Sadulla Z. JAFAROV

ON MODULI OF SMOOTHNESS IN ORLICZ CLASSES

Abstract

Let $T$ be the unit circle in the complex plane. In this work the relationship between the modulus of smoothness and the best approximation in Orlicz space $L_{M}(T)$, have been investigated.

1. Introduction, auxiliary and main result

A convex and continuous function $M : [0, \infty) \to [0, \infty)$ where $M(0) = 0$, $M(t) > 0$ for $t > 0$, and

$$\lim_{t \to 0} \frac{M(t)}{t} = \lim_{t \to \infty} \frac{t}{M(t)} = 0,$$

is called a Young function. The complementary Young function $N$ of $M$ is defined by

$$N(t) := \max_{y \geq 0} \{ty - M(y)\}, \quad t \geq 0$$

Let $N$ be the complementary Young function for a Young function $M$, then

$$t \leq M^{-1}(t)N^{-1}(t) \leq 2t, \quad t \geq 0,$$

where $M^{-1}$ is the inverse function of $M$.

Suppose that $T$ denotes the unit circle, $\mathbb{C}$ the complex plane, and $L_{p}(T)$, $1 \leq p \leq \infty$, the Lebesgue space of measurable complex-valued functions on $T$.

Assume that $M$ is the N-function and $N$ is its complementary function. $L_{M}(T)$ denotes the linear space of Lebesque measurable functions $f : T \to \mathbb{C}$ satisfying the condition

$$\int_{T} M(\alpha |f(z)|) \, dz < \infty$$

for some $\alpha > 0$.

The linear span of $L_{M}(T)$ equipped with the Orlicz norm

$$\|f\|_{L_{M}(T)} := \sup \left\{ \int_{T} |f(z)g(z)| \, dz : g \in L_{N}(T), \quad \rho(g, N) \leq 1 \right\},$$

where

$$\rho(g, N) := \int_{T} N(|g(z)|) \, dz,$$

or with the Luxemburg norm

$$\|f\|_{L_{M}(T)}^{\ast} := \inf \left\{ x > 0 : \rho \left( \frac{f}{x}, M \right) \leq 1 \right\}$$

is Banach space which is called the Orlicz space $L_{M}(T)$ [18, p.69].

Since

$$\|f\|_{L_{M}(T)} \leq \|f\|_{L_{M}(T)}^{\ast} \leq 2 \|f\|_{L_{M}(T)}^{\ast},$$

these norms are equivalent [18, p.80].

In addition, the Orlicz norm can be determined by means of the Luxemburg norm [18, pp.79-80].

\[ \|f\|_{L^M(T)} := \sup \left\{ \int_T |f(x)g(x)|\,dx : \|g\|_{L^N(T)}^* \leq 1 \right\} \]

and then the Hölder inequalities

\[ \int_T |f(x)g(x)|\,dx \leq \|f\|_{L^M(T)} \cdot \|g\|_{L^N(T)}^* \]

\[ \int_T |f(x)g(x)|\,dx \leq \|f\|_{L^M(T)}^* \cdot \|g\|_{L^N(T)} \]

hold for every \( f \in L^M(T) \) and \( g \in L^N(T) \) [18, p.80].

Every function in \( L^M(T) \) is integrable on \( T \) [18, p.50], i.e.

\[ L^M(T) \subset L^1(T). \]

If we take the Young function \( M(t) = \frac{t^p}{p} \), the Lebesque space \( L^p(T) \), \( 1 < p < \infty \) is isomorphic to the Orlicz space \( L^M(T) \).

If

\[ \lim_{x \to \infty} \sup \frac{M(2x)}{M(x)} < \infty, \]

then \( N \)-function \( M \) holds the \( \Delta_2 \)-condition.

The Orlicz space \( L^M(T) \) is reflexive if and only if the \( N \)-function \( M \) and its complementary function \( N \) both satisfy the \( \Delta_2 \)-condition [27, p.113].

For \( r = 1, 2, 3, \cdots \) the \( r \)-th modulus of smoothness of a function \( f \in L^M(T) \) defines with

\[ w^r_M(\delta, f) := \sup_{|h| \leq \delta} \|\Delta^r_h f\|_{L^M(T)} \cdot \delta > 0, \]

where

\[ \Delta^r_h f(\cdot) := \sum_{\nu=0}^{r} (-1)^{r-\nu} \binom{r}{\nu} f(\cdot + \nu h). \]

Now, we give some properties of \( w_r(\delta, f) \).

1) \( w_r(\delta, f) \) is a monotone non-decreasing function of \( \delta \geq 0 \).
2) \( w_r(0, f) = 0 \).
3) If \( M(u) \) satisfies the \( \Delta_2 \)-condition and \( f \in L^M \), then \( w_p(\delta, f) \to 0 \) as \( \delta \to 0 \).
4) If \( f^{(n-1)}(x) \) is absolutely continuous and \( f^{(n)} \in L^M \) then we have

\[ w_{r+n}(\delta, f) \leq \delta^n w_r(\delta, f^{(n)}). \]

5) For any non-negative integer \( m \) we get

\[ w_r(m\delta, f) \leq m^r w_r(\delta, f). \]

Let

\[ E_n(f)_M := \inf_{T \in T_n} \|f - T\|_{L^M(T)}, f \in L^M(T), \]
where \( T_n \) is the class of trigonometric polynomials of degree not greater than \( n \geq 1 \).

An upper bound for \( E_n(f)_M \), in terms of the modulus of smoothness of arbitrary order \( w(f,n^{-1})_M \), has been studied in [4, 26]. In this paper we formulate a theorem (see Theorem 3) that gives a condition under which \( E_n(f)_M \) can easily be estimated from below. This theorem is a generalization of the corresponding Yu.S. Kolomoitsev’s result [23] to the case of the spaces \( L_M(T) \). Similar problems in the Lebesgue spaces have been studied in [23, 28].

In terms of the usual modulus of smoothness, these problems in the Lebesgue, Smirnov and Orlicz spaces defined on the complex domains with the various boundary conditions were investigated by Walsh-Russel [29], Al’per [1], Kokilashvili [19,20], Andersson [2], Dyn’kin [7], Ibragimov-Mamedkhanov [9], Israfilov [10,11,12], Israfilov-Guven [13,15], Israfilov-Akgn [14], Akgn-Israfilov [3,4,5], Mamedkhanov [24], Mhaskar [25], Ramazanov [26], Jafarov [17] and other mathematicians.

Throughout this paper we shall denote \( c_1, c_2, ... \) constants depending only on numbers that are not important for the questions of interest.

We shall also employ the symbol \( A \preceq B \), denoting that \( A \leq CB \), where \( C = \text{const} > 0 \) does not depend on \( A \) or \( B \); and \( A \equiv B \), if simultaneously \( A \preceq B \) and \( B \preceq A \).

We will use the following auxiliary results

**Theorem 1** [4, 26]. Let \( L_M(T) \) be a reflexive Orlicz space on \( T \), \( f \in L_M(T) \) and \( n,r \in \mathbb{N} \). Then we have

\[
E_n(f)_M \leq c_1 w_r(f,\frac{1}{n})_M,
\]

where the constant \( c_1 > 0 \) depends on \( n \).

**Theorem 2** [4, 22]. Suppose that \( L_M(T) \) be a reflexive Orlicz space on \( T \), \( f \in L_M(T) \), and \( n,r \in \mathbb{N} \). Then we get

\[
w_r(f,\frac{1}{n})_M \leq \frac{c_2}{n^r} \sum_{k=1}^{n} k^{r-1} E_k(f)_M,
\]

where the constant \( c_2 > 0 \) depends only on \( r \) and \( M \).

The main results of this work are the following:

**Theorem 3.** Assume that \( f \in L_M(T) \) and \( r \in \mathbb{N} \). There exists a constant \( D > 0 \) such that

\[
w_r(f,\frac{1}{n})_M \leq DE_n(f)_M \quad \forall n \in \mathbb{N},
\]

if and only if for a certain \( k > r \), there exist a constant \( B > 0 \) such that

\[
w_r(f,h)_M \leq Bw_k(f,h)_M \quad \forall h \in (0,1],
\]

2. Proof of main result

**Proof of Theorem 3.** Suppose that the condition (4) is satisfied. Then, properties (1) and (2) of the modulus of smoothness, we obtain that the following inequality satisfies for all \( n \in \mathbb{N} \) and \( h \in (0,1] \):

\[
w_k(f,nh)_M \leq c_3 n^r w_k(f,h)_M,
\]
where $c_3$ is a constant that depends on $r$ and $B$

Also, we prove that

$$\frac{1}{n^k} \sum_{\nu=1}^{n} \nu^{k-1} E_{\nu}(f)_M \leq c_4 w_k(f, \frac{1}{n})_M$$

(6)

where $c_4$ is a constant that depends only on $r$ and $B$.

By Theorem 1 and inequality (5), we have

$$\frac{1}{n^k} \sum_{\nu=1}^{n} \nu^{k-1} E_{\nu}(f)_M \leq \frac{1}{n^k} \sum_{\nu=1}^{n} \nu^{k-1} w_k(f, \frac{1}{n})_M \leq \frac{1}{n^k} \sum_{\nu=1}^{n} \nu^{k-1} w_k(f, \frac{n}{\nu})_M$$

$$\leq \frac{1}{n^k} \sum_{\nu=1}^{n} \nu^{k-1} \nu^{r} w_k(f, \frac{1}{n})_M \leq c_5 \frac{n}{n^{r-k}} \sum_{\nu=1}^{n} \nu^{k-1-r} w_k(f, \frac{1}{n})_M \leq c_6 w_k(f, \frac{1}{n})_M.$$ 

For all $m, n \in \mathbb{N}$, from Theorem 2 we get

$$w_k(f, \frac{1}{mn}) \leq \frac{c_7}{(mn)^k} \sum_{\nu=1}^{mn} \nu^{k-1} E_{\nu}(f)_M$$

$$= \frac{c_8}{(mn)^k} \left\{ \sum_{\nu=n+1}^{mn} \nu^{k-1} E_{\nu}(f)_M + \sum_{\nu=1}^{n} \nu^{k-1} E_{\nu}(f)_M \right\}$$

(7)

$$\leq c_9 \left\{ \frac{1}{(mn)^k} \sum_{\nu=n+1}^{mn} \nu^{k-1} E_{\nu}(f)_M + \frac{1}{mn} w_k(f, \frac{1}{n})_M \right\}$$

Thus, according to (7) we get

$$\sum_{\nu=n+1}^{mn} \nu^{k-1} E_{\nu}(f)_M \geq \frac{(mn)^k}{c_{10}} w_k \left( f, \frac{1}{mn} \right)_M - n^k w_k \left( f, \frac{1}{n} \right)_M.$$ 

Also we obtain

$$E_n(f)_M \sum_{\nu=n+1}^{mn} \nu^{k-1} \geq \left( c_{11} m^{k-r} - 1 \right) n^k w_k \left( f, \frac{1}{n} \right)_M$$

by the monotonicity of $E_n(f)_M$ and property (5). Properly choosing $m$ and performing simple transformations, we have a positive constant $C_{12}$ such that,

$$E_n(f)_M \geq c_{12} w_k \left( f, \frac{1}{n} \right)_M$$

(8)

From relation (8) and inequality (4), we obtain (3).

Assume that the condition (3) holds. From Theorem 1, we have (4). The Theorem 3 is proved.

**Corollary 1.** Suppose $f \in L_M(T)$ and $r \in \mathbb{N}$. The relation

$$E_n(f)_M \approx w_r \left( f, \frac{1}{n} \right)_M, n \to \infty,$$
is true if and only if, for a certain \( k > r \), we have

\[ w_r(f, h)_M \approx w_k(f, h)_M, \quad h \to +0 \]

(two-sided inequalities with positive constants).

References


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Sadulla Z. Jafarov
Department of Mathematics, Faculty of Art and Sciences, Pamukkale University, 20017, Denizli, Turkey, e-mail: sjafarov@pau.edu.tr

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