi) \frac{\triangle_{j1}}{\triangle} \right\} y^{(n)}(\xi)d\xi -$$

$$-\int_0^1 \sum_{n=0}^1 \left\{ \left[\int_0^1 \beta_0(x) e(x-t)dx \right] \sum_{j=1}^4 K_{jn}(\xi) \frac{\triangle_{j4}}{\triangle} \right\} y^{(n)}(\xi)d\xi -$$

$$-\int_0^1 \sum_{n=0}^1 \left\{ \left[\int_0^1 \alpha_0(x) e(x-t)dx \right] \sum_{j=1}^4 K_{jn}(\xi) \frac{\triangle_{j2}}{\triangle} \right\} y^{(n)}(\xi)d\xi +$$

$$+\left\{ -\int_0^1 f(x) e(x-t)dx + \sum_{j=1}^4 \gamma_j \frac{\triangle_{j3}}{\triangle} \left[e(1-t) + \int_0^1 \beta_1(x) e(x-t)dx \right] + \right.$$

$$+\sum_{j=1}^4 \gamma_j \frac{\triangle_{j1}}{\triangle} \left[-e(-t) + \int_0^1 \alpha_1(x) e(x-t)dx \right] +$$

$$+\sum_{j=1}^{4}\gamma_j\frac{\triangle_{j4}}{\triangle}\int_0^1\beta_0(x)e(x-t)dx+$$

$$\left.+\sum_{j=1}^{4}\gamma_j\frac{\triangle_{j2}}{\triangle}\int_0^1\alpha_0(x)e(x-t)dx\right\}, \quad t\in(0,1)\,. \tag{7.2}$$

This proves the following statement [4]:

Theorem 3. *Under conditions of Theorems 1 and 2 the boundary-value problem (2.1), (2.2) is equivalent to the system of second kind Fredholm integral equations (7.1) and (7.2).*

8. Main results

In this present paper, sufficient condition is found for Fredholm property of the problem under boundary condition containing general linear non-local and global term for general form second order integral-differential loaded equation.

The kernel of the second kind integral equation where the boundary value problem is reduced, is regular. In the paper, the problem is considered under the boundary condition (2.2) for second order equation (2.1). Though the considered equation is linear, -point problem is easily obtained from the integral given in the boundary condition. So, some of these points are Dirichlet type points, the remanings are Neumann type points. Finally, we notice that unlike [4]-[8], the equation of problem (2.1)-(2.2) is an integro-differential loaded equation. The boundary condition contains nonhomogeneous and global (containing integrals) terms.

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