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WEIGHTED APPROXIMATION BY POSITIVE LINEAR OPERATORS

Abstract

In this paper we introduce a modified Gadjiev-Ibragimov operators in weighted spaces and obtain its Korovkin type approximation properties.The rates of convergence of this generalization are obtained by means of modulus of continuity.

1. Introduction

Approximation theory is an area of mathematical analysis, which, at its core, is concerned with the approximation of functions by simpler and more easily calculated functions. Korovkin discovered a simple and easily applicable criterion to check if a sequence of positive linear operators converges uniformly to the function to be approximated. This criterion says that the necessary and sufficient condition for the uniform convergence of the sequence L_n of positive linear operators to the continuous function f on the interval $[a,b]$, is the uniform convergence of the sequence $L_n f$ to f for the only three functions t^v, $v=0,1,2$.

In expensive paper [11] defined a general sequence of positive operators and called later Gadjiev Ibragimov operators. Some properties of Gadjiev Ibragimov operators were established in different papers, and we refer to [1], [3], [4], [5], [6], [7] and [8].

In this paper we will give a modification of the Gadjiev-Ibragimov operators in weighted spaces.

There are many approximating operators that their Korovkin type approximation properties and rates of convergence are investigated. We will use the weighted Korovkin type theorems, proved by A. D. Gadjiev [9], [10]. Now,we give Gadjiev 's results in weighted spaces.

We consider the weighted spaces of the functions which are defined on positive semi- axis $[0,\infty)$ and satisfy the inequality $|f(x)| \leq M_f \rho(x)$. Here $\rho(x) = 1+x^2$ is a weight function and M_f is a positive constant depending only on f. We denote the set of functions that satisfy this inequality by B_ρ. B_ρ is a normed space with the norm $\|f\|_\rho = \sup\limits_{x\in\mathbb{R}} \dfrac{|f(x)|}{\rho(x)}$. C_ρ denotes the subspace of all continuous functions in $B_\rho[0,\infty)$ and $C_\rho^k[0,\infty)$ denotes the subspace of all functions $f \in C_\rho[0,\infty)$ with $\lim\limits_{|x|\longrightarrow\infty} \dfrac{f(x)}{\rho(x)} = k_f < \infty$ where k_f is a constant depending only on f.

Lemma 1 ([9], [10]). *In order that the sequence of positive linear operators $\{L_n\}_{n\in\mathbb{N}}$ act from C_ρ to B_ρ, it is necessary and sufficient that the inequality*

$$L_n(\rho,x) \leq M\rho(x)$$

is fulfilled with some positive constant M.

Theorem 1 ([9], [10]). *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of linear positive operators taking C_ρ into B_ρ and satisfying the conditions*

$$\lim_{n\longrightarrow\infty}\|L_n(t^m,x)-x^m\|_\rho=0,\quad m=0,1,2$$

Then for any function $f\in C_\rho^k$,

$$\lim_{n\longrightarrow\infty}\|L_n(f,x)-f(x)\|_\rho=0,$$

and there exists a function $f^*\in C_\rho\setminus C_\rho^k$ such that

$$\limsup_{n\longrightarrow\infty}\|L_n(f^*,x)-f^*(x)\|_\rho>0.$$

Note that for every $f\in C_\rho^k$ weighted modulus of continuity Ω introduced in [2], [12] by the formula

$$\Omega(f,\delta)=\sup_{\substack{x\in[0,\infty)\\|h|\le\delta}}\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}$$

Some elementary properties of $\Omega(f,\delta)$are collected in the following.

Lemma 2 ([2], [8], [12]). *For every $f\in C_\rho^k$,*

(i) $\Omega(f,\delta)$ is a monotonically increasing function of $\delta,\delta\ge 0$.

(ii) $\lim_{\delta\longrightarrow 0}\Omega(f,\delta)=0$.

(iii) For each positive value of λ

$$\Omega(f,\lambda\delta)\le 2(1+\lambda)\left(1+\delta^2\right)\Omega(f,\delta).$$

From the *(iii)* and the definition of $\Omega(f,\delta)$ we get

$$|f(t)-f(x)|\le 2\left(1+\delta^2\right)\Omega(f,\delta)\left(1+x^2\right)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)$$

for every $f\in C_\rho^k$ and $x,t\in[0,\infty)$.

The rest of the paper is organized as follows.

In Section 2 we introduce the modified Gadjiev-Ibragimov operators in $C_\rho[0,\infty)$ and give some approximation properties of this new operator. In Section 3 we compute the rate of convergence for the $\sup_{x\in[0,\infty)}\frac{|L_n(f,x)-f(x)|}{(1+x^2)^3}$ with the help of weighted modulus of continuity.

2. Construction of the operators

We introduce a sequence of linear positive operators

$$L_n(f,x)=\sum_{v=0}^{\infty}f\left(\frac{v}{\beta_n}\right)K_{n,v}(x)\frac{(-\alpha_n)^v}{v!}. \tag{2.1}$$

Here $(\alpha_n), (\beta_n)$; $\lim\limits_{n\longrightarrow\infty}\beta_n=\infty$, $\lim\limits_{n\longrightarrow\infty}\dfrac{\alpha_n}{\beta_n}=0$ and $\lim\limits_{n\longrightarrow\infty}n\dfrac{\alpha_n}{\beta_n}=1$ are the terms of the real number sequences; $K_{n,v}(x)$ the function depending on the parameters v and n to get the following conditions:

1) For any natural n any $v=0,1,2,...$ and for any $x\in[0,\infty)$

$$(-1)^v K_{n,v}(x)\geq 0;$$

2) For any $x\in[0,\infty)$

$$\sum_{v=0}^{\infty}K_{n,v}(x)\frac{(-\alpha_n)^v}{v!}=1; \tag{2.2}$$

3)

$$K_{n,v}(x)=-nxK_{n+m,v-1}(x). \tag{2.3}$$

for any $x\in[0,\infty)$ where $n+m$ is natural number and m is a constant independent of v.

Remark 1. Taking

$$K_{n,v}(x)=\begin{cases}(-1)^v\frac{n!}{(n-v)!}x^v(1-x)^{n-v}, & x\in[0,1], v\leq n\\ 0, & otherwise\end{cases}$$

and choosing $\alpha_n=1$ and $\beta_n=n$; (2.1) became the classical Bernstein polynomials.If we take $\beta_n=\dfrac{n}{b_n}$where β_n is the increasing sequences of positive number, such that $\lim\limits_{n\longrightarrow\infty}b_n=\infty$ and $\lim\limits_{n\longrightarrow\infty}\dfrac{b_n}{n}=0$ then substituting in representation $K_{n,v}(x)$given above $\dfrac{x}{b_n}$replace x we obtain for $0\leq x\leq b_n$ well known Bernstein- Cholodowsky polynomials. In booth these cases $m=-1$.Choosing $\alpha_n=1$ and $\beta_n=n$ and

$$K_{n,v}(x)=(-1)^v(nx)^v e^{-nx},$$

we obtain Szasz operators $(m=0)$.

Lemma 3. *L_n defined by* (2.1) *is a linear positive operator from $C_\rho[0,\infty)$ into $B_\rho[0,\infty)$ and having the properties*

$$L_n(1,x)=1 \tag{2.4}$$

$$L_n(t,x)=\frac{\alpha_n}{\beta_n}nx \tag{2.5}$$

$$L_n(t^2,x)=\left(\frac{\alpha_n}{\beta_n}x\right)^2 n(n+m)+\frac{\alpha_n}{\beta_n^2}nx. \tag{2.6}$$

Proof. Since

$$\sum_{v=0}^{\infty}K_{n,v}(x)\frac{(-\alpha_n)^v}{v!}=1,$$

we get (2.4) immediately. It is obvious that

$$L_n(t,x)=\sum_{v=0}^{\infty}\frac{v}{\beta_n}K_{n,v}(x)\frac{(-\alpha_n)^v}{v!}=$$

$$= \frac{\alpha_n}{\beta_n} nx \sum_{v=1}^{\infty} K_{n+m,v-1}(x) \frac{(-\alpha_n)^{v-1}}{(v-1)!} =$$

$$= \frac{\alpha_n}{\beta_n} nx \sum_{v=0}^{\infty} K_{n+m,v}(x) \frac{(-\alpha_n)^{v}}{v!} = \frac{\alpha_n}{\beta_n} nx.$$

Then we can write

$$L_n\left(t^2, x\right) = \sum_{v=0}^{\infty} \left(\frac{v}{\beta_n}\right)^2 K_{n,v}(x) \frac{(-\alpha_n)^{v}}{v!} =$$

$$= \sum_{v=2}^{\infty} \frac{v(v-1)}{\beta_n^2} K_{n,v}(x) \frac{(-\alpha_n)^{v}}{v!} + \sum_{v=1}^{\infty} \frac{v}{\beta_n^2} K_{n,v}(x) \frac{(-\alpha_n)^{v}}{v!} =$$

$$= \sum_{v=2}^{\infty} \left(\frac{\alpha_n}{\beta_n}\right)^2 K_{n,v}(x) \frac{(-\alpha_n)^{v-2}}{(v-2)!} + \frac{1}{\beta_n} L_n(t, x).$$

From (2.3), we obtain

$$L_n\left(t^2, x\right) = \left(\frac{\alpha_n}{\beta_n} x\right)^2 n(n+m) + \frac{\alpha_n}{\beta_n^2} nx.$$

Now let us show that L_n is acting from $C_\rho[0,\infty)$ into $B_\rho[0,\infty)$. Using Lemma 1.1 we show that there is a positive constant M_ρ such that

$$\|L_n(\rho, x)\|_\rho \leq M_\rho.$$

Since $\rho(x) = 1 + x^2$,

$$L_n(\rho, x) = L_n\left(1 + t^2, x\right) = 1 + \left(\frac{\alpha_n}{\beta_n} x\right)^2 n(n+m) + \frac{\alpha_n}{\beta_n^2} nx.$$

Using definition of norm in $C_\rho[0,\infty)$ and definition of $(\alpha_n), (\beta_n)$;

$$\|L_n(\rho, x)\|_\rho = \sup_{x\in\mathbb{R}} \frac{1 + \left(\frac{\alpha_n}{\beta_n} x\right)^2 n(n+m) + \frac{\alpha_n}{\beta_n^2} nx}{1+x^2} \leq$$

$$\leq \sup_{x\in\mathbb{R}} \frac{1}{1+x^2} + \left(\frac{\alpha_n}{\beta_n}\right)^2 n(n+m) \sup_{x\in\mathbb{R}} \frac{x^2}{1+x^2} + \frac{\alpha_n}{\beta_n^2} n \sup_{x\in\mathbb{R}} \frac{x}{1+x^2}.$$

So we get

$$\|L_n(\rho, x)\|_\rho \leq M_\rho.$$

The properties of the sequences$(\alpha_n), (\beta_n)$,formula(2.6) and Lemma 1.1 show that L_n acts from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$. The Lemma 2.1 is proven.

We are ready to give the following first result for the operators L_n introduced by (2.1).

Theorem 2. *Let $\{L_n\}$ be sequence of linear positive operators defined by (2.1). Then for every function $f \in C_\rho^k[0,\infty)$,*

$$\lim_{n\longrightarrow\infty} \|L_n(f, x) - f(x)\|_\rho = 0.$$

Proof. Clearly, on $[0,\infty)$ $\lim\limits_{n\longrightarrow\infty}\|L_n(1,x)-1\|_\rho=0$.

From the assumption $\lim\limits_{n\longrightarrow\infty} n\dfrac{\alpha_n}{\beta_n}=1$ and equality (2.5) we can write

$$\sup_{x\in[0,\infty)}\frac{|L_n(t,x)-x|}{\rho(x)}=\left|\frac{\alpha_n}{\beta_n}n-1\right|\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\ .$$

Hence we obtain

$$\lim_{n\longrightarrow\infty}\|L_n(t,x)-x\|_\rho=0\ .$$

Also by the equality (2.6) and assumption $\lim\limits_{n\longrightarrow\infty} n\dfrac{\alpha_n}{\beta_n}=1$, $\lim\limits_{n\longrightarrow\infty}\dfrac{\alpha_n}{\beta_n}=0$ we have

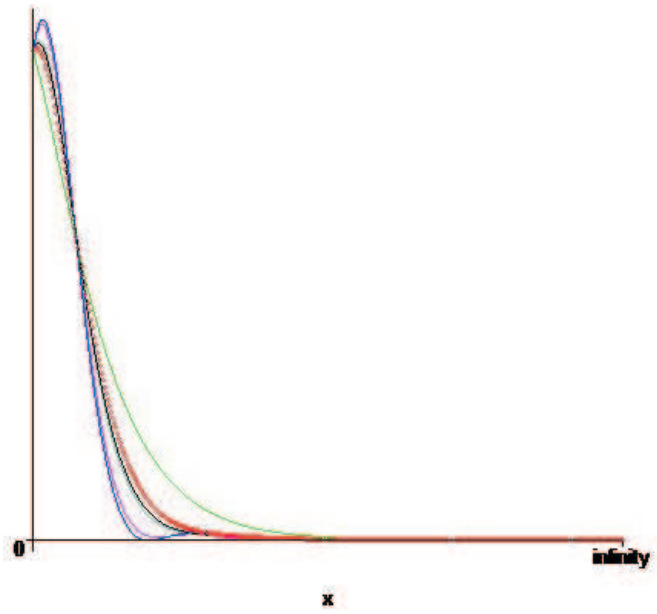
$$\sup_{x\in[0,\infty)}\frac{\left|L_n\left(t^2,x\right)-x^2\right|}{\rho(x)}=\sup_{x\in[0,\infty)}\frac{\left|x^2\left(\frac{\alpha_n^2}{\beta_n^2}n(n+m)-1\right)+\frac{\alpha_n}{\beta_n^2}nx\right|}{1+x^2}\leq$$

$$\leq\left|\frac{\alpha_n^2}{\beta_n^2}n(n+m)-1\right|\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}+\left|\frac{\alpha_n}{\beta_n^2}n\right|\sup_{x\in[0,\infty)}\frac{x}{1+x^2}.$$

Thus we get

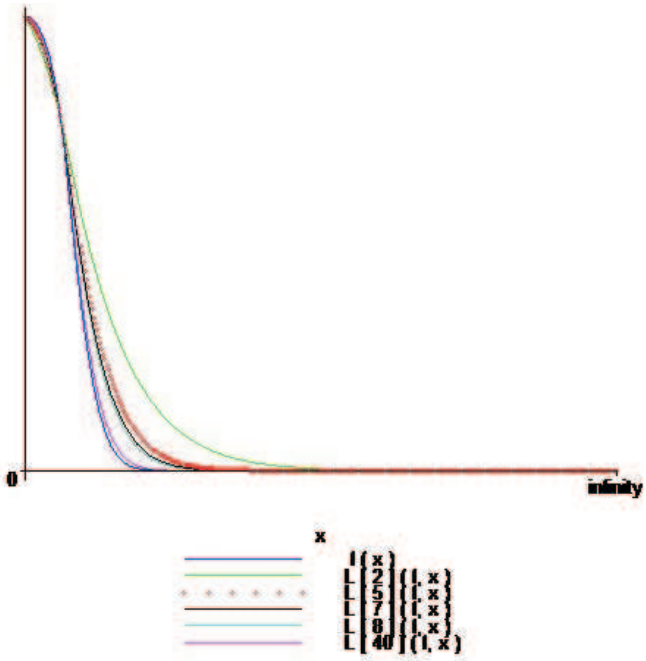
$$\lim_{n\longrightarrow\infty}\left\|L_n\left(t^2,x\right)-x^2\right\|_\rho=0\ .$$

Therefore the proof follows from Theorem 1.



Approximation of $f(x)=\frac{1+\sin 3x}{e^{2x+1}}$ by $L_n(f,x)$.

We give the graphics of approximation of functions $f(x)=\dfrac{1+\sin 3x}{e^{2x+1}}$ and $f(x)=\dfrac{x^2+1}{e^{3x^2-2}}$ by modified Gadjiev-İbragimov operators (2.1) (see Figs. 1 and 2 respectively).



Approximation of $f(x)=\frac{x^2+1}{e^{3x^2-2}}$ by $L_n(f,x)$.

3. Rates of convergence

In this section we want to find the rate of convergence of the sequence of operators $\{L_n\}$.

Theorem 3. *If $f\in C_\rho^k[0,\infty)$ then the inequality*

$$\sup_{x\in[0,\infty)}\frac{|L_n(f,x)-f(x)|}{(1+x^2)^3}\leq K\Omega\left(f\ ,\sqrt[4]{\left(\frac{\alpha_n}{\beta_n}n-1\right)^4+\frac{\alpha_n}{\beta_n}+\frac{1}{\beta_n}}\right) \tag{3.1}$$

holds for a sufficiently large n, where K is a constant independent of n.

Proof. Using linearity and positivity of operators (2.1) we get by formula (2.4)

$$|L_n(f,x)-f(x)|\leq\sum_{v=0}^{\infty}\left|f\left(\frac{v}{\beta_n}\right)-f(x)\right|K_{n,v}(x)\frac{(-\alpha_n)^v}{v!}$$

From the definition of weighted modulus of continuity by choosing $t=\dfrac{v}{\beta_n}$ we have

$$|f(t)-f(x)|\leq 2\left(1+\delta_n^2\right)\Omega(f\ ,\delta_n)\left(1+x^2\right)\left(1+\left(\frac{v}{\beta_n}-x\right)^2\right)\left(1+\frac{\left|\frac{v}{\beta_n}-x\right|}{\delta_n}\right)$$

for every $\delta_n>0$. Thus we obtain

$$\begin{aligned}|L_n(f,x)-f(x)| \ &\leq\ 2\left(1+\delta_n^2\right)\Omega(f\ ,\delta_n)\left(1+x^2\right)\sum_{v=0}^{\infty}\left(1+\left(\frac{v}{\beta_n}-x\right)^2\right)\\ &\left(1+\frac{\left|\frac{v}{\beta_n}-x\right|}{\delta_n}\right)K_{n,v}(x)\frac{(-\alpha_n)^v}{v!}.\end{aligned} \tag{3.2}$$

It is easy to see that

$$\left(1+\left(\frac{v}{\beta_n}-x\right)^2\right)\left(1+\frac{\left|\frac{v}{\beta_n}-x\right|}{\delta_n}\right)\leq 2^4\left(1+\delta_n^2\right)\left(1+\frac{\left(\frac{v}{\beta_n}-x\right)^4}{\delta_n^4}\right)$$

and using (3.2) we obtain

$$\begin{aligned}|L_n(f,x)-f(x)| &\leq 32\left(1+\delta_n^2\right)^2\left(1+x^2\right)\Omega\left(f\ ,\delta_n\right)\\ &\left(1+\frac{1}{\delta_n^4}L_n\left((t-x)^4,x\right)\right).\end{aligned}\tag{3.3}$$

Using equalities

$$\begin{aligned}t^4 &= t(t-1)(t-2)(t-3)+6t^3-11t^2+6t\\ t^3 &= t(t-1)(t-2)+3t^2-2t\end{aligned}$$

and the properties (2.5) and (2.6) it is easy to calculate $L_n\left(t^3,x\right)$ and $L_n\left(t^4,x\right)$.

We have

$$L_n\left(t^3,x\right)=\sum_{v=0}^{\infty}\frac{v(v-1)(v-2)}{\beta_n^3}K_{n,v}(x)\frac{(-\alpha_n)^v}{v!}+\frac{3}{\beta_n}L_n\left(t^2,x\right)-\frac{2}{\beta_n^2}L_n(t,x)=$$

$$=n(n+m)(n+2m)\left(\frac{\alpha_n}{\beta_n}\right)^3x^3\sum_{v=3}^{\infty}K_{n+3m,v-3}(x)\frac{(-\alpha_n)^{v-3}}{(v-3)!}+$$

$$+\frac{3}{\beta_n}L_n\left(t^2,x\right)-\frac{2}{\beta_n^2}L_n(t,x)=$$

$$=n(n+m)(n+2m)\left(\frac{\alpha_n}{\beta_n}\right)^3x^3+\frac{3}{\beta_n}L_n\left(t^2,x\right)-\frac{2}{\beta_n^2}L_n(t,x)=$$

$$=\left\{\left(\frac{\alpha_n}{\beta_n}\right)^3n^3+3\left(\frac{\alpha_n}{\beta_n}\right)^3n^2m+2\left(\frac{\alpha_n}{\beta_n}\right)^3nm^2\right\}x^3+$$

$$+\left\{\frac{3}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^2n^2+\frac{3}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^2nm\right\}x^2+\frac{1}{\beta_n^2}\left(\frac{\alpha_n}{\beta_n}\right)nx.\tag{3.4}$$

Similarly, next equalities are true.

$$L_n\left(t^4,x\right)=\sum_{v=0}^{\infty}\frac{v(v-1)(v-2)(v-3)}{\beta_n^4}K_{n,v}(x)\frac{(-\alpha_n)^v}{v!}+\frac{6}{\beta_n}L_n\left(t^3,x\right)-$$

$$-\frac{11}{\beta_n^2}L_n\left(t^2,x\right)+\frac{6}{\beta_n^3}L_n(t,x)=$$

$$=\left\{\left(\frac{\alpha_n}{\beta_n}\right)^4n^4+6\left(\frac{\alpha_n}{\beta_n}\right)^4n^3m+11\left(\frac{\alpha_n}{\beta_n}\right)^4n^2m^2+6\left(\frac{\alpha_n}{\beta_n}\right)^4nm^3\right\}x^4+$$

$$+\left\{\frac{6}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3n^3+\frac{18}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3n^2m\right\}x^3+\frac{12}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3nm^2+$$

$$+\left\{\frac{7}{\beta_n^2}\left(\frac{\alpha_n}{\beta_n}\right)^2 n^2+\frac{7}{\beta_n^2}\left(\frac{\alpha_n}{\beta_n}\right)^2 nm\right\}x^2+\frac{1}{\beta_n^3}\left(\frac{\alpha_n}{\beta_n}\right)nx. \tag{3.5}$$

Using (2.4), (2.5), (2.6), (3.4)and (3.5) we can write

$$L_n\left((t-x)^4,x\right)=L_n\left(t^4,x\right)-4xL_n\left(t^3,x\right)+6x^2L_n\left(t^2,x\right)-4x^3L_n\left(t,x\right)+x^4=$$

$$=\left\{\left(\frac{\alpha_n}{\beta_n}n-1\right)^4+6\left(\frac{\alpha_n}{\beta_n}\right)^4 n^3m+11\left(\frac{\alpha_n}{\beta_n}\right)^4 n^2m^2+\right.$$

$$+6\left(\frac{\alpha_n}{\beta_n}\right)^4 nm^3-12\left(\frac{\alpha_n}{\beta_n}\right)^3 n^2m-8\left(\frac{\alpha_n}{\beta_n}\right)^3 nm^2+$$

$$\left.+6\left(\frac{\alpha_n}{\beta_n}\right)^2 nm\right\}x^4+\left\{\frac{6}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3 n^3+\frac{18}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3 n^2m+\right.$$

$$\left.+\frac{12}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3 nm^2-\frac{12}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^2 n^2-\frac{12}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^2 nm+\frac{6}{\beta_n}\frac{\alpha_n}{\beta_n}n\right\}x^3+$$

$$+\left\{\frac{7}{\beta_n^2}\left(\frac{\alpha_n}{\beta_n}\right)^2 n^2+\frac{7}{\beta_n^2}\left(\frac{\alpha_n}{\beta_n}\right)^2 nm-\frac{4}{\beta_n^2}\frac{\alpha_n}{\beta_n}n\right\}x^2+\frac{1}{\beta_n^3}\frac{\alpha_n}{\beta_n}nx\leq$$

$$\leq\left\{\left(\frac{\alpha_n}{\beta_n}n-1\right)^4+6\left(\frac{\alpha_n}{\beta_n}\right)^4 n^3m+11\left(\frac{\alpha_n}{\beta_n}\right)^4 n^2m^2+6\left(\frac{\alpha_n}{\beta_n}\right)^4 nm^3+\right.$$

$$\left.+6\left(\frac{\alpha_n}{\beta_n}\right)^2 nm\right\}x^4+\left\{\frac{6}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3 n^3+\frac{18}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3 n^2m+\right.$$

$$\left.+\frac{12}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3 nm^2+\frac{6}{\beta_n}\frac{\alpha_n}{\beta_n}n\right\}x^3+\left\{\frac{7}{\beta_n^2}\left(\frac{\alpha_n}{\beta_n}\right)^2 n^2+\frac{7}{\beta_n^2}\left(\frac{\alpha_n}{\beta_n}\right)^2 nm\right\}x^2+\frac{1}{\beta_n^3}\frac{\alpha_n}{\beta_n}nx.$$

Deviding by $\left(1+x^2\right)^2$ and using the inequalities

$$\left(\frac{\alpha_n}{\beta_n}\right)^3\leq\left(\frac{\alpha_n}{\beta_n}\right)^2\leq\frac{\alpha_n}{\beta_n}\leq 1,\frac{\alpha_n}{\beta_n}n\leq 2\ \text{ and }\ \left(\frac{1}{\beta_n}\right)^3\leq\left(\frac{1}{\beta_n}\right)^2\leq\frac{1}{\beta_n}$$

for a large n, we obtain

$$\frac{\left|L_n\left((t-x)^4,x\right)\right|}{(1+x^2)^3}\leq\left(\frac{\alpha_n}{\beta_n}n-1\right)^4+6\left(\frac{\alpha_n}{\beta_n}\right)^4 n^3m+11\left(\frac{\alpha_n}{\beta_n}\right)^4 n^2m^2+6\left(\frac{\alpha_n}{\beta_n}\right)^4 nm^3+$$

$$+6\left(\frac{\alpha_n}{\beta_n}\right)^2 nm+\frac{6}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3 n^3+\frac{18}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3 n^2m+\frac{12}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3 nm^2+$$

$$+\frac{6}{\beta_n}\frac{\alpha_n}{\beta_n}n+\frac{7}{\beta_n^2}\left(\frac{\alpha_n}{\beta_n}\right)^2 n^2+\frac{7}{\beta_n^2}\left(\frac{\alpha_n}{\beta_n}\right)^2 nm+\frac{1}{\beta_n^3}\frac{\alpha_n}{\beta_n}n\leq$$

$$\leq(4m)^4\left\{\left(\frac{\alpha_n}{\beta_n}n-1\right)^4+\frac{\alpha_n}{\beta_n}+\frac{1}{\beta_n}\right\}.$$

Therefore for a large n

$$\frac{|L_n(f,x)-f(x)|}{(1+x^2)^3} \leq 32\left(1+\delta_n^2\right)^2 \Omega\left(f\ ,\delta_n\right)\left(1+\frac{(4m)^4}{\delta_n^4}\left\{\left(\frac{\alpha_n}{\beta_n}n-1\right)^4+\frac{\alpha_n}{\beta_n}+\frac{1}{\beta_n}\right\}\right)$$

Choosing

$$\delta_n=\sqrt[4]{\left(\frac{\alpha_n}{\beta_n}n-1\right)^4+\frac{\alpha_n}{\beta_n}+\frac{1}{\beta_n}}$$

we have the following inequality with positive constant K independent on n

$$\sup_{x\in[0,\infty)}\frac{|L_n(f,x)-f(x)|}{(1+x^2)^3} \leq K\Omega\left(f\ ,\sqrt[4]{\left(\frac{\alpha_n}{\beta_n}n-1\right)^4+\frac{\alpha_n}{\beta_n}+\frac{1}{\beta_n}}\right).$$

The proof is complete.

4. Appendix

```
> restart;
> with (plots):
> f := x-> (x^2 + 1)/exp(2 * x^3 + 3);
> printf('Please enter the number of operator.\n');
> m := scanf('%d')[1];
> printf('Please enter the sum of the number of terms.\n');
> t := scanf('%d')[1];
> for n from 1 to m do
> alpha(n):=1:
> beta(n):=n:
> K[n](v,x):=(-1)^v*(n*x)^v*(exp(-n*x)):
> L[n](f,x):=sum(f(v/beta(n))*K[n](v,x)*(-alpha(n))^v/v!,v=0..t):
> end do:
> p1:=plot(f(x),x=0..infinity,color=blue):
> p2:=plot(L[2](f,x),x=0..infinity,color=green):
> p3:=plot(L[10](f,x),x=0..infinity,color=red):
> p4:=plot(L[20](f,x),x=0..infinity,color=black):
> p5:=plot(L[30](f,x),x=0..infinity,color=cyan):
> p6:=plot(L[55](f,x),x=0..infinity,color=magenta):
> display([p1,p2,p3,p4,p5,p6]);
```

Acknowledgment

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