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WEIGHTED INEQUALITY FOR SINGULAR INTEGRALS IN LEBESGUE SPACES, ASSOCIATED WITH THE LAPLACE-BESSEL DIFFERENTIAL OPERATORS

Abstract

In this paper, the author establish some theorem for the boundedness of singular integral operators, associated with the Laplace-Bessel differential operator on a weighted Lebesgue space. Sufficient condition on weighted function ω is given so that certain singular integral operator is bounded on the weighted Lebesgue spaces $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$.

1. Introduction

The singular integral operators that have been considered by Mihlin [11] and Calderon and Zygmund [5] are playing an important role in the theory Harmonic Analysis and in particular, in the theory partial differential equations. Klyuchantsev [8] and Kipriyanov and Klyuchantsev [9] have firstly introduced and investigated by the boundedness in L_p-spaces of multidimensional singular integrals, generated by the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i} + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad \gamma_1 > 0, \ldots, \gamma_k > 0, \quad k = 1, \ldots, n$$

($B \equiv B_{k,n}$ singular integrals). Aliev and Gadjiev [3] and Gadjiev and Guliyev [4] have studied the boundedness of $B_n \equiv B_{n-1,n}$ singular integrals in weighted L_p-spaces with radial and general weights consequently. The maximal functions, singular integrals, potentials and related topics associated with the Laplace-Bessel differential operator Δ_{B_n} which is known as an important differential operator in analysis and its applications, have been the research areas many mathematicans such as K. Stempak [16], I. Kipriyanov and M. Klyuchantsev [8,9], L. Lyakhov [13,14], A.D. Gadjiev and I.A. Aliev [2,3], V.S. Guliyev [6,7] and others.

In the paper, we shall prove the boundedness of singular integral operator, generated by the B Bessel differential operators on a weighted L_p spaces. Sufficient conditions on weighted function ω is given so that certain singular integral operator is bounded from the weighted Lebesgue spaces $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ into $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$.

2. Notations and Background

Let $\mathbb{R}^n$ be n–dimensional Euclidean space, $x = (x_1, ..., x_n)$, $\xi = (\xi_1, ..., \xi_n)$ are vectors in $\mathbb{R}^n$, $x \cdot \xi = x_1\xi_1 + \ldots + x_n\xi_n$, $|x| = (x \cdot x)^{1/2}$, $x = (x', x'')$, $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$, $\mathbb{R}^n_{k,+} = \{x \in \mathbb{R}^n; x_1 > 0, \ldots, x_k > 0\}$, $E(x, r) = \{y \in \mathbb{R}^n_{k,+} \ ; \ |x - y| < r\}$, $\gamma = (\gamma_1, \ldots, \gamma_k)$, $\gamma_1 > 0, \ldots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$, $(x')^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_k^{\gamma_k}$.

Let $S_{k,+} = \{x \in \mathbb{R}^n_{k,+} : \ |x| = 1\}$ and for measurable set $E \subset \mathbb{R}^n_{k,+}$ let $|E|_\gamma = \int\limits_E (x')^\gamma dx$, then

$$|E(0, r)|_\gamma = \omega(n, k, \gamma) r^{n+|\gamma|},$$

where $\omega(n,k,\gamma)=|E(0,1)|_\gamma$.

An almost everywhere positive and locally integrable function $\omega:\ \mathbb{R}^n_{k,+}\to\mathbb{R}$ will be called a weight. We shall denote by $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ the set of all measurable function f on $\mathbb{R}^n_{k,+}$ such that the norm

$$\|f\|_{L_{p,\omega,\gamma}}\equiv\|f\|_{p,\omega,\gamma}=\left(\int\limits_{\mathbb{R}^n_{k,+}}|f(x)|^p\omega(x)(x')^\gamma dx\right)^{1/p},\qquad 1\le p<\infty$$

is finite. For $\omega=1$ the space $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ is denoted by $L_{p,\gamma}(\mathbb{R}^n_{k,+})$, and the norm $\|f\|_{L_{p,\omega,\gamma}}$ by $\|f\|_{L_{p,\gamma}}$.

The operator of generalized shift (B shift operator) is defined by the following way (see [8], [12]):

$$T^y f(x)=C_{k,\gamma}\int\limits_0^\pi\ldots\int\limits_0^\pi f\left((x',y')_\alpha,x''-y''\right)\,d\nu(\alpha),$$

where

$$C_{k,\gamma}=\pi^{-\frac{k}{2}}\prod_{i=1}^k\frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)},\quad (x',x'')\in\mathbb{R}^k\times\mathbb{R}^{n-k},$$

$$(x',y')_\alpha=((x_1,y_1)_{\alpha_1},\ldots,(x_k,y_k)_{\alpha_k}),$$

$$(x_i,y_i)_{\alpha_i}=\sqrt{x_i^2-2x_iy_i\cos\alpha_i+y_i^2},\quad 1\le i\le k,$$

and

$$d\nu(\alpha)=\prod_{i=1}^k\sin^{\gamma_i-1}\alpha_i d\alpha_i,\quad 1\le k\le n.$$

It is well known (see, for example [10]) that the generalized shift operator T^y is closely related to the Laplace-Bessel differential operator Δ_B.

Note that this shift operator is closely connected with B_n Bessel's singular differential operators (see [8], [12]).

Definition 1. *The weight function ω belongs to the class $A_{p,\gamma}(\mathbb{R}^n_{k,+})$ for $1<p<\infty$, if there exists a positive constant C such that for any $x\in\mathbb{R}^n_{k,+}$ and $r>0$*

$$|E(x,r)|_\gamma^{-1}\int\limits_{E(x,r)}\omega(y)(y')^\gamma dy\left(|E(x,r)|_\gamma^{-1}\int\limits_{E(x,r)}\omega^{-\frac{1}{p-1}}(y)(y')^\gamma dy\right)^{p-1}\le C$$

and ω belongs to $A_{1,\gamma}(\mathbb{R}^n_{k,+})$, if there exists a positive constant C such that for any $x\in\mathbb{R}^n_{k,+}$ and $r>0$

$$|E(x,r)|_\gamma^{-1}\int\limits_{E(x,r)}w^{-\frac{1}{p-1}}(y)(y')^\gamma dy\le C\operatorname*{ess\,inf}_{y\in E(x,r)}\omega(y).$$

The properties of the class $A_{p,\gamma}(\mathbb{R}^n_{k,+})$ are analogous to those of the B.Muckenhoupt classes. In particular, if $w \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$, then $w \in A_{p-\varepsilon,\gamma}(\mathbb{R}^n_{k,+})$ for a certain sufficiently small $\varepsilon > 0$ and $w \in A_{p_1,\gamma}(\mathbb{R}^n_{k,+})$ for any $p_1 > p$.

Note that, $|x|^\alpha \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$, if and only if $-(n+|\gamma|) < \alpha < (n+|\gamma|)(p-1)$ and $|x|^\alpha \in A_{1,\gamma}(\mathbb{R}^n_{k,+})$, if and only if $-(n+|\gamma|) < \alpha \le 0$.

The main goal of this paper is to establish weighted L_p-estimates for the norms of the singular integral operator generated by a generalized shift operator (B singular integral operator):

$$Tf(x) = p.v. \int\limits_{\mathbb{R}^n_{k,+}} \frac{\Omega(\theta)}{|y|^{n+|\gamma|}} [T^y f(x)](y')^\gamma dy =$$

$$= \lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^n_{k,+}\setminus E(0,\varepsilon)} \frac{\Omega(\theta)}{|y|^{n+|\gamma|}} [T^y f(x)](y')^\gamma dy = \lim_{\varepsilon\to 0} T_\varepsilon f(x), \tag{1}$$

where $\theta = y/|y|$, and the characteristic $\Omega(\theta)$ belong to some function space on the hemisphere $S_{k,+} = \{x \in \mathbb{R}^n_{k,+} \ : \ |x| = 1\}$ and satisfying the "cancellation" condition

$$\int\limits_{S_{k,+}} \Omega(\theta)(\theta')^\gamma d\sigma(\theta) = 0$$

($d\sigma(\theta)$ is the area element of the sphere $|\theta| = 1$). The existence of the limit (1) for all $x \in \mathbb{R}^n_{k,+}$ and for Schwartz test functions $f(x)$ can be proved in the standard way if we take into account the well-known estimate $|T^y f(x) - f(x)| \le c(x)|y|$.

The theorem below is known about the behavior of the B singular integral operator T in $L_{p,\gamma}$ (see [8,9]).

Theorem 1. *Suppose that the characteristic $\Omega(\theta)$ of the B singular integral (1) satisfies the conditions*

$$\int\limits_{S_{k,+}} \Omega(\theta)(\theta')^\gamma d\sigma(\theta) = 0, \qquad C_0 = \sup_{\theta\in S_{k,+}} |\Omega(\theta)| < \infty. \tag{2}$$

Then

$$\|Tf\|_{L_{p,\gamma}} \le C\, C_0 \, \|f\|_{L_{p,\gamma}}, \qquad 1 < p < \infty,$$

where $C > 0$ depend only on p, k, γ and n.

The aim of this paper is the following assertion about the behavior of the B singular integral operator (1) in weighted spaces.

We establish the boundedness in weighted L_p spaces for the B singular integrals.

Theorem 2. *Suppose that the characteristic $\Omega(\theta)$ of the B singular integral (1) satisfies the conditions* (2).

Moreover, let ω be a positive function for which there exists a constant $c_1 > 0$ such that

$$\sup_{2^{m-2}\le|x|<2^{m+1}} \omega(x) \le c_1 \inf_{2^{m-2}\le|x|<2^{m+1}} \omega(x), \quad m \in \mathbb{Z} \tag{3}$$

and $\omega \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1 \le p < \infty$.

Then,

i_1) there exists a constant C_1, independent of f and ε such that for all $f \in L_{p,\omega}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$

$$\int\limits_{\mathbb{R}^n_{k,+}} |T_\varepsilon f(x)|^p \omega(x)(x')^\gamma dx \le C_1 \int\limits_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx. \tag{4}$$

i_2) the limit $\lim\limits_{\varepsilon\to 0} T_\varepsilon f$, which will be denoted by Tf, exists in the sense of $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$ and

$$\int\limits_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \omega(x)(x')^\gamma dx \le C_1 \int\limits_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx. \tag{5}$$

ii_1) there exists a constant C_2, independent of f and ε such that for all $f \in L_{1,\omega,\gamma}(\mathbb{R}^n_{k,+})$

$$\int\limits_{\{x\in\mathbb{R}^n_{k,+}:|T_\varepsilon f(x)|>t\}} \omega(x)(x')^\gamma dx \le \frac{C_2}{t} \int\limits_{\mathbb{R}^n_{k,+}} |f(x)|\omega(x)(x')^\gamma dx.$$

ii_2) the limit $\lim\limits_{\varepsilon\to 0} T_\varepsilon f$, which will be denoted by Tf, exists in the sense of $L_{1,\omega,\gamma}(\mathbb{R}^n_{k,+})$, and

$$\int\limits_{\{x\in\mathbb{R}^n_{k,+}:|Tf(x)|>t\}} \omega(x)(x')^\gamma dx \le \frac{C_2}{t} \int\limits_{\mathbb{R}^n_{k,+}} |f(x)|\omega(x)(x')^\gamma dx.$$

Proof. We note that the coefficients C_k in the estimates below depend in general on the parameters n, p and γ, but not the function f and the parameter $\varepsilon > 0$. We first prove parts i_1) and ii_1) of the theorem. Parts i_2) and ii_2) follows from parts i_1) and ii_1) consequently. Without loss of generality we assume that $f(x)$ is an infinitely differentiable function, because such functions are dense in $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$.

Note that

$$T_\varepsilon f(x) = \int\limits_{\mathbb{R}^n_{k,+}} \chi_\varepsilon(y) \frac{\Omega(y/|y|)}{|y|^{n+|\gamma|}} [T^y f(x)](y')^\gamma dy = \int\limits_{\mathbb{R}^n_{k,+}} T^y \left[\chi_\varepsilon(x) \frac{\Omega(x/|x|)}{|x|^{n+|\gamma|}}\right] f(y)(y')^\gamma dy,$$

where χ_ε is the characteristic function of the set $\mathbb{R}^n_{k,+} \setminus E_+(0,\varepsilon)$.

For simplicity we let $K_\varepsilon(x) = \chi_\varepsilon(x)\dfrac{\Omega(x/|x|)}{|x|^{n+|\gamma|}}$.

We proof this theorem along the same line as the proof of Theorem 2 in [15]. Throughout this paper, for $k \in \mathbb{Z}$ we define

$$E_m = \{x \in \mathbb{R}^n_{k,+} : \ 2^{m-1} \le |x| < 2^m\} \text{ and } E^*_m = \{x \in \mathbb{R}^n_{k,+} : \ 2^{m-2} \le |x| < 2^{m+1}\}.$$

If ω satisfies (3) and we set

$$p_m = \inf\{\omega(x) : x \in E^*_m\},$$

then

$$\omega(x) \sim p_m \quad \text{for every} \quad x \in E^*_m.$$

Here the expression $A \sim B$ means, as usual, that there are constants τ_0, τ_1 (independent of the main parameters involved) such that $\tau_0 \le A/B \le \tau_1$.

Given $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$, $1 \le p < \infty$ we write

$$|T_\varepsilon f(x)| = \sum_{m\in\mathbb{Z}} |T_\varepsilon f(x)|\chi_{E_m}(x) \le$$

$$\le \sum_{m\in\mathbb{Z}} |T_\varepsilon f_{m,1}(x)|\,\chi_{E_m}(x) + \sum_{m\in\mathbb{Z}} |T_\varepsilon f_{m,2}(x)|\,\chi_{E_m}(x) \equiv T_{1,\varepsilon} f(x) + T_{2,\varepsilon} f(x),$$

where χ_{E_m} is the characteristic function of the set E_m, $f_{m,1} = f\chi_{E_m^*}$ and $f_{m,2} = f - f_{m,1}$.

By the $L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$ boundedness of T and (3), on T_1, we have

$$\int\limits_{\mathbb{R}^n_{k,+}} |T_{1,\varepsilon} f(x)|^p \omega(x)(x')^\gamma dx = \sum_{m\in\mathbb{Z}} \int\limits_{E_m} |T_{1,\varepsilon} f(x)|^p \omega(x)(x')^\gamma dx \sim$$

$$\sim \sum_{m\in\mathbb{Z}} p_m \int\limits_{E_m} |T_{1,\varepsilon} f(x)|^p (x')^\gamma dx \le \sum_{m\in\mathbb{Z}} C_3 \int\limits_{E_m^*} |f(x)|^p \, p_m \, (x')^\gamma dx \sim$$

$$\sim \sum_{m\in\mathbb{Z}} C_4 \int\limits_{E_m^*} |f(x)|^p \, \omega(x) \, (x')^\gamma dx.$$

By the weak type $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ boundedness of T and (3), on T_1, we have

$$\omega(\{x \in \mathbb{R}^n_{k,+} : |T_{1,\varepsilon} f(x)| > \lambda\}) = \sum_{m\in\mathbb{Z}} \omega(\{x \in E_m : |T_{1,\varepsilon} f(x)| > \lambda\}) \sim$$

$$\sim \sum_{m\in\mathbb{Z}} p_m |\{x \in E_m : |T_{1,\varepsilon} f(x)| > \lambda\}| \le \sum_{m\in\mathbb{Z}} \frac{C_3}{\lambda} \int\limits_{E_m^*} |f(x)| \, p_m \, (x')^\gamma dx \sim$$

$$\sim \sum_{m\in\mathbb{Z}} \frac{C_4}{\lambda} \int\limits_{E_m^*} |f(x)| \, \omega(x) \, (x')^\gamma dx.$$

On $T_{2,\varepsilon}$, we first note that

$$\frac{1}{4}(|x| + |y|) \le |x - y| \le |x| + |y|, \quad x \in E_m, \quad \text{and} \quad y \notin E_m^*,$$

and by (3), we obtain

$$T_{2,\varepsilon} f(x) \le c_0 \sum_{m\in\mathbb{Z}} \left(\int\limits_{\mathbb{R}^n_{k,+} \setminus E(0,\varepsilon)} T^y |x|^{-n-|\gamma|} \, |f_{m,2}(y)| (y')^\gamma dy \right) \chi_{E_m}(x) \le$$

$$\le c_0 \sum_{m\in\mathbb{Z}} \left(\int\limits_{\mathbb{R}^n_{k,+} \setminus E_m^*} |x - y|^{-n-|\gamma|} |f(y)| \, (y')^\gamma dy \right) \chi_{E_m}(x) \le$$

$$\leq 4^{n+|\gamma|}c_0 \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}^n_{k,+}} (|x|+|y|)^{-n-|\gamma|}|f(y)|\ (y')^{\gamma}dy \leq$$

$$\leq 4^{n+|\gamma|}c_0|x|^{-n-|\gamma|} \int_{\{y\in\mathbb{R}^n_{k,+}:|y|\leq|x|\}} |f(y)|\ (y')^{\gamma}dy +$$

$$+4^{n+|\gamma|}c_0 \int_{\{y\in\mathbb{R}^n_{k,+}:|y|>|x|\}} |f(y)|\ |y|^{-n-|\gamma|}\ (y')^{\gamma}dy \equiv 4^{n+|\gamma|}c_0(A_1f(x)+A_2f(x)).$$

Let

$$M_{\mu}f(x)=\sup_{r>0}\mu(E(x,r))^{-1}\int_{E(x,r)}|f(y)|d\mu(y).$$

Here $E(x,r)=\{y\in\mathbb{R}^n_{k,+}\ :\ |x-y|<r\}$.

We have

$$A_1f(x)\leq |x|^{-n-|\gamma|}\int_{\{y\in\mathbb{R}^n_{k,+}:|x-y|\leq 2|x|\}}|f(y)|\ (y')^{\gamma}dy\leq 2^{n+|\gamma|}M_{\mu}f(x).$$

It is well known that the maximal function M_{μ} is weak type (1,1) and is bounded on $L_p(X,d\mu)$ for $1<p<\infty$ (see [1]). Here we are concerned with the maximal operator defined by $d\mu(x)=(x')^{\gamma}dx$. It is clear that this measure satisfies the doubling condition

$$\mu(E(x,2r))\leq C_0\mu(E(x,r))$$

with a constant C_0 independent of x and $r>0$.

Therefore A_1 satisfies the conclusion of the theorem. By a duality argument, A_2 satisfies the same conclusion if $p\in(1,\infty)$. It remains to show that A_2 is of weak type $L_{1,\omega,\gamma}(\mathbb{R}^n_{k,+})$, if $\omega\in A_{1,\gamma}(\mathbb{R}^n_{k,+})$. Given $\lambda>0$, let

$$R\equiv R_{\lambda}=\sup\Big\{r>0: \int_{\{y\in\mathbb{R}^n_{k,+}\,:\,|y|\geq r\}}|f(y)||y|^{-n-|\gamma|}\ (y')^{\gamma}dy>\frac{\lambda}{c_0}\Big\}.$$

Then

$$\omega(\{x\in\mathbb{R}^n_{k,+}:|A_2f(x)|>\lambda\})=\omega(\{x\in\mathbb{R}^n_{k,+}:|x|\leq R\})\leq$$

$$\leq\frac{c_0}{\lambda}\int_{\{y\in\mathbb{R}^n_{k,+}\,:\,|y|\geq r\}}|f(y)||y|^{-n-|\gamma|}\ (y')^{\gamma}dy\ \omega(\{x\in\mathbb{R}^n_{k,+}:|x|\leq R\})\leq$$

$$\leq\frac{c_0}{\lambda}\int_{\{y\in\mathbb{R}^n_{k,+}\,:\,|y|\geq r\}}|f(y)||y|^{-n-|\gamma|}\ \omega(\{x\in\mathbb{R}^n_{k,+}:|x|\leq|y|\})\ (y')^{\gamma}dy\leq$$

$$\leq\frac{c}{\lambda}\int_{\{y\in\mathbb{R}^n_{k,+}\,:\,|y|\geq r\}}|f(y)|\ \inf_{|x|\leq|y|}\omega(x)\ (y')^{\gamma}dy\leq\frac{c}{\lambda}\int_{\mathbb{R}^n_{k,+}}|f(y)|\ \omega(y)\ (y')^{\gamma}dy.$$

This finishes the proof of Theorem 2.

Now let us proceed to the parts $i_2)$ and $ii_2)$. At first proof the part $i_2)$. We prove that the limit $\lim\limits_{\varepsilon\to 0} T_\varepsilon f = Tf$ exists in the sense of $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ and hence the estimate (5) hold for Tf. It suffices to prove that the limit exists for functions that have compact support, are smooth, and are even with respect to the variable x_n. Indeed, representing any function f in $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ in the form of a sum $f = f_1 + f_2$, where f_1 is a function that has compact support, is smooth, and is even with respect to x_n and f_2 is such that $\|f\|_{L_{p,\omega,\gamma}}$ is sufficiently small, we have from the equality $T_\varepsilon f = T_\varepsilon f_1 + T_\varepsilon f_2$ and (4) that

$$\|T_\varepsilon f - T_\varepsilon f_1\|_{L_{p,\omega,\gamma}} \le c\|f\|_{L_{p,\omega,\gamma}} \le \delta,$$

where δ is a sufficiently small number.

Therefore, it suffices to prove the existence of the limit $\lim\limits_{\varepsilon\to 0} T_\varepsilon f = Tf$ (in the sense of $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$) for smooth compactly supported functions that is even with respect to the variable $x_1, \dots, x_k$. Taking $f(x)$ as such a function and using the "cancellation" condition (2), we have

$$T_{\varepsilon_2} f(x) - T_{\varepsilon_1} f(x) = \int\limits_{\{y\in\mathbb{R}^n_{k,+}:\varepsilon_1<|y|<\varepsilon_2\}} \frac{\Omega(y/|y|)}{|y|^{n+|\gamma|}} [T^y f(x)](y')^\gamma dy =$$

$$= \int\limits_{\{y\in\mathbb{R}^n_{k,+}:\varepsilon_1<|y|<\varepsilon_2\}} \frac{\Omega(y/|y|)}{|y|^{n+|\gamma|}} \left[T^y f(x) - f(x)\right] (y')^\gamma dy,$$

where $x \in \mathbb{R}^n_{k,+}$.

By using the Taylor-Delsarte formula [12] for $T^y f(x)$ is not hard to show that

$$\|T^y f(x) - f(x)\|_{L_{p,\omega,\gamma}} \le c|y|.$$

Therefore,

$$\|T_{\varepsilon_2} f - T_{\varepsilon_1} f\|_{L_{p,\omega,\gamma}} \le \int\limits_{\{y\in\mathbb{R}^n_{k,+}:\varepsilon_1<|y|<\varepsilon_2\}} \frac{c|y|}{|y|^{n+|\gamma|}} (y')^\gamma dy \le c(\varepsilon_2 - \varepsilon_1).$$

Since the space $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ is complete, this implies that the limit $\lim\limits_{\varepsilon\to 0} T_\varepsilon f = Tf$ exists and belongs to $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$. Analogously proved the part $ii_2)$. Thus the proof is complete.

Acknowledgements. The authors would like to express their thanks to Acad. A.D. Gadjiev and Prof. V.S. Guliyev for many helpful discussions on this subject.

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Received December 06, 2011; Revised February 23, 2012.