Elman J. IBRAHIMOV, Saadat A. JAFAROVA

ON APPROXIMATION OF FUNCTIONS BY GEGEN BAWER SINGULAR INTEGRALS

Abstract

In paper [1] we considered a problem on approximation of locally summable functions by Gegenbawer singular integrals. Here, we introduce the notion of Gegebawer singular integral with identical approximation kernel and clarify their convergence both in the nutric of the space $L_{p,\lambda}$ and at typical points of Lebesgue type. The obtained results are the analogies of appropriate results from the monograph [3].

Denote by

$$A_t^\lambda f(x) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int\limits_0^\pi f(xt - \sqrt{x^2 - 1}\sqrt{t^2 - 1}\cos\varphi)(\sin\varphi)^{2\lambda - 1}d\varphi, \quad 0 < \lambda < \frac{1}{2},$$

a generalized shift function [1], by $L_{p,\lambda}[1,\infty)$ $1 \le p \le \infty$, a class of functions with finite norm

$$\|f\|_{p,\lambda} = \left(\int\limits_1^\infty |f(x)|^p (x^2 - 1)^{\lambda - \frac{1}{2}} dx \right)^{\frac{1}{p}}, \quad \text{or} \quad 1 \le p < \infty.$$

$$\|f\|_{\infty,\lambda} = \|f\|_\infty = \underset{x \in [1,\infty)}{vrai \sup} |f(x)|.$$

Let $f, g \in L_{1,\lambda}[1,\infty)$. The operator

$$(f * g)(x) = \int\limits_1^\infty g(u)A_u^\lambda f(x)(u^2 - 1)^{\lambda - \frac{1}{2}} du.$$

is a convolution of these functions.

the convolution exists almost everywhere on $[1,\infty)$ moreover $(f * g) \in L_{1,\lambda}$.

1. Gegenbauer singular integrals and these convergence

Let the parameter $\tau \in G \subset R$ and τ_0 be a concentration point of this set.

Definition 1.1. *The function $K_\tau^\lambda(x)$ determined on $[1,\infty)$ and dependent on the parameter τ is said to be a kernel if $K_\tau^\lambda(x) \in L_{1,\lambda}[1,\infty)$ for any $\tau \in G$ and the following equality is fulfilled:*

$$\lim_{\tau \to \tau_0} \int\limits_1^\infty K_\tau^\lambda(x)(x^2 - 1)^{\lambda - \frac{1}{2}} dx = 1 \tag{1.1}$$

Definition 1.2. *It for the kernel $K_\tau^\lambda(x)$ the condition*

$$\left\| K_\tau^\lambda(\cdot) \right\|_{1,\lambda} \le M < \infty, \tag{1.2}$$

are fulfilled uniformly with respect to $\tau \in G$ *and*

$$\lim_{\tau \to \tau_0} \int\limits_{ch\delta}^{\infty} \left| K_\tau^\lambda(x) \right| (x^2 - 1)^{\lambda - \frac{1}{2}} dx = 0, \tag{1.3}$$

for any $\delta > 0$, *then it is called an identical approximation kernel.*

Definition 1.3. *It* $K_\tau^\lambda(x)$ *is a kernel, the expression*

$$L_\tau^\lambda f(x) = \int\limits_1^{\infty} (t^2 - 1)^{\lambda - \frac{1}{2}} A_t^\lambda f(x) K_\tau^\lambda(t) dt \tag{1.4}$$

is said to be a singular Gegenbauer integral.

Theorem 1.1. *For any* $f \in L_{p,\lambda}[1, \infty)$ $(1 \le p < \infty)$ *the equality*

$$\lim_{t \to 1+0} \left\| A_t^\lambda f - f \right\|_{p,\lambda} = 0 \tag{1.5}$$

is valid.

This results was obtained in [2].

Theorem 1.2. *Let* $f \in L_{p,\lambda}[1, \infty)$ *and* $K_\tau^\lambda(x)$ *be a kernel. Then* $L_\tau^\lambda f(x) \in L_{p,\lambda}[1, \infty)$ *and the inequality*

$$\left\| L_\tau^\lambda f \right\|_{p,\lambda} \le \|f\|_{p,\lambda} \left\| K_\tau^\lambda(\cdot) \right\|_{1,\lambda}. \tag{1.6}$$

is valid.

Inequality (1.6) follows the convolution property if we take into account that $L_\tau^\lambda f(x) = (f * K_\tau^\lambda)(x)$ (see [4], p. 1082).

Theorem 1.3. *It* $f \in L_{p,\lambda}[1, \infty)$, $1 \le p < \infty$ *then for Gegenbauer singular integral (1.4) with identical approximation the equality*

$$\lim_{\tau \to \tau_0} \left\| L_t^\lambda f - f \right\|_{p,\lambda} = 0$$

is valid.

Proof. From (1.4) we have

$$L_t^\lambda f(x) - f(x) = \int\limits_1^{\infty} \left[A_u^\lambda f(x) - f(x) \right] K_\tau^\lambda(u)(u^2 - 1)^{\lambda - \frac{1}{2}} du +$$

$$+ f(x) \left[\int\limits_1^{\infty} K_\tau^\lambda(u)(u^2 - 1)^{\lambda - \frac{1}{2}} du - 1 \right].$$

Using Minkovskiy's generalized inequality [5], p. 17)

$$\left\{ \int\limits_X \left[\int\limits_Y f(x,y) dy \right]^p dx \right\}^{\frac{1}{p}} < \int\limits_Y \left[\int\limits_X f(x,y)^p dx \right]^{\frac{1}{p}} dy,$$

we get

$$\left\| L_\tau^\lambda f - f \right\|_{p,\lambda} \leq$$

$$\leq \left\{ \int\limits_1^\infty \left| \int\limits_1^\infty \left[A_u^\lambda f(x) - f(x) \right] K_\tau^\lambda(u)(u^2-1)^{\lambda-\frac{1}{2}} du \right|^p (x^2-1)^{\lambda-\frac{1}{2}} dx \right\}^{\frac{1}{p}} +$$

$$+ \left| \int\limits_1^\infty K_\tau^\lambda(u)(u^2-1)^{\lambda-\frac{1}{2}} du - 1 \right| \left(\int\limits_1^\infty |f(x)|^p (x^2-1)^{\lambda-\frac{1}{2}} dx \right)^{1/p} \leq$$

$$\leq \int\limits_1^\infty \left\| A_u^\lambda f - f \right\|_{p,\lambda} \left| K_\tau^\lambda(u) \right| (u^2-1)^{\lambda-\frac{1}{2}} du + \|f\|_{p,\lambda} \left| \int\limits_1^\infty K_\tau^\lambda(u)(u^2-1)^{\lambda-\frac{1}{2}} du - 1 \right|.$$

The second summand tends to zero as $\tau \to \tau_0$ by (1.1). For the first one we have

$$A = \left(\int\limits_1^{ch\delta} + \int\limits_{ch\delta}^\infty \right) \left| K_\tau^\lambda(u) \right| \left\| A_u^\lambda f - f \right\|_{p,\lambda} (u^2-1)^{\lambda-\frac{1}{2}} du = A_1 + A_2.$$

By theorem 1.1 $\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0$ such that $\left\| A_u^\lambda f - f \right\|_{p,\lambda} < \varepsilon$ as soon as $u < 1 + \delta$ and therefore by (1.2)

$$A_1 < \varepsilon \int\limits_1^{ch\delta} \left| K_\tau^\lambda(u) \right| (u^2-1)^{\lambda-\frac{1}{2}} du < \varepsilon M.$$

By (1.3) $A_2 \to 0$ as $\tau \to \tau_0$ since by lemma 2 from [2]

$$\left\| A_u^\lambda f - f \right\|_{p,\lambda} \leq \left\| A_u^\lambda f \right\|_{p,\lambda} + \|f\|_{p,\lambda} \leq 2 \|f\|_{p,\lambda}.$$

This proves the theorem.

Weak convergence of Gegenbauer singular integrals to the function $L_{\infty,\lambda}[1,\infty)$ also holds in the space $f(x)$.

Theorem 1.4. *Let $f \in L_{\infty,\lambda}$ and the kernel $K_\tau^\lambda(x)$ of Gegenbauer singular integral is an identity approximation kernel. Then for any function $g \in L_{1,\lambda}$, the equality*

$$\lim_{\tau \to \tau_0} \int\limits_1^\infty [L_\tau^\lambda f(x) - f(x)]g(x)(x^2-1)^{\lambda-\frac{1}{2}} dx = 0.$$

is valid.

Proof. It is easy to show that

$$J(\tau) = \int\limits_1^\infty [L_\tau^\lambda f(x) - f(x)]g(x)(x^2-1)^{\lambda-\frac{1}{2}} dx =$$

$$\int\limits_1^\infty g(x) \left\{ \int\limits_1^\infty \left[A_u^\lambda f(x) - f(x) \right] K_\tau^\lambda(u)(u^2-1)^{\lambda-\frac{1}{2}} du + f(x) \times \right.$$

$$\times \left[\int\limits_1^\infty K_\tau^\lambda(u)(u^2-1)^{\lambda-\frac{1}{2}} du - 1 \right] \right\} (x^2-1)^{\lambda-\frac{1}{2}} dx = J_1(\tau) + J_2(\tau). \qquad (1.7)$$

Since

$$J_1(\tau) = \int\limits_1^\infty K_\tau^\lambda(u) \left\{ \int\limits_1^\infty g(x) \left[A_u^\lambda f(x) - f(x) \right] (x^2-1)^{\lambda-\frac{1}{2}} dx \right\} (u^2-1)^{\lambda-\frac{1}{2}} du,$$

then using the equality

$$\int\limits_1^\infty g(x) \left[A_u^\lambda f(x) - f(x) \right] (x^2-1)^{\lambda-\frac{1}{2}} dx =$$

$$= \int\limits_1^\infty f(x) \left[A_u^\lambda g(x) - g(x) \right] (x^2-1)^{\lambda-\frac{1}{2}} dx,$$

that follows from the symmetry property of convolution [2], lemma 3), we get

$$|J_1| \leq \|f\|_{\infty,\lambda} \int\limits_1^\infty \left| K_\tau^\lambda(u) \right| \left\| A_u^\lambda g - g \right\|_{1,\lambda} (u^2-1)^{\lambda-\frac{1}{2}} du =$$

$$= \|f\|_{\infty,\lambda} \left(\int\limits_1^{ch\delta} + \int\limits_{ch\delta}^\infty \right) \left| K_\tau^\lambda(u) \right| \left\| A_u^\lambda g - g \right\|_{1,\lambda} (u^2-1)^{\lambda-\frac{1}{2}} du \leq$$

$$\leq M(J_{1.1}(\tau) + J_{1.2}(\tau)). \qquad (1.8)$$

By theorem 1.1 $\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0$ such that inequality $\left\| A_u^\lambda g - g \right\|_{1,\lambda} < \varepsilon$ holds as soon as $u \in (1, ch\delta)$. But then

$$|J_{1.1}(\tau)| < \varepsilon. \qquad (1.9)$$

By lemma 2 from [2]

$$\left\| A_u^\lambda g - g \right\|_{1,\lambda} \leq 2 \|g\|_{1.\lambda}.$$

But then

$$|J_{1.2}(\tau)| \leq 2 \|g\|_{1,\lambda} \int\limits_{ch\delta}^\infty \left| K_\tau^\lambda(u) \right| (u^2-1)^{\lambda-\frac{1}{2}} du.$$

Hence and from (1.3) it follows that

$$\lim_{\tau \to \tau_0} J_{1.2.}(\tau) = 0. \qquad (1.10)$$

Concerning the integral J_2, by (1.1)

$$\lim_{\tau \to \tau_0} J_{.2}(\tau) = 0. \qquad (1.11)$$

Taking into account (1.9) and (1.10) in (1.8), we see that

$$\lim_{\tau \to \tau_0} J_{1.}(\tau) = 0. \tag{1.12}$$

The statement of the theorem follows from (1.7) if we take into account (1.11) and (1.12).

Remark 1. Condition (1.1) is equivalent to the condition

$$\lim_{\tau \to \tau_0} \int_0^\infty K_\tau^\lambda(chu) sh^{2\lambda} u \, du = 1, \tag{1.13}$$

the condition (1.3) to the condition

$$\lim_{\tau \to \tau_0} \int_\delta^\infty \left| K_\tau^\lambda(chu) \right| sh^{2\lambda} u \, du = 0. \tag{1.14}$$

As realization of the theorems proved above, we consider the average

$$S_r^\lambda f(x) = \frac{2^{\lambda+1}}{(shr)^{2\lambda+1}} \int_0^r A_{chu}^\lambda f(x) sh^{2\lambda} u \, du. \tag{1.15}$$

Let the parameter $r \in (0, \infty)$ and $r \to r_0 = 0$.

Assume

$$K_r^\lambda(chu) = \begin{cases} (2\lambda+1)(shr)^{-2\lambda-1}, & x \in (0, r) \\ 0, & x \in [r, \infty] \end{cases} \tag{1.16}$$

Show that the function (1.16) is an identity approximation kernel. Really, it follows from (1.13) and (1.16) that i.e. (1.3) holds for

$$\lim_{\tau \to 0} \int_0^\infty K_\tau^\lambda(chu) sh^{2\lambda} u \, du = \lim_{\tau \to 0} \frac{2\lambda+1}{(shr)^{2\lambda+1}} \int_0^r sh^{2\lambda} u \, du =$$

$$= \lim_{\tau \to 0} \frac{2\lambda+1}{(shr)^{2\lambda+1}} \int_0^r u^{2\lambda} du = \lim_{\tau \to 0} \left(\frac{r}{shr} \right)^{2\lambda+1} = 1.$$

The condition (1.14) follows from the definition of (1.16).

It remains to prove property (1.2).

$$\left\| K_r^\lambda(\cdot) \right\|_{1,\lambda} = \frac{2\lambda+1}{(shr)^{2\lambda+1}} \int_0^r sh^{2\lambda} u \, du \le \frac{(2\lambda+1) r sh^{2\lambda} r}{(shr)^{2\lambda+1}} = \frac{(2\lambda+1)r}{shr} \le 2\lambda+1 < 2.$$

Thus, integral (1.15) is Gegenbauer singular integral and according to theorems 1.2 and (13) the following conjecture is valid for it.

Conjecture 1. It $f \in L_{p,\lambda}$, $1 \le p < \infty$ then the relation

a) $\left\| S_r^\lambda f \right\|_{p,\lambda} \le (2\lambda + 1) \left\| f \right\|_{p,\lambda}$.

b) $\lim\limits_{r\to 0}\left\|S_r^\lambda f - f\right\|_{p,\lambda} = 0$ hold.

2. On convergence of Gegenbauer singular integrals at typical points

In the previous section we proved theorems on strong and weak convergence of Gegenbauer singular operators in the spaces $L_{p,\lambda}$. Here, we consider the problems on convergence at separate points reminding the Lebesgue points and Lebesgue points of order p.

Definition 2.4. *Let $f \in L_{p,\lambda}[1,\infty)$, $1 \le p \le \infty$. The point $x \in [1,\infty)$ is called $(L\text{-}Q)_p$ Lebesgue-Gegenbauer point of the function f if at this point the equality*

$$\lim_{\tau\to 0}\left(sh\frac{r}{2}\right)^{-(2\lambda+1)}\int\limits_0^r \left|A_{chu}^\lambda f(x) - f(x)\right|^p sh^{2\lambda}tdt = 0$$

is valid.

Theorem 2.5. *Let $f \in L_{1,\lambda}[1,\infty)$. The identity approximation kernel $K_\tau^\lambda(chu)$ have a derivative and satisfy the following conditions:*

$$\lim_{\tau\to\tau_0}\sup_{u\ge\delta}\left|K_\tau^\lambda(chu)\right| = 0 \tag{2.1}$$

and

$$\int\limits_0^\infty \left|\left[K_\tau^\lambda(chu)\right]'\right|(sh\ u)^{2\lambda+1}du \le C - conxt. \tag{2.2}$$

for any $0 < \delta$ and $\tau \in G$.

Then, at each $(L\text{-}Q)_1$-point $x \in [1,\infty)$ of the function $f(x)$ the equality

$$\lim_{\tau\to\tau_0}L_\tau^\lambda f(x) = f(x).$$

is valid.

Proof. From (1.4) we have

$$L_\tau^\lambda f(x) - f(x) = \int\limits_0^\infty \left|A_{chu}^\lambda f(x) - f(x)\right|K_\tau^\lambda(chu)sh^{2\lambda}udu +$$

$$+f(x)\left[\int\limits_0^\infty K_\tau^\lambda(chu)sh^{2\lambda}udu - 1\right] = J_1(\tau) + J_2(\tau). \tag{2.3}$$

It follows from (1.13) that

$$\lim_{\tau\to\tau_0}J_2(\tau) = 0. \tag{2.4}$$

Assume

$$\varphi(u) = \int\limits_0^u \left|A_{chu}^\lambda f(x) - f(x)\right|sh^{2\lambda}tdt.$$

By definition of $(L\text{-}Q)_p$-point $\forall\varepsilon > 0\ \exists\delta_0 > 0$ such that for all $u \le \delta < \delta_0$ the inequality

$$\varphi(u) < \varepsilon\cdot u^{2\lambda+1}. \tag{2.5}$$

will hold.

Now, expand J_1 by the scheme

$$J_1 = \left(\int\limits_0^\delta + \int\limits_\delta^\infty \right) \left[A_{chu}^\lambda f(x) - f(x) \right] K_\tau^\lambda(chu) sh^{2\lambda} u \, du = J_{1.1} + J_{1.2}. \qquad (2.6)$$

For $J_{1.1}$ we have

$$J_{1.1} = \int\limits_0^\delta \varphi'(u) K_\tau^\lambda(chu) du = \int\limits_0^\delta K_\tau^\lambda(chu) d\varphi(u) = \varphi(u) K_\tau^\lambda(chu) \Big|_0^\delta -$$

$$- \int\limits_0^\delta \varphi(u) \left[K_\tau^\lambda(chu) \right]' du = \varphi(\delta) K_\tau^\lambda(ch\delta) - \int\limits_0^\delta \varphi(u) \left[K_\tau^\lambda(chu) \right]' du.$$

It follows from (2.5) that

$$|J_{1.1}| \leq \varphi(\delta) \left| K_\tau^\lambda(chu) \right| + \varepsilon \int\limits_0^\delta u^{2\lambda+1} \left| \left[K_\tau^\lambda(chu) \right]' \right| du \leq$$

$$\leq \varepsilon \left\{ \delta^{2\lambda+1} \left| K_\tau^\lambda(chu) \right| + \int\limits_0^\delta u^{2\lambda+1} \left| \left[K_\tau^\lambda(chu) \right]' \right| du \right\}. \qquad (2.7)$$

Since

$$\int\limits_0^\delta t^{2\lambda} K_\tau^\lambda(cht) dt = \frac{1}{2\lambda+1} \int\limits_0^\delta K_\tau^\lambda(cht) dt^{2\lambda+1} =$$

$$= \frac{1}{2\lambda+1} \left(\delta^{2\lambda+1} K_\tau^\lambda(ch\delta) - \int\limits_0^\delta t^{2\lambda+1} \left[K_\tau^\lambda(cht) \right]' dt \right),$$

then

$$\delta^{2\lambda+1} K_\tau^\lambda(ch\delta) = (2\lambda+1) \int\limits_0^\delta t^{2\lambda} K_\tau^\lambda(cht) dt + \int\limits_0^\delta t^{2\lambda+1} \left[K_\tau^\lambda(cht) \right]' dt.$$

Hence and from (2.2) it follows that

$$\left| \delta^{2\lambda+1} K_\tau^\lambda(ch\delta) \right| \leq (2\lambda+1) \int\limits_0^\infty t^{2\lambda} \left| K_\tau^\lambda(cht) \right| dt + \int\limits_0^\infty t^{2\lambda+1} \left[K_\tau^\lambda(cht) \right]' dt \leq$$

$$\leq (2\lambda+1) \left\| K_\tau^\lambda(\cdot) \right\|_{1,\lambda} + C \leq (2\lambda+1) M + C. \qquad (2.8)$$

From (2.7) and (2.8) we get

$$|J_{1.1}| < \varepsilon(c_\lambda M + 2C). \qquad (2.9)$$

Consider the integral $J_{1.2}$.

$$|J_{1.2}| \leq \int_\delta^\infty \left|A_{chu}^\lambda f(x) - f(x)\right| \left|K_\tau^\lambda(chu)\right| sh^{2\lambda}udu \leq$$

$$\leq \int_\delta^\infty \left|A_{chu}^\lambda f(x)\right| \left|K_\tau^\lambda(chu)\right| sh^{2\lambda}udu + |f(x)| \int_\delta^\infty \left|K_\tau^\lambda(chu)\right| sh^{2\lambda}udu \leq$$

$$\leq \sup_{\delta \leq u < \infty} \left|K_\tau^\lambda(chu)\right| \int_0^\infty \left|A_{chu}^\lambda f(x)\right| sh^{2\lambda}udu + |f(x)| \int_\delta^\infty \left|K_\tau^\lambda(chu)\right| sh^{2\lambda}udu.$$

Whence, allowing for lemma 2, from [2] (1.14) and (2.1) we get:

$$\lim_{\tau \to \tau_0} J_{1,2} = 0. \tag{2.10}$$

It follows from (2.9), (2.10) and (2.6) that

$$\lim_{\tau \to \tau_0} J_1 = 0. \tag{2.11}$$

The statement of the lemma follow from (2.4) (2.11) and (2.1).

Theorem 2.6. *Let* $f \in L_{1,\lambda}[1, \infty)$ *and* $K_\tau^\lambda(chx)$ *be a positive identity approximation kernel satisfying the following conditions:*

a) The function $K_\tau^\lambda(chx)$ *monotonically increases on* $(0, \delta)$, $\delta > 0$ *and is bounded on it*

$$K_\tau^\lambda(chx) \leq C - const \quad x \in (0, \delta),$$

b) Monotonically decreases on $[\delta, \infty)$.

Then, at each $(L–Q)_1$point, for Gegenbauer singular integral (1.4) the equality

$$\lim_{\tau \to \tau_0} L_\tau^\lambda f(x) = f(x).$$

holds.

Proof. From (1.4) we have

$$L_\tau^\lambda f(x) - f(x) = \int_0^\infty \left[A_{chu}^\lambda f(x) - f(x)\right] K_\tau^\lambda(chu)du +$$

$$+ f(x) \left[\int_0^\delta K_\tau^\lambda(chu)sh^{2\lambda}udu - 1\right] = J_1(\tau) + J_2(\tau). \tag{2.12}$$

By (1.13)

$$\lim_{\tau \to \tau_0} J_2 = 0. \tag{2.13}$$

Consider the integral $J_1(\tau)$.

$$J_1(\tau) = \left(\int_0^\delta + \int_\delta^\infty\right) \left[A_{chu}^\lambda f(x) - f(x)\right] K_\tau^\lambda(chu)du = J_{1.1}(\tau) + J_{1.2}(\tau). \tag{2.14}$$

By condition b) of the theorem, for $\delta < \dfrac{x}{2} < \infty$ we have:

$$\int\limits_{\delta}^{\infty} K_{\tau}^{\lambda}(chu)sh^{2\lambda}udu \geq \int\limits_{\frac{x}{2}}^{\infty} K_{\tau}^{\lambda}(chu)sh^{2\lambda}udu \geq \int\limits_{\frac{x}{2}}^{x} K_{\tau}^{\lambda}(chu)sh^{2\lambda}udu >$$

$$> (sh^2\frac{x}{2})K_{\tau}^{\lambda}(chx)\frac{x}{2} \geq \left(\frac{x}{2}\right)^{2\lambda+1} K_{\tau}^{\lambda}(chx).$$

Hence and from (1.14) we have

$$\lim_{\tau\to\tau_0}\sup_{x>\delta}K_{\tau}^{\lambda}(chx) = 0. \tag{2.15}$$

Since,

$$|J_{1.2}| \leq \|f\|_{1,\lambda}\sup_{x>\delta}K_{\tau}^{\lambda}(chx) + |f(x)|\int\limits_{\delta}^{\infty} K_{\tau}^{\lambda}(chu)sh^{2\lambda}udu,$$

then it follows from (2.15) and (1.14) that

$$\lim_{\tau\to\tau_0} J_{1.2} = 0. \tag{2.16}$$

For the integral $J_{1.1}(\tau)$, by condition a) we have:

$$|J_{1.1}(\tau)| \leq \int\limits_{0}^{\delta} \left|A_{chu}^{\lambda}f(x) - f(x)\right| K_{\tau}^{\lambda}(chu)sh^{2\lambda}udu \leq \varepsilon\delta^{2\lambda+1}K_{\tau}^{\lambda}(ch\delta) \leq C_{\lambda}\varepsilon. \tag{2.17}$$

Allowing for (2.16) and (2.17), in (2.14) we get

$$\lim_{\tau\to\tau_0} J_1 = 0. \tag{2.18}$$

The statement of the theorem follows from (2.13), (2.18) and (2.12).

Theorem 2.7. *It the function f is continuous and summable on the segment $[1,\infty)$, and $K_{\tau}^{\lambda}(chx) \geq 0$ is an identity approximation kernel that monotonically increases on $(0,\delta)$, $\delta > 0$ and monotonically decreases on $[\delta,\infty)$ then the equality $x \in [1,\infty)$ holds as each point*

$$\lim_{\tau\to\tau_0} L_{\tau}^{\lambda}f(x) = f(x).$$

Proof. If follows from the continuity of the function $f(x)$ that $\forall \varepsilon > 0$ $\exists \delta > 0$ such that for all $u < \delta$ the inequality $\left|A_{chu}^{\lambda}f(x) - f(x)\right| < \varepsilon$ is fulfilled. But, then

$$|J_{1.1}(\tau)| \leq \int\limits_{0}^{\infty} \left|A_{chu}^{\lambda}f(x) - f(x)\right| K_{\tau}^{\lambda}(chu)sh^{2\lambda}udu \leq$$

$$\leq \varepsilon \int\limits_{0}^{\infty} K_{\tau}^{\lambda}(chu)sh^{2\lambda}udu < \varepsilon \left\|K_{\tau}^{\lambda}(\cdot)\right\|_{1,\lambda}.$$

All the remainings follow from the proof of the previous theorem.

Theorem 2.8. *Let* $f \in L_{p,\lambda}[1,\infty)$, $1 \le p < \infty$. *The kernel* $K_\tau^\lambda(x)$ *have a differentiable majorant* $\phi_\tau(x)$ $\left(\left|K_\tau^\lambda(x)\right| \le \phi_\tau(x)\right)$ *satisfying the conditions:*

a) $\|\phi_\tau\|_{1,\lambda} \le M < \infty$

b) $\int\limits_0^\delta \left|\phi_\tau'(chu)\right|(shu)^{2\lambda+1}du \le C < \infty$

c) $\lim\limits_{\tau \to \tau_0} \sup\limits_{u > \delta} \phi_\tau'(chu) = 0$

d) $\lim\limits_{\tau \to \tau_0} \int\limits_0^\delta \phi_\tau'(chu)sh^{2\lambda}udu = 0$ *for any* $0 < \delta < 1$.

Then at each $(L - Q)_p$-point $x \in [1, \infty)$ it holds the equality

$$\lim_{\tau \to \tau_0} L_\tau^\lambda f(x) = f(x).$$

Proof. From (1.4) we have

$$L_\tau^\lambda f(x) - f(x) = \int\limits_1^\infty \left[A_{chu}^\lambda f(x) - f(x)\right] K_\tau^\lambda(t)(t^2 - 1)^{\lambda - \frac{1}{2}}dt +$$

$$+f(x)\left[\int\limits_1^\infty K_\tau^\lambda(t)(t^2 - 1)^{\lambda - \frac{1}{2}}dt - 1\right] = J_1(\tau) + J_2(\tau). \tag{2.19}$$

By (1.1)

$$\lim_{\tau \to \tau_0} J_2(\tau) = 0. \tag{2.20}$$

By the Holder inequality, for the integral $J_1(\tau)$ we have

$$|J_1(\tau)| \le \int\limits_0^\infty \left|A_{chu}^\lambda f(x) - f(x)\right| K_\tau^\lambda(chu)sh^{2\lambda}udu \le$$

$$\le \int\limits_0^\infty \left|A_{chu}^\lambda f(x) - f(x)\right| \phi_\tau(chu)sh^{2\lambda}udu \le$$

$$\le \left(\int\limits_0^\infty \phi_\tau(chu)sh^{2\lambda}udu\right)^{1/q} \left(\int\limits_0^\infty \left|A_{chu}^\lambda f(x) - f(x)\right|^p \phi_\tau(chu)sh^{2\lambda}udu\right)^{1/p} \le$$

$$\le M^{1/q}\left(\int\limits_0^\infty \left|A_{chu}^\lambda f(x) - f(x)\right|^p \phi_\tau(chu)sh^{2\lambda}udu\right)^{1/p}. \tag{2.21}$$

Consider the integral

$$\left(\int\limits_0^\delta + \int\limits_\delta^\infty\right) \left|A_{chu}^\lambda f(x) - f(x)\right|^p \phi_\tau(chu)sh^{2\lambda}udu = J_{1.1}(\tau) + J_{1.2}(\tau). \tag{2.22}$$

For the estimation of the integral $J_{1.1}(\tau)$ we assume

$$\varphi(u) = \int\limits_0^u \left| A_{chu}^\lambda f(x) - f(x) \right|^p sh^{2\lambda} t \, dt.$$

By definition of $(L\text{–}Q)_p$point,$\forall \varepsilon > 0 \ \exists \delta > 0$ such that for all $u \leq \delta$ we'll have the inequality $\varphi(u) < \varepsilon u^{2\lambda+1}$. Taking into account this inequality, we get

$$|J_{1.1}(\tau)| = \left| \int\limits_0^\delta \phi_\tau(chu) d\varphi(u) \right| = \left| \varphi(\delta)\varphi_\tau(ch\delta) - \int\limits_0^\delta \varphi(u)\varphi_\tau'(chu) \, du \right| <$$

$$< \varepsilon \left[\delta^{2\lambda+1}\varphi_\tau(ch\delta) + \int\limits_0^\delta \left| \varphi_\tau'(chu) \right| (sh \, u)^{2\lambda+1} \, du \right] < \varepsilon(C_\lambda + C). \qquad (2.23)$$

For $J_{1.2}$ we have

$$J_{1.2}(\tau) = \int\limits_\delta^\infty |A_{chu}f(x) - f(x)|^p \, \phi_\tau(chu)sh^{2\lambda}u \, du \leq$$

$$\leq 2^p \int\limits_\delta^\infty |A_{chu}f(x)|^p \, \phi_\tau(chu)sh^{2\lambda}u \, du + 2^p |f(x)|^p \int\limits_\delta^\infty \phi_\tau(chu)sh^{2\lambda}u \, du \leq$$

$$\leq 2^p \sup_{u \geq \delta} \phi_\tau(chu) \int\limits_\delta^\infty \left| A_{chu}^\lambda f(x) \right|^p sh^{2\lambda}u \, du +$$

$$+2^p |f(x)|^p \int\limits_\delta^\infty \phi_\tau(chu)sh^{2\lambda}u \, du = 2^p \sup_{u \geq \delta} \phi_\tau(chu) \|f\|_{p,\lambda} + 2^p |f(x)|^p \int\limits_\delta^\infty \phi_\tau(chu)sh^{2\lambda}u \, du.$$

Taking into account conditions c) and d) of the theorem, we get

$$\lim_{\tau \to \tau_0} J_{1.2}(\tau) = 0. \qquad (2.24)$$

It follows from (2.23), (2.24), (2.22) and (2.21) that

$$\lim_{\tau \to \tau_0} J_1(\tau) = 0. \qquad (2.25)$$

Taking into account (2.20) and (2.25) in (2.19), we get the theorem statement.

References

[1]. Guliyev V.S., Ibrahimov E.J. *On estimating the approximation of locally summable functions by Gegenbauer singular integrals.* Georgian Mathematical Journal, 2008, vol. 15, No 2, pp. 251-262.

[2]. Guliyev V.S., Ibrahimov E.J. *Equivalent normalizations of the space of functions associated with Gegenbauer generalized schift*, Analysis Math. 2008, 34, pp. 83-103.

[3]. Mamedov R.G. *Mellin's transformation and approximation theory.* Baku, Elm, 1991, 272 p. (Russian).

[4]. Vagif S. G., Elman J.I. *Calderon reproducing formula associated with the Gegenbauer operator on the haldf-line,* J. Math. Anal. and Appl. 2007, 335, pp. 1079-1094.

[5]. Hardy H.H., Littlewood D.E., Polia G. *Ineqaulities IL,* Moscow, 1948, 456 p. (Russian).

Elman J. Ibrahimov
Azerbaijan State Oil Academy
20, Azadlig av. AZ 1601, Baku, Azerbaijan
Tel.: (99412) 472 82 96 (apt.).

Saadat A. Jafarova
Azerbaijan State Economic University,
6, Istiglaliyyat str., AZ1001, Baku, Azerbaijan;