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b-BASES IN BANACH SPACES

Abstract

The notion of $b-$basis generalizing the known classic notion of basis on Banach spaces is introduced in the paper. The criteria for b-basis are obtained.

As is known, in the paper [1], the criteria of basis in Hilbert space is given by means of the space of sequences of its coefficients. Generalization of these results to Banach space is given in [3], where the notion of the K-basis is introduced. In the present paper all these results are generalized for bilinear mappings.

1. Some auxiliary definition and statements. Give some notion and statements from [3-5].

Let X, Y, Y_1, Z and Z_1 be $B-$spaces. Consider the bilinear mappings $b\,(x,y)$: $X \times Y \to Z$ and $b_1\,(x,y) : X \times Y_1 \to Z_1$ such that

$$\exists m, M > 0 : m\left\|x\right\|_X \left\|y\right\|_Y \leq \left\|b\,(x,y)\right\|_Z \leq M\left\|x\right\|_X \left\|y\right\|_Y \ ,$$

$$\exists m_1, M_1 > 0 : m_1\left\|x\right\|_X \left\|y\right\|_{Y_1} \leq \left\|b_1\,(x,y)\right\|_{Z_1} \leq M_1\left\|x\right\|_X \left\|y\right\|_{Y_1} \ .$$

For convenience, in the sequel we assume $b\,(x,y) \equiv xy\ \forall x \in X,\ y \in Y,\ b_1\,(x,\varphi) \equiv x * \varphi\ \forall x \in X,\ \varphi \in Y_1$.

The aggregate $L_b\,(M)$ of all possible finite sums $\sum x_i m_i$, where $x_i \in X,\ m_i \in M$ is called a b-linear span of the set $M \subset Y$.

The system $\{y_n\}_{n\in N} \subset Y$ is said to be b-complete if the closure $L_b(\{y_n\}_{n\in N})$ coincides with space Z.

The systems $\{y_n\}_{n\in N} \subset Y$ and $\{y_n^*\}_{n\in N} \subset L\,(Z,X)$ are called b-biorthogonal if

$$\forall k,n \in N,\quad x \in X \quad y_n^*\,(xy_k) = \delta_{nk}x,$$

where δ_{nk} is Kreneker's symbol, and the system $\{y_n^*\}_{n\in N}$ will be called a b-biorthogonal system to the system $\{y_n\}_{n\in N}$.

The system $\{y_n\}_{n\in N} \subset Y$ is called b-basis in Z if

$$\forall z \in Z \quad \exists!\,\{x_n\}_{n\in N} \subset X : z = \sum_{n=1}^{\infty} x_n y_n,$$

and the space of sequences $\{x_n\}_{n\in N}$ is said to be a space of coefficients in b-basis $\{y_n\}_{n\in N}$.

Let $\widetilde{X}$ be some $B-$space of sequences $\widetilde{x} = \{x_n\}_{n\in N}\,,\ x_n \in X$ with coordinatewise linear operations.

If $\widetilde{E}_n = \left\{ \widetilde{x} = \{\delta_{in} x\}_{i\in N}\, ,\ x \in X \right\}$ form a basis in $\widetilde{X}$ and the convergence in $\widetilde{X}$ yields its coordinatewise convergences, then the B-space $\widetilde{X}$ is called a coordinate $B-$space (briefly $CB-$space). Obviously, $\forall \widetilde{x} \in \widetilde{X}$, $\widetilde{x} = \{x_n\}_{n\in N}\,,$

$$\left\| \widetilde{x} - \sum_{k=1}^{n} \{\delta_{ik} x_k\}_{i\in N} \right\|_{\widetilde{X}} \to 0, \quad n \to \infty.$$

For $\forall n \in N$ consider the operator $e_n : \widetilde{X} \to \widetilde{X}$ determined by the expression $e_n(\widetilde{x}) = \{\delta_{in} x_n\}_{i\in N}\,,\ \widetilde{x} = \{x_n\}_{n\in N}$. From the basicity of $\widetilde{E}_n$ in $\widetilde{X}$, $\forall \widetilde{x} \in \widetilde{X}$, $\widetilde{x} = \{x_n\}_{n\in N}$, we get $\widetilde{x} = \sum\limits_{n=1}^{\infty} e_n(\widetilde{x})$. The system $\{e_n\}_{n\in N}$ is called a canonical basis of the CB-space $\widetilde{X}$.

Let $\{y_n\}_{n\in N} \subset Y$ and $\{y_n^*\}_{n\in N} \subset L(Z, X)$ be b-orthogonal systems.

The pair $\{y_n; y_n^*\}_{n\in N}$ is said to be $b_{\widetilde{X}}$ -Bessel in Z if $\forall z \in Z \quad \{y_n^*(z)\}_{n\in N} \in \widetilde{X}$. When the system $\{y_n\}_{n\in N}$ is b-complete, the system $\{y_n\}_{n\in N} \subset Y$ is simply called $b_{\widetilde{X}}$-Bessel in Z.

The pair $\{y_n; y_n^*\}_{n\in N}$ is called $b_{\widetilde{X}}$ -Hilbert in Z if $\forall \widetilde{x} \in \widetilde{X} \quad \exists z \in Z : \{y_n^*(z)\}_{n\in N} = \widetilde{x}$. When the system $\{y_n\}_{n\in N}$ is b-complete, the system $\{y_n\}_{n\in N} \subset Y$ is simply called $b_{\widetilde{X}}$-Hilbert in Z.

Cite the criteria of $b_{\widetilde{X}}$-Bessel property and $b_{\widetilde{X}}$-Hilbert property from the paper [4,5]. The following statements are true.

Statement 1. *Let X, Y and Z be $B-$spaces, $\widetilde{X}$ be a CB-space of sequences $\widetilde{x} = \{x_n\}_{n\in N}\,,\ x_n \in X$, with canonic basis $\{e_n\}_{n\in N}\,,\ \{y_n\}_{n\in N} \subset Y$ have the b-biorthogonal system $\{y_n^*\}_{n\in N}$. Then, for the pair $\{y_n; y_n^*\}_{n\in N}$ to be $b_{\widetilde{X}}$-Bassel in Z, it is necessary, in the case of b-completeness of the system $\{y_n\}_{n\in N}$, is sufficient the existence of the operator $T \in L\left(Z, \widetilde{X}\right) : T(xy_n) = \{\delta_{in} x\}_{i\in N}\,,\ \forall x \in X,\ n \in N$.*

By dire application of statement 1, we get that $\{y_n\}_{n\in N}$ is $b_{\widetilde{X}}$-Bessel in Z iff for any finite sequence $\{x_n\}$ from $\widetilde{X}$: $\|\{x_n\}\|_{\widetilde{X}} \le c_1 \left\| \sum x_n y_n \right\|_Z$, where c_1 is a constant.

Statement 2. *Let X, Y and Z be $B-$spaces, $\widetilde{X}$ be a CB-space of sequences $\widetilde{x} = \{x_n\}_{n\in N}\,,\ x_n \in X$, with canonic basis $\{e_n\}_{n\in N}\,,\ \{y_n\}_{n\in N} \subset Y$ and have the $b-$biorthogonal system $\{y_n^*\}_{n\in N}$. For the pair $\{y_n; y_n^*\}_{n\in N}$ to be $b_{\widetilde{X}}$-Hilbert in Z, it is sufficient, in the case of completeness of the system $\{y_n^*\}_{n\in N}$ in $L(Z, X)$ is necessary $\exists S \in L\left(\widetilde{X}, Z\right) : S\left(\{\delta_{in} x\}_{i\in N}\right) = xy_n\ \forall x \in X,\ n \in N$.*

From statement 2 it follows that the system $\{y_n\}_{n\in N}$ is $b_{\widetilde{X}}$-Hilbert in Z iff for any finite sequence $\{x_n\}$ from $\widetilde{X}$: $\left\| \sum x_n y_n \right\|_Z \le c_2 \|\{x_n\}\|_{\widetilde{X}}$, where c_2 is a constant.

In the case when $\widetilde{X}$ is a space of coefficients in some b-basis, the following statements are true.

Statement 3. *Let X, Y, Y_1, Z and Z_1 be B-spaces, $\{y_n\}_{n\in N} \subset Y$ have the b-biorthogonal system $\{y_n^*\}_{n\in N}$, $\Phi = \{\varphi_n\}_{n\in N} \subset Y_1$ be some b_1-basis in Z_1 with a* *space of coefficients $\widetilde{X}_\Phi$. Then, for the pair $\{y_n; y_n^*\}_{n\in N}$ to be $b_{\widetilde{X}_\Phi}$-Bassel in Z it is necessary, and in the case b-completeness of the system $\{y_n\}_{n\in N}$ it is sufficient the existence of the operator $T \in L(Z, Z_1) : T(xy_n) = x * \varphi_n \ \forall x \in X, \ n \in N$.*

Statement 4. *Let X, Y, Y_1, Z and Z_1, be B-spaces, $\Phi = \{\varphi_n\}_{n\in N} \subset Y_1$ form a b_1-basis in Z_1 with a space of coefficients $\widetilde{X}_\Phi$, $\{y_n\}_{n\in N} \subset Y$ have the b-biorthogonal system $\{y_n^*\}_{n\in N}$. Then for the pair $\{y_n; y_n^*\}_{n\in N}$ to be $b_{\widetilde{X}_\Phi}$-Hilbert in Z it is sufficient and in the case of completeness $\{y_n^*\}_{n\in N}$ in $L(Z, X)$ of $\exists T \in L(Z_1, Z)$: $T(x * \varphi_n) = xy_n \quad x \in X, \ n \in N$ it is necessary.*

2. The notion of b-basis.

b-basis in $\{y_n\}_{n\in N} \subset Y$ in Z is called $b_{\widetilde{X}}$-basis in Z if its corresponding space of coefficients is $\widetilde{X}$.

Theorem 1. *Let X, Y and Z be B-spaces, $\widetilde{X}$ be a CB-space of sequences $\widetilde{x} = \{x_n\}_{n\in N}$, $x_n \in X$ with canonic basis $\{e_n\}_{n\in N}$, and $\{y_n\}_{n\in N}$ be b-complete and have the b-biorthogonal system $\{y_n^*\}_{n\in N}$. Then, for the system $\{y_n\}_{n\in N}$ to be simultaneously $b_{\widetilde{X}}$-Bessel and $b_{\widetilde{X}}$-Hilbert, it is sufficient and necessasy that $\{y_n\}_{n\in N}$ to be $b_{\widetilde{X}}$-basis in Z.*

Proof. Necessity. Let the systen $\{y_n\}_{n\in N}$ simultaneously be $b_{\widetilde{X}}$-Bessel and $b_{\widetilde{X}}$-Hilbert. Then from statements 1 and 2 it follows that there exist the operators $T \in L\left(Z, \widetilde{X}\right)$ and $S \in L\left(\widetilde{X}, Z\right) : T(xy_n) = \{\delta_{in}x\}_{i\in N}$, $S\{\delta_{in}x\}_{i\in N} = xy_n \ \forall x \in X, \ n \in N$. Obviously, that $(ST)(xy_n) = xy_n$. Hence, since $\{y_n\}_{n\in N}$ is b-complete, then $\forall z \in Z \ (ST)z = z$. Take $\forall z \in Z$. Let $T(z) = \widetilde{x}$, $\widetilde{x} = \{x_n\}_{n\in N}$. Since $z = \sum_{n=1}^{\infty} e_n(\widetilde{x})$, then $S(\widetilde{x}) = \sum_{n=1}^{\infty} S(e_n(\widetilde{x})) = \sum_{n=1}^{\infty} x_n y_n$. Since $\widetilde{x} = \sum_{n=1}^{\infty} e_n(\widetilde{x})$, then $S(\widetilde{x}) = \sum_{n=1}^{\infty} S(e(\widetilde{x})) = \sum_{n=1}^{\infty} x_n y_n$. So, $z = \sum_{n=1}^{\infty} x_n y_n$. The uniqueness of the representation z follows from the fact that $\{y_n\}_{n\in N}$ has the b-biorthogonal system $\{y_n^*\}_{n\in N}$. Thus, $\{y_n\}_{n\in N}$ forms a $b_{\widetilde{X}}$-basis in Z with space of coefficients $\widetilde{X}$.

Sufficiency. Let $\{y_n\}_{n\in N}$ form a $b_{\widetilde{X}}$-basis in Z with space of coefficients $\widetilde{X}$. Consider the operator $T : Z \to \widetilde{X}$ determined according to the expression $T(z) = \widetilde{x}$, $\widetilde{x} = \{x_n\}_{n\in N}$, where $z = \sum_{n=1}^{\infty} x_n y_n$. It is clear that the operator T is well-defined, linear and $T(xy_n) = \{\delta_{in}x\}_{i\in N}$. Let $z^{(m)} \to z$ as $m \to \infty$, $T\left(z^{(m)}\right) = \widetilde{x}^{(m)}$, $\widetilde{x}^{(m)} = \left\{x_n^{(m)}\right\}_{n\in N}$, $T\left(z^{(m)}\right) \to \widetilde{w}$ as $m \to \infty$, $\widetilde{w} = \{w_n\}_{n\in N}$. Show that $T(z) = \widetilde{w}$. Since $z^{(m)} \to z, m \to \infty$, then $x_n^{(m)} \to x_n$ as $m \to \infty$. On the other hand, from $\widetilde{x}^{(m)} \to \widetilde{w}$ as $m \to \infty$ it follows that $x_n^{(m)} \to w_n$. So, $\widetilde{w} = \widetilde{x}$, i.e. $T\left(z^{(m)}\right) \to T(z)$ as $m \to \infty$. Applying the theorem on a closed graph, we get the boundedness of the operator T. By statement 1, the system $\{y_n\}_{n\in N}$ is $b_{\widetilde{X}}$-Bessel. Linearity and boundedness of the operator $S : \widetilde{X} \to Z$ determined by the expression $S(\widetilde{x}) = z$, $\widetilde{x} = \{x_n\}_{n\in N}$, where $z = \sum_{n=1}^{\infty} x_n y_n$, is shown in the same way. By virtue of this we apply statement 4 and get $b_{\widetilde{X}}$-Hillbert property of the system $\{y_n\}_{n\in N}$.

The theorem is proved.

Validity of the following statement follows from the proof of the theorem.

Corollory 1. Let X, Y and Z be B-spaces, $\widetilde{X}$ be a CB-space of sequences $\widetilde{x} = \{x_n\}_{n\in N}$, $x_n \in X$ with canonical basis $\{e_n\}_{n\in N}$, and $\{y_n\}_{n\in N}$ be b-complete and have the b-biorthogonal system $\{y_n^*\}_{n\in N}$. Then, for the system $\{y_n\}_{n\in N}$ to be $b_{\widetilde{X}}$ -bases in Z it is necessary and sufficient the existence of the boundedly invertible operator $T \in L\left(Z, \widetilde{X}\right) : T(xy_n) = \{\delta_{in} x\}_{i\in N}$ $\forall x \in X,\ n \in N$.

Corollary 2. Let X, Y and Z be B-spaces, $\widetilde{X}$ be a CB-space of sequences $\widetilde{x} = \{x_n\}_{n\in N}$, $x_n \in X$ with canoncal basis $\{e_n\}_{n\in N}$, and $\{y_n\}_{n\in N} \subset Y$ be b-complete and have the b-biorthogonal system $\{y_n^*\}_{n\in N}$. Then for the system $\{y_n\}_{n\in N}$ to be a $b_{\widetilde{X}}$-basis in Z it is necessary and sufficient that for any finite sequence $\{x_n\}$ from $\widetilde{X}$:

$$a_1 \|\{x_n\}\|_{\widetilde{X}} \le \left\|\sum x_n y_n\right\|_Z \le a_2 \|\{x_n\}\|_{\widetilde{X}} \ ,$$

where a_1 and a_2 are some constants.

Proof. The proof of the theorem directly follows from the theorem proved above with using statements 1 and 2.

Let $\widetilde{X}_p = \left\{ \widetilde{x} = \{x_n\}_{n\in N},\ x_n \in X : \sum_{n=1}^{\infty} \|x_n\|^p < +\infty \right\}$. Having determined coordinatewisely the linear operations in $\widetilde{X}_p$, it is easy to show that $\left\|\{x_n\}_{n\in N}\right\|_{\widetilde{X}_p} = \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p}$ satisfies the axioms of the norm according to which $\widetilde{X}_p$ is a B-space.

In the special case, if in place of the space $\widetilde{X}$ we take the space $\widetilde{X}_p$, then from Corollary 1 we get validity of the following statement.

Corollary 3. Let X, Y and Z be B-spaces, be $\{y_n\}_{n\in N} \subset Y$ b-complete and have the b-biorthogonal system $\{y_n^*\}_{n\in N}$. Then, for the system $\{y_n\}_{n\in N}$ to be a $b_{\widetilde{X}_p}$-basis in Z, it is necessary and sufficient that for any finite sequence $\{x_n\}$ from $\widetilde{X}_p$:

$$a_1 \|\{x_n\}\|_{\widetilde{X}^p} \le \left\|\sum x_n y_n\right\|_Z \le a_2 \|\{x_n\}\|_{\widetilde{X}^p} \ ,$$

where a_1 and a_2 are some constants.

Theorem 2. *Let X, Y, Y_1, Z and Z_1 be Banach spaces, $\{y_n\}_{n\in N} \subset Y$ have the b-biorthogonal system $\{y_n^*\}_{n\in N}$, $\Phi = \{\varphi_n\}_{n\in N} \subset Y_1$ be some b_1-basis in Z_1 with space of coefficients $\widetilde{X}_\Phi$. Then, for the system $\{y_n\}_{n\in N}$ to be simultaneously $b_{\widetilde{X}_\Phi}$-Bessel and $b_{\widetilde{X}_\Phi}$-Hilbert in Z, it is necessary and sufficients that the system $\{y_n\}_{n\in N}$ to be $b_{\widetilde{X}_\Phi}$-basis in Z.*

Proof. Necessity. Let $\{y_n\}_{n\in N}$ be simultaneously $b_{\widetilde{X}_\Phi}$-Bassel and $b_{\widetilde{X}_\Phi}$-Hilbert in Z. Then on the base of statements 3 and 4, there exist the operators $T \in L(Z, Z_1)$

and $S \in L(Z_1, Z)$ such that $T(xy_n) = x * \varphi_n,\ S(x * \varphi_n) = xy_n, \forall x \in X,\ n \in N$. Obviously, $\forall z \in Z$, take $ST(z) = z$. Take $\forall z \in Z$ and let $T(z) = z_1$ and $z_1 = \sum_{n=1}^{\infty} x_n * \varphi_n$. We have $z = S(z_1) = \sum_{n=1}^{\infty} S(x_n * \varphi_n) = \sum_{n=1}^{\infty} x_n y_n$. So, $z = \sum_{n=1}^{\infty} x_n y_n$. The uniqueness of the representation follows from the existence of the b-biorthofonal system $\{y_n^*\}_{n\in N}$. So, $\{y_n\}_{n\in N}$ is a $b_{\widetilde{X}_\Phi}$- basis in Z.

Sufficiency. Let $\{y_n\}_{n\in N}$ be a $b_{\widetilde{X}_\Phi}$-basis in Z with space of coefficients $\widetilde{X}_\Phi$. Show that $\{y_n\}_{n\in N}$ is similtaneously $b_{\widetilde{X}_\Phi}$-Bessel and $b_{\widetilde{X}_\Phi}$-Hilbert in Z. Consider the operators $T : Z \to Z_1$ and $S : Z_1 \to Z$ determined by the expressions, $T(z) = z_1$, $z = S(z_1)$ respectively, where $z = \sum_{n=1}^{\infty} x_n y_n,\ z_1 = \sum_{n=1}^{\infty} x_n * \varphi_n$.

Linearity of the operators T and S is obvious. Boundedness of the operators T and S follows from the fact that if $z^{(m)} \to z, T\left(z^{(m)}\right) = z_1^{(m)} \to z_1$, then $T(z) = z_1$ and vice versa. On the other hand, it is clear that $T(xy_n) = x*\varphi_n,\ S(x * \varphi_n) = xy_n$. Applying statements 3 and 4, we get the proof of the theorem.

The theorem is proved.

Corollary 4. Let X, Y, Y_1, Z and Z_1 be Banach spaces, $\{y_n\}_{n\in N} \subset Y$ have the b-biorthogonal system $\{y_n^*\}_{n\in N}$, $\Phi = \{\varphi_n\}_{n\in N} \subset Y_1$ be some b_1-basis in Z_1 with space of coefficients $\widetilde{X}_\Phi$. Then, for the system $\{y_n\}_{n\in N}$ to be a $b_{\widetilde{X}_\Phi}$-basis in Z if there exists a boundedly invertible operator $T \in L(Z_1, Z) : T(xy_n) = x * \varphi_n\ \forall x \in X,\ n \in N$.

Thus, from the proved theorems we get the following statement.

Corollary 5. Let $\widetilde{X}$ be a CB-space of sequences $\widetilde{x} = \{x_n\}_{n\in N},\ x_n \in X$ with canonical basis $\{e_n\}_{n\in N}$. Then all the $b_{\widetilde{X}}$ bases in Z are isomorphic

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