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ON THE PROPERTIES OF PASSAGE COEFFICIENT OF SCHRODINGER ONE-DIMENSIONAL EQUATION WITH DISCONTINUITY CONDITIONS

Abstract

In the paper we study the properties of passage coefficient of Schrodinger one-dimensional equation with discontinuity conditions.

Consider the differential equation

$$-y'' + q(x)\, y = \lambda^2 y, \quad -\infty < x < +\infty, \tag{1}$$

with discontinuity conditions at some point $a \in (-\infty, +\infty)$

$$y(a-0) = \alpha y(a+0),$$

$$y'(a-0) = \alpha^{-1} y'(a+0), \tag{2}$$

where $\alpha > 0$, $\alpha \neq 1$, λ is a complex parameter, $q(x)$ is a real valued function satisfying the condition

$$\int_{-\infty}^{+\infty} (1+|x|)\, |q(x)|\, dx < +\infty. \tag{3}$$

It is known that equation (1) has the solution $e^{\pm}(x,\lambda)$ (Jost solution) representable in the form [1]

$$e^{\pm}(x,\lambda) = e_0^{\pm}(x,\lambda) \pm \int_x^{+\infty} K^{\pm}(x,t)\, e^{\pm i\lambda t} dt, \tag{4}$$

and the kernels $K^{\pm}(x,t)$ satisfy the inequalities

$$\left|K^{\pm}(x,t)\right| \leq \frac{C}{2}\sigma^{\pm}\left(\frac{x+t}{2}\right) e^{C\sigma_1^{\pm}(x)}, \quad 0 < |x-a| < \pm(t-a),$$

$$\left|K^{\pm}(x,t)\right| \leq \left\{\frac{C}{2}\sigma^{\pm}\left(\frac{x+t}{2}\right) + \frac{|B|}{2}\sigma^{\pm}\left(\frac{2a+x-t}{2}\right)\right\} e^{C\sigma_1^{\pm}(x)},$$

$$|t-a| < \pm(a-x), \tag{5}$$

where $C = A + |B|$, $A = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right)$, $B = \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right)$, $\sigma^{\pm}(x) = \pm\int_x^{+\infty} |q(s)|\, ds$, $\sigma_1^{\pm}(x) = \pm\int_x^{+\infty} \sigma(s)\, ds$, $e_0^{\pm}(x,\lambda) = \begin{cases} e^{\pm i\lambda x}, & \pm x > \pm a, \\ Ae^{\pm i\lambda x} \pm Be^{\pm i\lambda(2a-x)}, & \pm x < \pm a \end{cases}$.

From the real properties of $q(x)$ and the number α it follows that for $\lambda \in R^* = R\backslash\{0\}$ it holds the representation

$$e^{\pm}(x,\lambda) = \pm b(\pm\lambda)\, e^{\mp}(x,\lambda) + a(\lambda)\, e^{\mp}(x,-\lambda), \tag{6}$$

moreover

$$a(\lambda) = \frac{1}{2i\lambda} W\left[e^{+}(x,\lambda), e^{-}(x,\lambda)\right],$$

$$b(\lambda) = -\frac{1}{2i\lambda} W\left[e^{+}(x,\lambda), e^{-}(x,-\lambda)\right], \tag{7}$$

where $W\{y,z\} = y'z - yz'$ is a wronskian of the functions y and z.

The functions $r^{-}(\lambda) = \frac{b(\lambda)}{a(\lambda)}$, $r^{+}(\lambda) = -\frac{b(-\lambda)}{a(\lambda)}$ and $t(\lambda) = \frac{1}{a(\lambda)}$ are said to be left and right reflection factor and passage factor of problem (1)-(2).

Note that the passage factor doesn't enter into the scattering data but it plays an essential part by investigating their properties and by solving the inverse problem.

The goal of the present paper is to investigate the properties of the function $a(\lambda)$. In the no-discontinuity condition, i.e. $\alpha = 1$, there properties are known well [2].

From formula (7) for the function $a(\lambda)$, it follows that it is regular in the upper half-plane. At first investigate the zeros of the function $a(\lambda)$.

Lemma 1. *In the half-plane $a(\lambda)$, the function $\operatorname{Im}\lambda > 0$ may have only a finite number of zeros. All these zeros are prime and lie on an imaginary axis of the semi-axis.*

Proof. At complex values of λ from the upper half-plane, the function $e^{+}(x,\lambda)\,(e^{-}(x,\lambda))$ is a unique solution of problem (1)-(2) belonging to the space $L_2(0,\infty)$ $(L_2(-\infty,0))$. Therefore, problem (1)-(2) has a solution belonging to the space $L_2(-\infty,+\infty)$ only under these values of $\lambda\,(\operatorname{Im}\lambda > 0)$ under which the functions $e^{+}(x,\lambda)$, $e^{-}(x,\lambda)$ are linear independent, i.e. when $W\{e^{+}(x,\lambda), e^{-}(x,\lambda)\} = 2i\lambda \cdot a(\lambda) = 0$. Since the function $a(\lambda)$ is regular in the half-plane $\operatorname{Im}\lambda > 0$, then its zeros λ_k form discrete sets.

By the standard method, based on the formal self-adjointness of problem (1)-(2) ($q(x)$ is real), we get that the eigen values $\mu_k = \lambda_k^2\,(a(\lambda_k) = 0)$ are real, consequently $\lambda_k = i\chi_k$, $\chi_k > 0$, and the zeros of the function may lie only on the imaginary semi-axis.

Further, from (7) and (4) it follows that as $|\lambda| \to \infty$, $\operatorname{Im}\lambda \geq 0$, $a(\lambda) \to A$ consequently, the roots $i\chi_k$ of the function $a\lambda$ lie on a finite interval of the imaginary axis: $0 \leq \chi_k \leq M$. For proving finiteness of the zeros $\lambda_k = i\chi_k$ denote by δ the greatest lower bound of distances between the neighboring zeros, and show that $\delta > 0$. Assuming contrary, we can distinguish such sequences of zeros $i\hat{\chi}_k$, $i\chi_k$, that $\hat{\chi}_k > \chi_k$ and $\lim\limits_{k\to\infty}\hat{\chi}_k = \lim\limits_{k\to\infty}\chi_k = \chi_0$. From (4) and (5) it follows that for sufficiently large K uniform with respect to $\chi \in [0,\infty)$ the inequalities $e^{+}(x,i\chi_k) > \frac{1}{2}e^{-\chi x}$, $x \in [K,\infty)$ are fulfilled such that $e^{-}(x,i\chi) > \frac{1}{2}e^{\chi x}$, $x \in (-\infty,-K]$

$$\int\limits_{K}^{+\infty} e^{+}(x,i\hat{\chi}_m)\, e^{+}(x,i\chi_m)\, dx > \frac{e^{-K(\hat{\chi}_m+\chi_m)}}{4\hat{\chi}_m + \chi_m} > \frac{e^{-2KM}}{8M},$$

$$\int_{-\infty}^{-K} e^{-}\left(x, i\hat{\chi}_m\right) e^{-}\left(x, i\chi_m\right) dx > \frac{e^{-2KM}}{8M}.$$

From the orthoganality of eigen functions of the discrete spectrum and the inequalities

$$e^{+}\left(x, i\hat{\chi}_k\right) = c^{+}\left(\hat{\chi}_k\right) e^{-}\left(x, i\hat{\chi}_k\right),$$
$$e^{+}\left(x, i\chi_k\right) = c\left(\chi_k\right) e^{-}\left(x, i\chi_k\right)$$

it follows that

$$0 = \int_{-\infty}^{+\infty} e^{+}\left(x, i\hat{\chi}_m\right) e^{+}\left(x, i\chi_m\right) dx = \int_{K}^{+\infty} e^{+}\left(x, i\hat{\chi}_m\right) e^{+}\left(x, i\chi_m\right) dx +$$

$$+ \int_{-K}^{K} e^{+}\left(x, i\hat{\chi}_m\right) \left[e^{+}\left(x, i\chi_m\right) - e^{+}\left(x, i\hat{\chi}_m\right)\right] dx +$$

$$+ \int_{-K}^{K} \left[e^{+}\left(x, i\hat{\chi}_m\right)\right]^2 dx + \int_{-\infty}^{-K} e^{-}\left(x, i\hat{\chi}_m\right) e^{-}\left(x, i\chi_m\right) dx \cdot c^{+}\left(\hat{\chi}_m\right) c^{+}\left(\chi_m\right). \quad (8)$$

Since $\lim\limits_{m\to\infty} e^{\pm}\left(x, i\hat{\chi}_m\right) = \lim\limits_{m\to\infty} e^{\pm}\left(x, i\chi_m\right) = e^{\pm}\left(x, i\chi_0\right)$, then

$$\lim_{m\to\infty} c^{+}\left(\hat{\chi}_m\right) c^{+}\left(\chi_m\right) = \left[e^{+}\left(x, i\chi_0\right)\right]^2 \left[e^{-}\left(x, i\chi_0\right)\right]^{-2} > 0,$$

$$\lim_{m\to\infty} \int_{-K}^{K} \left[e^{+}\left(x, i\hat{\chi}_m\right)\right]^2 dx = \int_{-K}^{K} \left[e^{+}\left(x, i\chi_0\right)\right]^2 dx > 0.$$

Therefore, passing to limit in (8) as $k \to \infty$, we get an absurd inequality $0 > \dfrac{e^{-2KM}}{8M}$ showing that the assumption made ($\delta = 0$) is not true, and the function $a(z)$ may have only a finite number of zeros $i\chi_1, ..., i\chi_n$.

Primality of the zeros $\lambda_k = i\chi_k$ follows from the equality

$$\int_{-\infty}^{+\infty} \left|e^{+}\left(x, i\chi_k\right)\right|^2 dx = ic_k^{+} \dot{a}\left(i\chi_k\right) \quad \left(c_k^{+} = c^{+}\left(i\chi_k\right)\right).$$

The lemma is proved.

Corollary. *In space $L_2(-\infty, +\infty)$ problem (1)-(2) may have a finite number of eigen values.*

Lemma 2. *The function $[a(\lambda)]^{-1}$ is bounded in a semi-circle $D_\rho = \{\lambda : |\lambda| \le \rho, \operatorname{Im}\lambda \ge 0\}$ for sufficiently small ρ.*

Proof. It suffices to show that for $\operatorname{Im} z \ge 0$ the following inequality is true:

$$\left|[a(z)]^{-1} \prod_{k=1}^{n} \frac{z - i\chi_k}{z + i\chi_k}\right| \le 1 \quad (\operatorname{Im} z \ge 0). \quad (9)$$

Consider the family of equations

$$-y'' + q_\beta(x)\, y = \lambda^2 y$$

in which

$$q_\beta(x) = \begin{cases} q(x), & for \quad |x| < \beta, \\ 0, & for \quad |x| > \beta, \end{cases}$$

and their solutions $e_\beta^\pm(x,\lambda)$. Obviously, $e_\beta^\pm(x,\lambda) = e^\pm(x,\lambda)$ for $|x| < \beta$ and according to (4), (5)

$$e_\beta^\pm(x,\lambda) = e_0^\pm(x,\lambda) \pm \int_x^{+\infty} K_\beta^\pm(x,t)\, e^{\pm i\lambda t} dt,$$

where $K^\pm(x,t) = 0$ for $\pm(x+t) > 2\beta$ (β is a sufficiently large number), moreover, $\lim\limits_{\beta\to\infty} K_\beta^\pm(x,t) = K^\pm(x,t)$ and $K_\beta^\pm(x,t)$, also satisfy estimation (5) uniformly with respect to β. Obviously, $za_\beta(z)$ is entire with respect to z and $|a_\beta(z)| \geq 1$ for $\operatorname{Im} z = 0$. Then the function

$$a_\beta^{-1}(z) \prod_{k=1}^{n_\beta} \frac{z - i\chi_{k\beta}}{z + i\chi_{k\beta}},$$

where $i\chi_{k\beta}$, $k = \overline{1, n_\beta}$ are the zeros $a_\beta(z)$ at the upper half-plane $\operatorname{Im} z > 0$, is regular for $\operatorname{Im} z \geq 0$ and

$$\left| a_\beta^{-1}(z) \prod_{k=1}^{n_\beta} \frac{z - i\chi_{k\beta}}{z + i\chi_{k\beta}} \right| \leq 1, \quad \operatorname{Im} z \geq 0. \tag{10}$$

Using the representations for $e_\beta^\pm(x,\lambda)$ and $e^\pm(x,\lambda)$ and formulae for $a_\beta(z)$, $a(z)$ it is easy to show that uniformly in $\operatorname{Im} z \geq 0$

$$\lim_{\beta\to\infty} za_\beta(z) = za(z). \tag{11}$$

In particular, hence it follows that $0 < \chi_{k\beta} \leq \widetilde{M}$ for all k and β. Let δ_β be a lower bound of distances between the zeros $i\chi_{k\beta}$ of the functions $a_\beta(z)$ at the half-plane $\operatorname{Im} z > 0$. Then, as above (see the proof of lemma 1) we can show that

$$\delta^* = \inf_{\beta>0} \delta_\beta > 0.$$

Now, let $D_{\delta,R} = \{z : \delta < |z| < R, \ \operatorname{Im} z > 0\}$, where $0 < \delta < \min(\delta^*, \chi_1)$, $R > \tau_n$. Using (11), we can show that uniformly in $\overline{D_{\delta,R}}$

$$\lim_{\beta\to\infty} a_\beta(z) = a(z). \tag{12}$$

From (10) and (12) it follows that the inequality in (9) is valid for $z \in \overline{D_{\delta,R}}$, by arbitrariness of δ and R we get (9).

The lemma is proved.

Lemma 3. *The following relation is valid*

$$a(z) = e^{-\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\ln\ 1-\left|r^{+}(\lambda)\right|^{2}}{\lambda-z}d\lambda}\cdot\prod_{k=1}^{n}\frac{z-i\chi_k}{z+i\chi_k}.$$

Proof. Denote

$$g_\beta(z) = Aa_\beta^{-1}(z)\prod_{k=1}^{n_\beta}\frac{z-i\chi_{k\beta}}{z+i\chi_{k\beta}},$$

where the variables $a_\beta(z)$ and $\chi_{k\beta}$ were introduced in the proof of lemma 2. Obviously, the functions $\ln g_\beta(z)$ are univalent and regular at the upper half-plane $\operatorname{Im} z > 0$, moreover $\ln g_\beta(z) = O\left(\frac{1}{z}\right)$ for $|z| \to \infty$, $\operatorname{Im} z \geq 0$. Further, in the vicinity of $z = 0$ $|\ln g_\beta(z)| \leq A_\beta \ln|z| + B_\beta$, $(\operatorname{Im} z \geq 0)$- since , $za_\beta(z)$ is entire. Therefore, $\ln g_\beta(z)$ may be represented by the Cauchy integral

$$\ln g_\beta(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\ln g_\beta(\lambda)}{\lambda - z}d\lambda \quad (\operatorname{Im} z > 0).$$

On the other hand, using the Cauchy theorem we have

$$0 = \frac{1}{2\pi i}\int\frac{\ln g_\beta(\lambda)}{\lambda - z}d\lambda \quad (\operatorname{Im} z > 0).$$

From the last two formulae (Poisson-Schwarts formula), we have

$$\ln g_\beta(z) = \frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{\ln|g_\beta(\lambda)|}{\lambda - z}d\lambda \quad (\operatorname{Im} z > 0), \tag{13}$$

or since

$$|g_\beta(\lambda)| = |a_\beta(\lambda)|^{-1} = \sqrt{1 - |r^{+}(\lambda)|^{2}}, \quad \lambda \in R^{*},$$

from (13) we get the equality

$$a_\beta(z) = Ae^{-\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\ln\left(1-r^{+}(\lambda)\right)}{\lambda-z}d\lambda}\cdot\prod_{k=1}^{n_\beta}\frac{z-i\chi_{k\beta}}{z+i\chi_{k\beta}}. \tag{14}$$

The estimations $1 > 1 - \left|r_\beta^{+}(\lambda)\right|^{2} > 1 - c_1\cdot\frac{\lambda^2}{1+\lambda^2}$, $\left|r_\beta^{+}(\lambda)\right|^{2} \leq c_2\cdot\frac{1}{\lambda^2}$ allow in equality (14) to make a limit passage as $\beta \to \infty$ under the sign of integral. Here, passing to zero as $\beta \to \infty$ and using some facts from the proof of lemma 2, we get a relation in lemma 3. The lemma is proved.

References

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