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ON SOME PROPERTIES OF CAUCHY SINGULAR INTEGRAL OF FINITE COMPLEX MEASURES

Abstract

In the paper the Cauchy singular integral of finite complex measures is investigated. It is proved that the Cauchy singular integral of finite measures is Q' - integrable and preserves some properties that hold in the case when the density is an absolutely continuous measure.

Let ν be a complex measure on a unit circle $T = \{t \in C : |t| = 1\}$. The function

$$(S\nu)(t) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0+} \int_{T_{\varepsilon,t}} \frac{d\nu(\tau)}{\tau - t}$$

is called the Cauchy singular integral of the measure ν, where $T_{\varepsilon,t} = \{\tau \in T : |\tau - t| \geq \varepsilon\}$; in particular, when ν is absolutely continuous, and $d\nu(t) = f(t)\,dt$, the function $(S\nu)(t)$ is also called the Cauchy singular integral of the function $f(t)$ and is denoted by $(Sf)(t)$. It is known that when the measure ν is absolutely continuous, the Cauchy singular integral is A -integrable and this case was investigated well in the papers [1]-[3], but if the singular part of the measure ν differs from zero, then $(S\nu)(t)$ is not integrable even in the sense of A-integral. For example, for the discrete measure

$$\nu(X) = \begin{cases} 2\pi i & for \;\; 1 \in X \\ 0 & for \;\; 1 \notin X \end{cases}$$

the Cauchy singular integral

$$(S\nu)(t) = \frac{1}{1-t}$$

is not A - integrable. For the Cauchy singular integral of finite complex measures, only the following fact is known

$$\lim_{\lambda \to \infty} \lambda m \{t \in T : |(S\nu)(t)| \geq \lambda\} = \frac{1}{\pi} \|\nu_s\|,$$

where ν_s is a singular part of the measure ν, $\|\nu_s\| = \int_0^{2\pi} |d\nu_s|$ is a total variation of the measure ν_s that may be obtained from one result of [4] (see below theorem B).

In the paper, the Cauchy singular integral of finite complex measures is investigated. It is proved that $(S\nu)(t)$ is Q' -integrable and preserves some properties that hold when the density is an absolutely continuous measure.

Following E. Titchmarsh [5], introduce the definition of Q' -integral.

Definition 1. *The measurable function f is called Q'-integrable on the segment $[a, b]$ if there exists a finite limit*

$$\lim_{\lambda \to \infty} \int_{\{t \in [a,b] : |f(t)| \leq \lambda\}} f(t)\,dt, \tag{1}$$

that is called Q'-integral of the function f in the segment $[a,b]$ and is denoted by $(Q')\int\limits_a^b f(t)\,dt$.

If besides the existence of limit (1) it is necessary to fulfill the conditions

$$m\{t\in[a,b]:|f(t)|\geq\lambda\}=o\left(\frac{1}{\lambda}\right)\quad as\quad \lambda\to+\infty, \tag{2}$$

where m is the Lebesque linear measure, the function f is called A-integrable on $[a,b]$, and limit (1) A -integral of the function f in the segment $[a,b]$ and denoted by $(A)\int\limits_a^b f(t)\,dt$.

The Q' -integral and A-integral of the function determined on Jordan's rectifiable curve γ are determined in the same way.

E. Titchmarsh [5] showed that the A-integral possesses the additivity property, the Q'- integral has not this property (see also [6]).

For absolutely continuous measures $d\nu(t)=f(t)\,dt$ P.L. Ulyanov [1] proved the following statement: if $g(t)$ is a bounded function such that $(Sg)(t)$ also is bounded, then the following equality holds

$$\int_T (Sg)(t)\,f(t)\,dt=-(A)\int_T g(t)\,(Sf)(t)\,dt. \tag{3}$$

Further, G.A. Khuskivadze [2] proved the validity of equality (3) for smooth curves satisfying some additional condition, T. S. Salimov [3] for regular curves.

Definition 2. *Let μ be a finite complex measure on a unit circle T. The function $\widetilde{\mu}$ conjugated to the measure μ is determined in the following way:*

$$\widetilde{\mu}\left(e^{i\theta}\right)=\frac{1}{2\pi}\lim_{\{s\in[0,2\pi):|s-\theta|>\varepsilon\}}ctg\frac{\theta-s}{2}d\mu\left(e^{is}\right),\quad \theta\in[0,2\pi).$$

It holds the following

Theorem. *If ν is a finite complex measure on a unit circle T, the function $g(t)$ is Holder continuous on T, then there exists $(Q')\int\limits_T g(t)\,(S\nu)(t)\,dt$, and the following equality is valid*

$$\int_T (Sg)(t)\,d\nu(t)\,dt=-\left(Q'\right)\int_T g(t)\,(S\nu)(t)\,dt. \tag{4}$$

For the proof we'll need the following theorems.

Theorem A [6]. *If the function f is Q' -integrable on the segment $[a,b]$, and there exists a finite limit*

$$\lim_{\lambda\to\infty}\lambda m\{t\in[a,b]:|f(t)|\geq\lambda\}=\alpha,$$

and the function g is Lebesgue integrable on $[a,b]$, then the function $f+g$ is Q' -integrable on $[a,b]$, and the following equality is valid

$$\left(Q'\right)\int_a^b[f(t)+g(t)]\,dt=\left(Q'\right)\int_a^b f(t)\,dt+(L)\int_a^b g(t)\,dt. \tag{5}$$

Theorem B [4]. *For any finite complex measure μ on T it is valid the equality*

$$\lim_{\lambda\to\infty}\lambda m\left\{\theta\in[0,2\pi]:\left|\widetilde{\mu}\left(e^{i\theta}\right)\right|\geq\lambda\right\}=\frac{2}{\pi}\left\|\mu_s\right\|,$$

where μ_s is a singular part of the measure μ, and $\|\mu_s\|=\int\limits_0^{2\pi}|d\mu_s|$ is a total variation of the measure μ_s.

Theorem C [7]. *On any finite complex measure μ on T the following equality is fulfilled:*

$$(Q')\int_0^{2\pi}\widetilde{\mu}\left(e^{i\theta}\right)d\theta=0.$$

Proof of the theorem. Since

$$(Sg)(t)=\frac{1}{2\pi i}\lim_{\varepsilon\to0+}\int_{T_{\varepsilon,t}}\frac{g(\tau)\,d\tau}{\tau-t}=\frac{1}{2\pi i}\nu.p.\int_0^{2\pi}\frac{g\left(e^{is}\right)de^{is}}{e^{is}-e^{i\theta}}=$$

$$=\frac{1}{2\pi}\nu\cdot p\cdot\int_0^{2\pi}\frac{e^{is}}{e^{is}-e^{i\theta}}g\left(e^{is}\right)ds,\quad t=e^{i\theta}$$

from the equality

$$\frac{e^{is}}{e^{is}-e^{i\theta}}=\frac{e^{is}}{2i\sin\frac{s-\theta}{2}e^{i\frac{s+\theta}{2}}}=\frac{e^{i\frac{s-\theta}{2}}}{2i\sin\frac{s-\theta}{2}}=\frac{1}{2i}\left[ctg\frac{s-\theta}{2}+i\right]$$

we have

$$(Sg)(t)=\frac{1}{4\pi i}\nu.p.\int_0^{2\pi}\left[ctg\frac{s-\theta}{2}+i\right]g\left(e^{is}\right)ds=$$

$$=\frac{1}{4\pi i}\nu.p.\int_0^{2\pi}ctg\frac{s-\theta}{2}g\left(e^{is}\right)ds+\frac{1}{4\pi}\int_0^{2\pi}g\left(e^{is}\right)ds=$$

$$=\frac{1}{4\pi i}\int_0^{2\pi}ctg\frac{s-\theta}{2}\left[g\left(e^{is}\right)-g\left(e^{i\theta}\right)\right]ds+g\left(e^{i\theta}\right)\frac{1}{4\pi i}\nu.p.\int_0^{2\pi}ctg\frac{s-\theta}{2}ds+$$

$$+\frac{1}{4\pi}\int_0^{2\pi}g\left(e^{is}\right)ds=\frac{1}{4\pi i}\int_0^{2\pi}ctg\frac{s-\theta}{2}\left[g\left(e^{is}\right)-g\left(e^{i\theta}\right)\right]ds+\frac{1}{4\pi i}\int_0^{2\pi}g\left(e^{is}\right)ds.\qquad(6)$$

Since the function $K(s,\theta)=ctg\frac{s-\theta}{2}\left[g\left(e^{is}\right)-g\left(e^{i\theta}\right)\right]$ is absolutely integrable in s, and there exists an integral

$$\int_0^{2\pi}\left(\int_0^{2\pi}|K(s,\theta)|\,ds\right)d\nu\left(e^{i\theta}\right),$$

then (see [8], ch. 5, p. 6), from equality (6) it follows that

$$\int_T (Sg)(t)\, d\nu(t) = \int_0^{2\pi} (Sg)\left(e^{i\theta}\right) d\nu\left(e^{i\theta}\right) =$$

$$= \int_0^{2\pi} \left\{ \frac{1}{4\pi i} \int_0^{2\pi} ctg\frac{s-\theta}{2} \left[g\left(e^{is}\right) - g\left(e^{i\theta}\right) \right] ds + \frac{1}{4\pi} \int_0^{2\pi} g\left(e^{is}\right) ds \right\} d\nu\left(e^{i\theta}\right) =$$

$$= \frac{1}{4\pi i} \int_0^{2\pi} \left\{ \int_0^{2\pi} ctg\frac{s-\theta}{2} \left[g\left(e^{is}\right) - g\left(e^{i\theta}\right) \right] d\nu\left(e^{i\theta}\right) \right\} ds +$$

$$+ \frac{1}{4\pi} \int_0^{2\pi} g\left(e^{is}\right) ds \int_0^{2\pi} d\nu\left(e^{i\theta}\right). \tag{7}$$

Consider the measure $d\mu\left(e^{i\theta}\right) = g\left(e^{i\theta}\right) d\nu\left(e^{i\theta}\right)$, $\theta \in [0, 2\pi)$ on a unit cricle T. From the theorem C we have

$$(Q') \int_0^{2\pi} \widetilde{\mu}\left(e^{i\theta}\right) d\theta = 0, \tag{8}$$

where

$$\widetilde{\mu}\left(e^{i\theta}\right) = \frac{1}{2\pi} \nu.p. \int_0^{2\pi} ctg\frac{\theta-s}{2} g\left(e^{is}\right) d\nu\left(e^{is}\right) =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} ctg\frac{\theta-s}{2} \left[g\left(e^{is}\right) - g\left(e^{i\theta}\right) \right] d\nu\left(e^{is}\right) + g\left(e^{i\theta}\right) \frac{1}{2\pi} \nu.p. \int_0^{2\pi} ctg\frac{\theta-s}{2} d\nu\left(e^{is}\right).$$

Considering the indentity

$$ctg\frac{\theta-s}{2} = \left(ctg\frac{\theta-s}{2} - i + i \right) \left(1 - e^{i(\theta-s)} + e^{i(\theta-s)} \right) =$$

$$= \left(ctg\frac{\theta-s}{2} - i \right) \left(1 - e^{i(\theta-s)} \right) + i + \left(ctg\frac{\theta-s}{2} - i \right) e^{i(\theta-s)} =$$

$$= -2i + i + \left(ctg\frac{\theta-s}{2} - i \right) e^{i(\theta-s)} = -i + \left(ctg\frac{\theta-s}{2} - i \right) e^{i(\theta-s)},$$

we get

$$\widetilde{\mu}\left(e^{i\theta}\right) = \frac{1}{2\pi} \int_0^{2\pi} ctg\frac{\theta-s}{2} \left[g\left(e^{is}\right) - g\left(e^{i\theta}\right) \right] d\nu\left(e^{is}\right) + \frac{1}{2\pi i} g\left(e^{i\theta}\right) \int_0^{2\pi} d\nu\left(e^{is}\right) +$$

$$+ g\left(e^{i\theta}\right) e^{i\theta} \frac{1}{2\pi} \nu.p. \int_0^{2\pi} \left(ctg\frac{\theta-s}{2} - i \right) e^{-is} d\nu\left(e^{is}\right).$$

Hence and from the equality

$$(S\nu)\left(e^{i\theta}\right)=\frac{1}{2\pi i}\nu.p.\int_0^{2\pi}\frac{d\nu\left(e^{is}\right)}{e^{is}-e^{i\theta}}=\frac{1}{2\pi i}\nu.p.\int_0^{2\pi}\frac{e^{is}}{e^{is}-e^{i\theta}}e^{-is}d\nu\left(e^{is}\right)=$$

$$=-\frac{1}{4\pi}\nu.p.\int_0^{2\pi}\left(ctg\frac{s-\theta}{2}+i\right)e^{-is}d\nu\left(e^{is}\right)=$$

$$=\frac{1}{4\pi}\nu.p.\int_0^{2\pi}\left(ctg\frac{\theta-s}{2}-i\right)e^{-is}d\nu\left(e^{is}\right)$$

we have

$$\widetilde{\mu}\left(e^{i\theta}\right)=J\left(\theta\right)+2e^{i\theta}g\left(e^{i\theta}\right)(S\nu)\left(e^{i\theta}\right),\tag{9}$$

where

$$J\left(\theta\right)=\frac{1}{2\pi}\int_0^{2\pi}ctg\frac{\theta-s}{2}\left[g\left(e^{is}\right)-g\left(e^{i\theta}\right)\right]d\nu\left(e^{is}\right)+\frac{1}{2\pi i}g\left(e^{i\theta}\right)\int_0^{2\pi}d\nu\left(e^{is}\right).$$

From equality (7) it follows that the function $J\left(\theta\right)$ is Lebesgue integrable on the interval $[0,2\pi)$, and from theorems B and C it follows that the function $\widetilde{\mu}\left(e^{i\theta}\right)$ is Q' - integrable on the interval $[0,2\pi)$ and satisfies the condition of the theorem A. From theorem A and equality (8), (9) we get

$$\int_0^{2\pi}J\left(\theta\right)d\theta+2\left(Q'\right)\int_0^{2\pi}e^{i\theta}g\left(e^{i\theta}\right)(S\nu)\left(e^{i\theta}\right)d\theta=0.\tag{10}$$

Since (see equality (7))

$$\int_0^{2\pi}J\left(\theta\right)d\theta=-2i\int_T\left(Sg\right)\left(t\right)d\nu\left(t\right)$$

and

$$\left(Q'\right)\int_T g\left(t\right)\left(S\nu\right)\left(t\right)dt=i\left(Q'\right)\int_0^{2\pi}g\left(e^{i\theta}\right)(S\nu)\left(e^{i\theta}\right)e^{i\theta}d\theta$$

equality (4) follows from equality (10). The theorem is proved.

Assuming $g\left(t\right)=1$, in the proved theorem we get

Corollary. *It ν is a finite complex measure on a unit circle T the Cauchy singular integral $\left(S\nu\right)\left(t\right)$ is Q' - integrable on T and the following equality*

$$\left(Q'\right)\int_T\left(S\nu\right)\left(t\right)dt=-\frac{1}{2}\int_T d\nu\left(t\right)$$

is valid.

References

[1]. Ulyanov P.L. *A - integral and conjugated functions* // Ucheniye zap. Mosk. Univ. 1956, vol. 8, No 181 (Matematika) pp. 139-157 (Russian).

[2]. Khuskivadze G.A. *On Cauchy type A -integrals* // Soob. AN. Gruz. SSR. 1961, vol. 27, No 6, pp. 663-670 (Russian).

[3]. Salimov T.S. *A - integral and boundary values of analytic functions* // Mat. Sbor. 1988, vol. 136, No 1, pp. 24-40 (Russian).

[4]. Hruscev S.V., Vinogradov S.A. *Free interpolation in the space of uniformly convergent Taylor series* // Lect. Notes in Math., 1981, vol. 864, pp. 171-213.

[5]. Titchmarsh E. C. *On conjugate functions* // Proc. London Math. Soc., 1929, vol. 29, No 1, pp. 49-80.

[6]. Aliev R.A. *On additivity of Q - integral and convergence of Fourier series* (Q) // Proceedings of NAS of Azerbaijan, 2010, vol. XXXII (XL), pp. 47-55.

[7]. Salimov T.S. *To Titchmarsh theorem on conjugated function* // Proceedings of A. Razmadze Math. Inst., 1993, vol. 102, pp. 99-114.

[8]. Kolmogrov A.N., Fomin S.V. *Elements of theory of functions and functional analysis.* M.: Nauka, 1989 (Russian).

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