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ON SOLVABILITY OF A BOUNDARY VALUE PROBLEM WITH AN OPERATOR IN THE BOUNDARY CONDITIONS FOR A CLASS OF THIRD ORDER OPERATOR-DIFFERENTIAL EQUATIONS

Abstract

In the paper we investigate a boundary value problem for a third order operator-differential equation with a discontinuous coefficient in one of the boundary conditions where some linear operator participates. The conditions expressed by means of the properties of operator coefficients are found. Under these conditions the considered boundary value problem is uniquely and well defined.

Consider in separable space H the boundary value problem

$$-u'''(t)+\rho(t)A^3u(t)=f(t),\quad t\in R_+=[0,+\infty), \tag{1}$$

$$u(0)=Ku''(0),\quad u'(0)=0, \tag{2}$$

where $f(t)\in L_2(R_+;H)$, $u(t)\in W_2^3(R_+;H)$, $\rho(t)=\alpha$, if $0\le t\le 1$, $\rho(t)=\beta$ if $1<t<+\infty$ moreover, α,β are positive, generally speaking, numbers not equal to each other, the operator coefficients satisfy the following conditions:

1) A is a positive-definite self-adjoint operator (i.e. $A=A^*\ge cE$, $c>0$, E is a unit operator);

2) the operator $K\in L\left(H_{1/2},H_{5/2}\right)$.

Here $H_\gamma=D(A^\gamma)$, $(x,y)_{H_\gamma}=(A^\gamma x,A^\gamma y)$, $x,y\in D(A^\gamma)$, $\gamma\ge 0$, $L(X,Y)$ is the space of linear bounded operators acting from the space X to the space Y,

$$L_2(R_+;H)=\left\{f(t):\|f\|_{L_2(R_+;H)}=\left(\int_0^{+\infty}\|f(t)\|_H^2\,dt\right)^{1/2}<+\infty\right\},$$

$$W_2^3(R_+;H)=\Big\{u(t):u'''(t),A^3u(t)\in L_2(R_+;H),$$
$$\|u\|_{W_2^3(R_+;H)}=\left(\|u'''\|^2_{L_2(R_+;H)}+\left\|A^3u\right\|^2_{L_2(R_+;H)}\right)^{1/2}\Big\}$$

(see [1]).

Definition 1. *If the vector-function $u(t)\in W_2^3(R_+;H)$ satisfies equation (1) always everywhere in R_+, then it is said to be a regular solution of equation (1).*

Definition 2. *If for any $f(t)\in L_2(R_+;H)$ there exists a unique regular solution of equation (1) that satisfies boundary conditions (2) in the sense*

$$\lim_{t\to 0}\left\|u(t)-Ku''(t)\right\|_{H_{5/2}}=0,\quad \lim_{t\to 0}\left\|u'(t)\right\|_{H_{3/2}}=0,$$

then we say that boundary value problem (1), (2) is regularly solvable.

The boundary value problems an a semi-axis for second order elliptic operator differential equations with an operator in the boundary condition were studied in [2,3] (see also their references). As it is seen, in boundary value problem (1), (2), one of the boundary conditions in zero contains some linear operator. Such problems for equations (1) for $\alpha = \beta = 1$ were investigated on the finite segment in [4], on the semi-axis in [5]. Note that boundary value problem (1), (2) for $K = 0$ was investigated in [6], [7].

In the present paper, we obtain the conditions expressed by the operator coefficients of boundary value problem (1), (2) that provide its regular solvability.

To this end, denote the subspace of the space $W_2^3(R_+;H)$ dictated by boundary conditions (2) by

$$W_{2,K}^3(R_+;H) = \left\{u(t) : u(t) \in W_2^3(R_+;H),\ u(0) = Ku''(0),\ u'(0) = 0\right\},$$

and by P_0 the operator acting from the space $W_{2,K}^3(R_+;H)$ to the space $L_2(R_+;H)$ by the rule

$$P_0 u(t) = -u'''(t) + \rho(t) A^3 u(t), \quad u(t) \in W_{2,K}^3(R_+;H).$$

Assume $\kappa(c_1,c_2,c_3) = c_1\sqrt[3]{\beta^2} + c_2\sqrt[3]{\alpha\beta} + c_3\sqrt[3]{\alpha^2}$. Denote by $\sigma(B)$ the spectrum of the operator B.

Theorem 1. *Let conditions 1), 2) be fulfilled,* $-\frac{1}{\sqrt[3]{\alpha^2}\omega_2} \notin \sigma\left(A^{5/2}KA^{-1/2}\right)$ *and the operator*

$$K_{\alpha,\beta} = \left(E + \sqrt[3]{\alpha^2}KA^2\right)\left(\kappa(\omega_1,1,\omega_2)\, e^{\sqrt[3]{\alpha}(\omega_2-1)A} - \kappa(1,1,1)\,E\right) -$$

$$-\left(E + \sqrt[3]{\alpha^2}\omega_2 KA^2\right)\left(\kappa(\omega_1,\omega_2,1)\, e^{\sqrt[3]{\alpha}(\omega_1-1)A} - \kappa(1,\omega_2,\omega_1)\, e^{\sqrt[3]{\alpha}(\omega_2-1)A}\right)$$

be boundedly invertible in the space $H_{5/2}$, *where* $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ *and* $\omega_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. *Then the equation* $P_0 u = 0$ *has a unique zero solution from the space* $W_{2,K}^3(R_+;H)$.

Proof. The general solution of the equation $P_0 u(t) = 0$ from the space $W_2^3(R_+;H)$ has the form [7]

$$u_0(t) = \begin{cases} u_{0,1}(t) = e^{\sqrt[3]{\alpha}\omega_1 tA}\varphi_0 + e^{\sqrt[3]{\alpha}\omega_2 tA}\varphi_1 + e^{-\sqrt[3]{\alpha}(1-t)A}\varphi_2, & if \quad 0 \le t \le 1, \\ u_{0,2}(t) = e^{\sqrt[3]{\beta}\omega_1(t-1)A}\varphi_3 + e^{\sqrt[3]{\beta}\omega_2(t-1)A}\varphi_4, & if \quad 1 < t < +\infty, \end{cases}$$

where $\varphi_0,\ \varphi_1,\ \varphi_2,\ \varphi_3,\ \varphi_4 \in H_{5/2}$. From the condition $u_0(t) \in W_{2,K}^3(R_+;H)$ we have:

$$\begin{cases} u_{0,1}(0) = Ku''_{0,1}(0), \\ u'_{0,1}(0) = 0, \\ u_{0,1}(1) = u_{0,2}(1), \\ u'_{0,1}(1) = u'_{0,2}(1), \\ u''_{0,1}(1) = u''_{0,2}(1). \end{cases}$$

Therefore we get the following system of equations with respect to $\varphi_0,\ \varphi_1,\ \varphi_2,\ \varphi_3,\ \varphi_4$:

$$\begin{cases} \varphi_0+\varphi_1+e^{-\sqrt[3]{\alpha}A}\varphi_2=\sqrt[3]{\alpha^2}KA^2\left(\omega_1^2\varphi_0+\omega_2^2\varphi_1+e^{-\sqrt[3]{\alpha}A}\varphi_2\right), \\ \omega_1\varphi_0+\omega_2\varphi_1+e^{-\sqrt[3]{\alpha}A}\varphi_2=0, \\ e^{\sqrt[3]{\alpha}\omega_1 A}\varphi_0+e^{\sqrt[3]{\alpha}\omega_2 A}\varphi_1+\varphi_2=\varphi_3+\varphi_4, \\ \sqrt[3]{\alpha}\omega_1 e^{\sqrt[3]{\alpha}\omega_1 A}\varphi_0+\sqrt[3]{\alpha}\omega_2 e^{\sqrt[3]{\alpha}\omega_2 A}\varphi_1+\sqrt[3]{\alpha}\varphi_2=\sqrt[3]{\beta}\omega_1\varphi_3+\sqrt[3]{\beta}\omega_2\varphi_4, \\ \sqrt[3]{\alpha^2}\omega_1^2 e^{\sqrt[3]{\alpha}\omega_1 A}\varphi_0+\sqrt[3]{\alpha^2}\omega_2^2 e^{\sqrt[3]{\alpha}\omega_2 A}\varphi_1+\sqrt[3]{\alpha^2}\varphi_2=\sqrt[3]{\beta^2}\omega_1^2\varphi_3+\sqrt[3]{\beta^2}\omega_2^2\varphi_4. \end{cases} \quad (3)$$

Since $\omega_1\omega_2=1,\ \omega_1+\omega_2=-1,\ \omega_1^2=\omega_2,\ \omega_2^2=\omega_1$, from system (3) after simple transformations we have:

$$\varphi_1=-\omega_2\varphi_0-\omega_1 e^{-\sqrt[3]{\alpha}A}\varphi_2, \quad (4)$$

$$\left(E+\sqrt[3]{\alpha^2}KA^2\right)\varphi_0-\omega_1\left(E+\sqrt[3]{\alpha^2}\omega_2 KA^2\right)e^{-\sqrt[3]{\alpha}A}\varphi_2=0, \quad (5)$$

$$\left(E+\sqrt[3]{\alpha^2}KA^2\right)\left(\kappa\left(\omega_1,1,\omega_2\right)e^{\sqrt[3]{\alpha}(\omega_2-1)A}-\kappa\left(1,1,1\right)E\right)\varphi_0-$$
$$-\left(E+\sqrt[3]{\alpha^2}\omega_2 KA^2\right)\left(\kappa\left(\omega_1,\omega_2,1\right)e^{\sqrt[3]{\alpha}(\omega_1-1)A}-\right.$$
$$\left.-\kappa\left(1,\omega_2,\omega_1\right)e^{\sqrt[3]{\alpha}(\omega_2-1)A}\right)\varphi_0=K_{\alpha,\beta}\varphi_0=0. \quad (6)$$

By the condition of the theorem, $K_{\alpha,\beta}$ is boundedly invertible in the space $H_{5/2}$. Then it follows from equation (6) that $\varphi_0=0$. In this case, equation (5) under the condition $-\frac{1}{\sqrt[3]{\alpha^2}\omega_2}\notin\sigma\left(A^{5/2}KA^{-1/2}\right)$ yields that $\varphi_2=0$. Consequently, from equation (4) $\varphi_1=0$. Taking into account $\varphi_0=\varphi_1=\varphi_2=0$, in the last two equations of system (3), we get $\varphi_3=\varphi_4=0$, i.e. $u_0\left(t\right)=0$. The theorem is proved.

Now, consider boundary value problem (1), (2).

Theorem 2. *Subject to the conditions of theorem 1, problem (1), (2) is regularly solvable.*

Proof. At first show that the equation $P_0 u\left(t\right)=f\left(t\right)$ has the solution $u\left(t\right)\in W_{2,K}^3\left(R_+;H\right)$ for any $f\left(t\right)\in L_2\left(R_+;H\right)$.

Denote by

$$\upsilon_1\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(i\xi^3E+\alpha A^3\right)^{-1}\left(\int_0^{+\infty}f\left(s\right)e^{-i\xi s}ds\right)e^{it\xi}d\xi,\ t\in R,$$

and

$$\upsilon_2\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(i\xi^3E+\beta A^3\right)^{-1}\left(\int_0^{+\infty}f\left(s\right)e^{-i\xi s}ds\right)e^{it\xi}d\xi,\ t\in R.$$

It is clear that the functions $\upsilon_1\left(t\right)$ and $\upsilon_2\left(t\right)$ satisfy the equations

$$-\frac{d^3\upsilon\left(t\right)}{dt^3}+\alpha A^3\upsilon\left(t\right)=f\left(t\right)$$

and

$$-\frac{d^3\upsilon\left(t\right)}{dt^3}+\beta A^3\upsilon\left(t\right)=f\left(t\right),$$

respectively, in R_+ almost everywhere. Show that $\upsilon_1(t)$ and $\upsilon_2(t)$ belong to $W_2^3(R;H)$. Note that from Plancharel's theorem it follows that it suffices to show the belonging to the space $L_2(R;H)$ of the quantities $A^3\hat{\upsilon}_j(\xi)$, $\xi^3\hat{\upsilon}_j(\xi)$, $j=1,2$, where $\hat{\upsilon}_1(\xi)$ and $\hat{\upsilon}_2(\xi)$ are Fourier transformations of the functions $\upsilon_1(\xi)$ and $\upsilon_2(\xi)$, respectively. From the spectral theory of self-adjoint operators it follows that

$$\left\|A^3\hat{\upsilon}_1(\xi)\right\|_{L_2(R;H)} = \left\|A^3\left(i\xi^3E+\alpha A^3\right)^{-1}\hat{f}(\xi)\right\|_{L_2(R;H)} \le$$
$$\le \sup_{\xi\in R}\left\|A^3\left(i\xi^3E+\alpha A^3\right)^{-1}\right\|_{H\to H}\left\|\hat{f}(\xi)\right\|_{L_2(R;H)} \le$$
$$\le \sup_{\xi\in R}\sup_{\sigma\in\sigma(A)}\left|\sigma^3\left(i\xi^3+\alpha\sigma^3\right)^{-1}\right|\left\|\hat{f}(\xi)\right\|_{L_2(R;H)} =$$
$$= const\left\|\hat{f}(\xi)\right\|_{L_2(R;H)} = const\left\|f\right\|_{L_2(R;H)}.$$

Here $\hat{f}(\xi)$ is the Fourier transformation of the function $f(t)$. Similarly, it is proved that $\xi^3\hat{\upsilon}_1(\xi)\in L_2(R;H)$. Consequently, $\upsilon_1(\xi)\in W_2^3(R;H)$. Thus, $\upsilon_2(\xi)\in W_2^3(R;H)$. Now, denote the contractions of the functions $\upsilon_1(t)$ on $[0,1)$ and $\upsilon_2(t)$ on $(1,+\infty)$ by $u_\alpha(t)$ and $u_\beta(t)$, respectively. Obviously, $u_\alpha(t)\in W_2^3([0,1);H)$, $u_\beta(t)\in W_2^3((1,+\infty);H)$. Continuing, denote by

$$u(t)=\begin{cases} u_1(t)=u_\alpha(t)+e^{\sqrt[3]{\alpha}\omega_1 tA}\psi_0+e^{\sqrt[3]{\alpha}\omega_2 tA}\psi_1+e^{-\sqrt[3]{\alpha}(1-t)A}\psi_2, & \text{if } 0\le t\le 1,\\ u_2(t)=u_\beta(t)+e^{\sqrt[3]{\beta}\omega_1(t-1)A}\psi_3+e^{\sqrt[3]{\beta}\omega_2(t-1)A}\psi_4, & \text{if } 1<t<+\infty,\end{cases}$$

where the vectors ψ_0, ψ_1, ψ_2, ψ_3, $\psi_4\in H_{5/2}$ and we determine them from the conditions that $u(t)\in W_{2,K}^3(R_+;H)$, i.e. the equalities of the following system should we fulfilled:

$$\begin{cases} u_1(0)=Ku_1''(0),\\ u_1'(0)=0,\\ u_1(1)=u_2(1),\\ u_1'(1)=u_2'(1),\\ u_1''(1)=u_2''(1).\end{cases}$$

Consequently, we get the following system of equations with respect to ψ_0, ψ_1, ψ_2, ψ_3, ψ_4 :

$$\begin{cases} u_\alpha(0)+\varphi_0+\varphi_1+e^{-\sqrt[3]{\alpha}A}\varphi_2=Ku_\alpha''(0)+\\ \qquad +\sqrt[3]{\alpha^2}KA^2\left(\omega_1^2\varphi_0+\omega_2^2\varphi_1+e^{-\sqrt[3]{\alpha}A}\varphi_2\right),\\ u_\alpha'(0)+\sqrt[3]{\alpha}\omega_1A\varphi_0+\sqrt[3]{\alpha}\omega_2A\varphi_1+\sqrt[3]{\alpha}Ae^{-\sqrt[3]{\alpha}A}\varphi_2=0,\\ u_\alpha(1)+e^{\sqrt[3]{\alpha}\omega_1A}\varphi_0+e^{\sqrt[3]{\alpha}\omega_2A}\varphi_1+\varphi_2=u_\beta(1)+\varphi_3+\varphi_4,\\ u_\alpha'(1)+\sqrt[3]{\alpha}\omega_1Ae^{\sqrt[3]{\alpha}\omega_1A}\varphi_0+\sqrt[3]{\alpha}\omega_2Ae^{\sqrt[3]{\alpha}\omega_2A}\varphi_1+\sqrt[3]{\alpha}A\varphi_2=\\ \qquad =u_\beta'(1)+\sqrt[3]{\beta}\omega_1A\varphi_3+\sqrt[3]{\beta}\omega_2A\varphi_4,\\ u_\alpha''(1)+\sqrt[3]{\alpha^2}\omega_1^2A^2e^{\sqrt[3]{\alpha}\omega_1A}\varphi_0+\sqrt[3]{\alpha^2}\omega_2^2A^2e^{\sqrt[3]{\alpha}\omega_2A}\varphi_1+\sqrt[3]{\alpha^2}A^2\varphi_2=\\ \qquad =u_\beta''(1)+\sqrt[3]{\beta^2}\omega_1^2A^2\varphi_3+\sqrt[3]{\beta^2}\omega_2^2A^2\varphi_4.\end{cases}$$

In its turn, hence we get:

$$\begin{cases} \varphi_0+\varphi_1+e^{-\sqrt[3]{\alpha}A}\varphi_2-\sqrt[3]{\alpha^2}KA^2\left(\omega_1^2\varphi_0+\omega_2^2\varphi_1+e^{-\sqrt[3]{\alpha}A}\varphi_2\right)= \\ \qquad =Ku''_\alpha(0)-u_\alpha(0), \\ \sqrt[3]{\alpha}\omega_1\varphi_0+\sqrt[3]{\alpha}\omega_2\varphi_1+\sqrt[3]{\alpha}e^{-\sqrt[3]{\alpha}A}\varphi_2=-A^{-1}u'_\alpha(0), \\ e^{\sqrt[3]{\alpha}\omega_1 A}\varphi_0+e^{\sqrt[3]{\alpha}\omega_2 A}\varphi_1+\varphi_2-\varphi_3-\varphi_4=u_\beta(1)-u_\alpha(1), \\ \sqrt[3]{\alpha}\omega_1 e^{\sqrt[3]{\alpha}\omega_1 A}\varphi_0+\sqrt[3]{\alpha}\omega_2 e^{\sqrt[3]{\alpha}\omega_2 A}\varphi_1+\sqrt[3]{\alpha}\varphi_2-\sqrt[3]{\beta}\omega_1\varphi_3-\sqrt[3]{\beta}\omega_2\varphi_4= \\ \qquad =A^{-1}\left(u'_\beta(1)-u'_\alpha(1)\right), \\ \sqrt[3]{\alpha^2}\omega_1^2 e^{\sqrt[3]{\alpha}\omega_1 A}\varphi_0+\sqrt[3]{\alpha^2}\omega_2^2 e^{\sqrt[3]{\alpha}\omega_2 A}\varphi_1+\sqrt[3]{\alpha^2}\varphi_2-\sqrt[3]{\beta^2}\omega_1^2\varphi_3- \\ \qquad -\sqrt[3]{\beta^2}\omega_2^2\varphi_4=A^{-2}\left(u''_\beta(1)-u''_\alpha(1)\right). \end{cases} \tag{7}$$

Since $u_\alpha(t)\in W_2^3([0,1);H)$ and $u_\beta(t)\in W_2^3((1,+\infty);H)$, then from the theorem on traces [1, chp.1] it follows that

$$Ku''_\alpha(0)-u_\alpha(0),\quad A^{-1}u'_\alpha(0),\quad u_\beta(1)-u_\alpha(1),$$

$$A^{-1}\left(u'_\beta(1)-u'_\alpha(1)\right),\quad A^{-2}\left(u''_\beta(1)-u''_\alpha(1)\right)$$

belong to $H_{5/2}$. Then from system (7), through these quantities, behaving as in system (3), and taking into account that $K_{\alpha,\beta}$ is boundedly invertible in the space $H_{5/2}$ and $-\frac{1}{\sqrt[3]{\alpha^2}\omega_2}\notin\sigma\left(A^{5/2}KA^{-1/2}\right)$, we obviously get that all the vectors ψ_0, ψ_1, ψ_2, ψ_3, $\psi_4\in H_{5/2}$. Consequently, $u(t)\in W_{2,K}^3(R_+;H)$.

On the other hand, for $u(t)\in W_{2,K}^3(R_+;H)$ it holds the inequality

$$\|P_0u\|_{L_2(R_+;H)}^2=\left\|-\frac{d^3u}{dt^3}+\rho A^3u\right\|_{L_2(R_+;H)}^2\le 2\max\left(1;\alpha^2;\beta^2\right)\|u\|_{W_2^3(R_+;H)}^2.$$

As the problem

$$-u'''(t)+\rho(t)A^3u(t)=0,\quad t\in R_+,$$

$$u(0)=Ku''(0),\quad u'(0)=0$$

by theorem 1 has only a zero solution from the space $W_{2,K}^3(R_+;H)$, then from the Banach theorem on an inverse operator, there exists $P_0^{-1}:L_2(R_+;H)\to W_{2,K}^3(R_+;H)$ and it is bounded. Hence it follows that

$$\|u\|_{W_2^3(R_+;H)}\le const\,\|f\|_{L_2(R_+;H)}.$$

The theorem is proved.

From theorems 1 and 2 it follows that the operator P_0 under conditions of theorem 1 realizes isomorphism between the spaces $W_{2,K}^3(R_+;H)$ and $L_2(R_+;H)$.

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Received November 19, 2012; Revised February 21, 2013.