Proceedings of IMM of NAS of Azerbaijan, 2010, vol. XXXIII (XLI), pp. 95-100. _ 95

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ON MODULI OF SMOOTHNESS IN ORLICZ CLASSES

Abstract

Let T be the unit circle in the complex plane. In this work the relationship between the modulus of smoothness and the best approximation in Orlicz space $L_M(T)$, have been investigated.

1. Introduction, auxiliary and main result

A convex and continuous function $M : [0, \infty) \to [0, \infty)$ where $M(0) = 0$, $M(t) > 0$ for $t > 0$, and

$$\lim_{t\to 0} \frac{M(t)}{t} = \lim_{t\to\infty} \frac{t}{M(t)} = 0,$$

is called a Young function. The complementary Young function N of M is defined by

$$N(t) := \max_{y\geq 0} \{ty - M(y)\}, \ t \geq 0$$

Let N be the complementary Young function for a Young function M, then

$$t \leq M^{-1}(t)N^{-1}(t) \leq 2t, \quad t \geq 0,$$

where M^{-1} is the inverse function of M.

Suppose that T denotes the unit circle, $\mathbb{C}$ the complex plane, and $L_p(T)$, $1 \leq p \leq \infty$, the Lebesque space of measurable complex-valued functions on T.

Assume that M is the N-function and N is its complementary function. $L_M(T)$ denotes the linear space of Lebesque measurable functions $f : T \to \mathbb{C}$ satisfying the condition

$$\int_T M(\alpha \, |f(z)| \, dz) < \infty$$

for some $\alpha > 0$.

The linear span of $L_M(T)$ equipped with the Orlicz norm

$$\|f\|_{L_M(T)} := \sup \left\{ \int_T |f(z)g(z)| \, |dz| : g \in L_N(T), \ \rho(g, N) \leq 1 \right\},$$

where

$$\rho(g, N) := \int_T N(|g(z)|) \, |dz| \, ,$$

or with the Luxemburg norm

$$\|f\|^*_{L_M(T)} := \inf \left\{ x > 0 : \rho\left(\frac{f}{\lambda}, M\right) \leq 1 \right\}$$

is Banach space which is called the Orlicz space $L_M(T)$ [18, p.69].

Since

$$\|f\|^*_{L_M(T)} \leq \|f\|_{L_M(T)} \leq 2 \, \|f\|^*_{L_M(T)} \, .$$

these norms are equivalent [18, p.80].

In addition, the Orlicz norm can be determined by means of the Luxemburg norm [18. pp.79-80].

$$\|f\|_{L_M(T)} := \sup\left\{\int_T |f(x)g(x)|\,dx : \|g\|^*_{L_N(T)} \le 1\right\}$$

and then the Hölder inequalities

$$\int_T |f(x)g(x)|\,dx \le \|f\|_{L_M(T)} \cdot \|g\|^*_{L_N(T)},$$

$$\int_T |f(x)g(x)|\,dx \le \|f\|^*_{L_M(T)} \cdot \|g\|_{L_N(T)}$$

hold for every $f \in L_M(T)$ and $g \in L_N(T)$ [18, p.80].

Every function in $L_M(T)$ is integrable on T [18 , p.50], i.e.

$$L_M(T) \subset L^1(T).$$

If we take the Young function M (t) $= \frac{t^p}{p}$, the Lebesque space $L_p(T)$, $1 < p < \infty$ is isomorphic to the Orlich space $L_M(T)$.

If

$$\lim_{x\to\infty} \sup \frac{M(2x)}{M(x)} < \infty,$$

then N-function M holds the Δ_2-condition.

The Orlicz space $L_M(T)$ is reflexive if and only if the N-function M and its complementary function N both satisfy the Δ_2-condition [27, p.113].

For $r = 1, 2, 3, \cdots$ the r-th modulus of smoothness of a function $f \in L_M(T)$ defines with

$$w^r_M(\delta, f) := \sup_{|h|\le\delta} \|\Delta^r_h f\|_{L_M(T)}, \delta > 0,$$

where

$$\Delta^r_h f(.) := \sum_{\nu=0}^{r} (-1)^{r-\nu} \binom{r}{\nu} f(. + \nu h).$$

Now, we give some properties of $w_r(\delta, f)_M$.

1) $w_r(\delta, f)_M$ is a monotone non-decreasing function of $\delta \ge 0$.

2) $w_r(0, f)_M = 0$.

3) If M(u) satisfies the Δ_2-condition and $f \in L_M$, then $w_p(\delta, f)_M \to 0$ as $\delta \to 0$.

4) If $f^{(n-1)}(x)$is absolutely continuous and $f^{(n)} \in L_M$ then we have

$$w_{r+n}(\delta, f)_M \le \delta^n w_r(\delta, f^{(n)})_M. \tag{1}$$

5) For any non-negative integer m we get

$$w_r(m\delta, f)_M \le m^r w_r(\delta, f)_M. \tag{2}$$

Let

$$E_n(f)_M := \inf_{T\in\mathcal{T}_n} \|f - T\|_{L_M(T)}, f \in L_M(T),$$

where $\mathcal{T}_n$ is the class of trigonometric polynomials of degree not greater than $n \geq 1$.

An upper bound for $E_n(f)_M$, in terms of the modulus of smoothness of arbitrary order $w(f, n^{-1})_M$, has been studied in [4, 26]. In this paper we formulate a theorem (see Theorem 3) that gives a condition under which $E_n(f)_M$ can easily be estimated from below. This theorem is a generalization of the corresponding Yu.S. Kolomoitsev's result [23] to the case of the spaces $L_M(T)$. Simillar problems in the Lebesque spaces have been studied in [23],[28].

In terms of the usual modulus of smoothness, these problems in the Lebesque, Smirnov and Orlicz spaces defined on the complex domains with the various boundary conditions were investigated by Walsh-Russel [29], Al'per [1], Kokilashvili [19,20], Andersson [2], Dyn'kin [7], Ibragimav-Mamedkhanov [9], Israfilov [10,11,12], Israfilov-Guven [13,15], Israfilov-Akgn [14], Akgn- Israfilov [3,4,5], Mamedkhanov [24], Mhaskar [25], Ramazanov [26], Jafarov [17] and other mathematicians.

Throughout this paper we shall denote $c_1, c_2, ...$ constants depending only on numbers that are not important for the questions of interest.

We shall also employ the symbol $A \preceq B$, denoting that $A \leq CB$, where $C = const > 0$ does not depend on A or B; and $A \asymp B$, if simultaneously $A \preceq B$ and $B \preceq A$.

We will use the following auxiliary results

Theorem 1 [4, 26]. *Let $L_M(T)$ be a reflexive Orlicz space on $T, f \in L_M(T)$ and $n, r \in N$. Then we have*

$$E_n(f)_M \leq c_1 w_r(f, \frac{1}{n})_M,$$

where the constant $c_1 > 0$ independs on n.

Theorem 2 [4, 22]. *Suppose that $L_M(T)$ be a reflexive Orlicz space on T, $f \in L_M(T)$, and $n, r \in N$. Then we get*

$$w_r(f, \frac{1}{n})_M \leq \frac{c_2}{n^r} \sum_{k=1}^{n} k^{r-1} E_k(f)_M,$$

where the constant $c_2 > 0$ depens only on r and M.

The main results of this work are the following:

Theorem 3. *Assume that $f \in L_M(T)$ and $r \in N$. There exists a constant $D > 0$ such that*

$$w_r(f, \frac{1}{n})_M \leq D E_n(f)_M \; mboxforall \; n \in N, \tag{3}$$

if and only if for a certain $k > r$, there exist a constant $B > 0$ such that

$$w_r(f, h)_M \leq B w_k(f, h)_M \quad for \quad all \quad h \in (0, 1]. \tag{4}$$

2. Proof of main result

Proof of Theorem 3. Suppose that the condition (4) is satisfied. Then, properties (1) and (2) of the modulus of smoothness, we obtain that the following inequality satisfies for all $n \in N$ and $h \in (0, 1]$:

$$w_k(f, nh)_M \leq c_3 n^r w_k(f, h)_M, \tag{5}$$

where c_3 is a constant that depends on r and B

Also, we prove that

$$\frac{1}{n^k}\sum_{\nu=1}^{n}\nu^{k-1}E_\nu(f)_M \le c_4 w_k(f,\frac{1}{n})_M \tag{6}$$

where c_4 is a constant that depends only on r and B.

By Theorem 1 and inequality (5), we have

$$\frac{1}{n^k}\sum_{\nu=1}^{n}\nu^{k-1}E_\nu(f)_M \le \frac{1}{n^k}\sum_{\nu=1}^{n}\nu^{k-1}w_k(f,\frac{1}{\nu})_M \le \frac{1}{n^k}\sum_{\nu=1}^{n}\nu^{k-1}w_k(f,\frac{n}{n}.\frac{1}{\nu})_M$$

$$\le \frac{1}{n^k}\sum_{\nu=1}^{n}\nu^{k-1}\frac{n^r}{\nu^r}w_k(f,\frac{1}{n})_M \le \frac{c_5}{n^{k-r}}\sum_{\nu=1}^{n}\nu^{k-1-r}w_k(f,\frac{1}{n})_M \le c_6 w_k(f,\frac{1}{n})_M.$$

For all m, $n \in N$, from Theorem 2 we get

$$w_k(f,\frac{1}{mn}) \le \frac{c_7}{(mn)^k}\sum_{\nu=1}^{mn}\nu^{k-1}E_\nu(f)_M$$

$$= \frac{c_8}{(mn)^k}\left\{\sum_{\nu=n+1}^{mn}\nu^{k-1}E_\nu(f)_M + \sum_{\nu=1}^{n}\nu^{k-1}E_\nu(f)_M\right\} \tag{7}$$

$$\le c_9\left\{\frac{1}{(mn)^k}\sum_{\nu=n+1}^{mn}\nu^{k-1}E_\nu(f)_M + \frac{1}{m^k}w_k(f,\frac{1}{n})_M\right\}$$

Thus, according to (7) we get

$$\sum_{\nu=n+1}^{mn}\nu^{k-1}E_\nu(f)_M \ge \frac{(mn)^k}{c_{10}}w_k\left(f,\frac{1}{mn}\right)_M - n^k w_k\left(f,\frac{1}{n}\right)_M.$$

Also we obtain

$$E_n(f)_M\sum_{\nu=n+1}^{mn}\nu^{k-1} \ge \left(c_{11}m^{k-r}-1\right)n^k w_k\left(f,\frac{1}{n}\right)_M$$

by the monotonicity of $E_n(f)_M$ and property (5). Properly choosing m and performing simple transformations, we have a positive constant C_{12} such that,

$$E_n(f)_M \ge c_{12}w_k\left(f,\frac{1}{n}\right)_M \tag{8}$$

From relation (8) and inequality (4) , we obtain (3).

Assume that the condition (3) holds. From Theorem 1, we have (4). The Theorem 3 is proved.

Corollary 1. *Suppose $f \in L_M(T)$ and $r \in N$. The relation*

$$E_n(f)_M \asymp w_r\left(f,\frac{1}{n}\right)_M, n\to\infty,$$

is true if and only if, for a certain $k > r$, we have

$$w_r(f,h)_M \asymp w_k(f,h)_M, \quad h \to +0$$

(two-sided inequalities with positive constants).

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Received July 14, 2009; Revised January 14, 2010.