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Rafiq K. TAGIYEV, Sadig A. HASHIMOV

ON OPTIMAL CONTROL OF THE COEFFICIENTS OF A PARABOLIC EQUATION INVOLING PHASE CONSTRAINTS

Abstract

For optimal control of the coefficients of a parabolic equation, the well-posedness of the problem statement is studied, and necessary optimality condition is established.

1. Introduction

The optimal control problems described by partial differential equations may be found at various applications. The problems where the controls are contained in the coefficients of the state equations including in the coefficients at higher derivatives, are of great interest among these problems. By investigating such optimal control problems, there arise some difficulties connected with their strong non-linearity and ill-posedness [1-5].

The optimization problems with controls in the coefficients of parabolic equations are found in optimization of heat physics, diffusion, filtration and other processes. Such problems also arise by solving the coefficient inverse problems for parabolic equations considered in variational statements [6-8].

In the papers [1, 4, 8-20] and others the problems of optimal control of the coefficients of parabolic equations are studied. Such problems involving phase restraints [12] and when the control coefficients belong to Sobolev spaces and Lebesgue spaces with finite summability index [19, 20] have not been studied enough.

In the present paper, an optimal control problem with a nonlinear quality criterion for a linear parabolic equation involving phase constraints is considered. The control functions enter to the coefficients of the state equation, and they are the elements of Sobolev space and Lebesgue space with finite summability indices. The well-posedness of the problem statement is investigated and a necessary optimality condition in the form of the generalized principle of Lagrange multiplies is established.

2. Problem statement

Let $\Omega \subset E_n$ be a bounded domain, S be the boundary of domain Ω and that is supposed to be Lipschits continuous, $x = (x_1, ..., x_n)$ be an arbitrary point of domain $\overline{\Omega}$, $0 < T$ be a given number, $0 \leq t \leq T$, $Q_T = \Omega \times (0, T]$, $S_T = S \times (0, T]$ The denotation used in the paper for functional spaces, correspond to [21, pp. 12-17]. The norms in the spaces $L_s(\Omega)$, $L_s(Q_T)$ $(s \geq 1)$, $W_2^1(\Omega)$, $W_p^{1,1}(Q_T)$ $(1 \leq p \leq \infty)$, $W_2^{2,1}(Q_T)$ are denoted by $\|\cdot\|_{s,\Omega}$, $\|\cdot\|_{s,Q_T}$, $\|\cdot\|_{2,\Omega}^{(1)}$, $\|\cdot\|_{p,Q_T}^{(1,1)}$, $\|\cdot\|_{2,Q_T}^{(2,1)}$, respectively. The positive constants independent of estimated quantities and admissible controls are denoted by M_j $(j = 1, 2, ...)$.

Let the controlled system be described in Q_T by the following initial-boundary value problem for the linear parabolic equation

$$u_t - \sum_{i=1}^{n} \left(k_i\left(x,t\right) u_{x_i}\right)_{x_i} + q\left(x,t\right) u = f\left(x,t\right), \quad (x,t) \in Q_T, \tag{1}$$

$$u|_{s_T} = 0, \qquad u|_{t=0} = \varphi\left(x\right), \quad x \in \Omega, \tag{2}$$

where $\varphi\left(x\right) \in \overset{\circ}{W}{}_2^1\left(\Omega\right)$, $f\left(x,t\right) \in L_2\left(Q_T\right)$ are the given functions, $k\left(x,t\right) = \left(k_1\left(x,t\right),...,k_n\left(x,t\right)\right)$, $q\left(x,t\right)$ are control functions, $\nu\left(x,t\right) = \left(k\left(x,t\right), q\left(x,t\right)\right)$, is a control, $u = u\left(x,t\right) = u\left(x,t,\nu\right)$, is the solution of problem (1), (2) state of the process corresponding to the control $\nu = \nu\left(x,t\right)$.

Let

$$B = \prod_{i=1}^{n} W_{p_i}^{1,1}\left(Q_T\right) \times L_s\left(Q_T\right)$$

be a control space, where $p_i \geq n+2\left(i=\overline{1,n}\right)$ for $n \geq 1$, $s > 2$ for $n = 1$ and $n = 2$, $s > \left(n+2\right)/2$ for $n \geq 3$. In space B we introduce the set of admissible controls

$$V = \prod_{i=1}^{n} K_i \times Q, \tag{3}$$

where

$$\begin{gathered} K_i = \Big\{k_i\left(x,t\right) \in W_{p_i}^{1,1}\left(Q_T\right) : 0 < \nu_i \leq k_i\left(x,t\right) \leq \mu \quad \text{a.e. on} \quad Q_T \\ \left\|k_{ix_j}\right\|_{p_i,Q_T} \leq d_j^{(i)}\left(j=\overline{1,n}\right) \ , \ \left\|k_{it}\right\|_{p_i,Q_T} \leq d_i\Big\} \qquad \left(i=\overline{1,n}\right) \end{gathered} \tag{4}$$

$$Q = \left\{q\left(x,t\right) \in L_s\left(Q_T\right) : \left|q\left(x,t\right)\right| \leq \rho \quad \text{a.e. on} \quad Q_T\right\}. \tag{5}$$

Here ν, μ_i, $d_{j,d_i}^{(i)}$, $\rho > 0$ $\left(i,j=\overline{1,n}\right)$ are the given numbers almost everywhere.

State the following optimal control problem: among all possible controls $\nu\left(x,t\right) = \left(k\left(x,t\right), q\left(x,t\right)\right) \in V$ satisfying the constraints

$$J_l\left(\nu\right) = \int\limits_{Q_T} F_l\left(x,t,u\left(x,t,\nu\right), k\left(x,t\right)\right) dxdt +$$

$$+ \int\limits_{\Omega} G_l\left(x, u\left(x,T,\nu\right)\right) dx \begin{cases} \leq 0, & l = \overline{1,l_0}, \\ = 0, & l = \overline{l_0+1,\ l_0+l_1}, \end{cases} \tag{6}$$

find the control $\nu_*\left(x,t\right) = \left(k_*\left(x,t\right), q_*\left(x,t\right)\right) \in V$ minimizing the functional

$$J_0\left(\nu\right) = \int\limits_{Q_T} F_0\left(x,t,u\left(x,t,\nu\right), k\left(x,t\right)\right) dxdt + \int\limits_{\Omega} G_0\left(x, u\left(x,T,\nu\right)\right) dx, \tag{7}$$

where $F_l\left(x,t,u,p\right)$ $G_l\left(x,u\right)$ $\left(l=\overline{0,\ l_0+l_1}\right)$ are the given functions satisfying Caratheodory conditions in domains $Q_T \times R \times U$, $\Omega \times R$, respectively, i.e. measurable with respect to $\left(x,t\right) \in Q_T$ $x \in \Omega$, and continuous with respect to $u \in R$, $p \in U$, here

$$U = \left\{k = \left(k_l,...,k_n\right) \in E_n : \nu_i \leq k_i \leq \mu_i\left(i=\overline{1,n}\right)\right\}$$

is the range of the control $k(x,t) = (k_1(x,t), ..., k_n(x,t))$.

Under the solution of problem (1), (2) corresponding to the control $\nu = \nu(x,t) \in V$ we'll understand a generalized solution from the space $V_2^{1,1/2}(Q_T)$, i.e. the function $u = u(x,t,\nu)$ from $\overset{\circ}{V}{}_2^{1,1/2}(Q_T)$ satisfying the integral identity

$$\int\limits_{Q_T} \left(-u\eta_t + \sum_{i=1}^{n} k_i u_{x_i} \eta_{x_i} + qu\eta \right) dxdt = \int\limits_{\Omega} \varphi(x)\, \eta(x,0)\, dx + \int\limits_{Q_T} f\eta dxdt \tag{8}$$

for any function $\eta = \eta(x,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T)$ equal to zero for $t = T$.

3. Well-posedness of the problem statement

Let's consider the well-posedness of boundary value problem (1), (2) for the given control $\nu(x,t) \in V$. From the results of [21, pp. 181-189, pp. 203-212] it follows that under the made assumptions, boundary value problem (1), (2) for $\nu(x,t) \in V$ has a unique generalized solution from $V_2^{1,1/2}(Q_T)$ and it belongs to the space $W_{2,0}^{2,1}(Q_T) = W_2^{2,1}(Q_T) \cap \overset{\circ}{W}{}_2^{1,0}(Q_T)$, satisfies equation (1) for almost all $(x,t) \in Q_T$, its trace for $t = 0$ coincides with $\varphi(x)$, and the following a priori estimation is valid:

$$\|u\|_{2,Q_T}^{(2,1)} \leq M_1 \left[\|f\|_{2,Q_T} + \|\varphi\|_{2,\Omega}^{(1)} \right]. \tag{9}$$

Using estimation (9) and arguing as in the paper [19], we can show that for the solution of boundary value problem (1), (2) it is valid the inequality

$$\|u\|_{r_1,Q_T} + \|u_x\|_{r_3,Q_T} + \|u(x,T,\nu)\|_{r_2,\Omega} \leq M_2 \left[\|f\|_{2,Q_\Gamma} + \|\varphi\|_{2,\Omega}^{(1)} \right], \tag{10}$$

where the numbers r_1, r_2, r_3 satisfy the conditions

$$\begin{cases} r_1 = \infty, \quad r_2 = \infty, \quad \text{for} \quad n = 1, \\ r_1 \in [2,\infty) \quad r_2 \in [2,\infty) \quad \text{for} \quad n = 2, \\ r_1 = \dfrac{2(n+2)}{n-2}, \quad r_2 = \dfrac{2n}{n-2} \quad \text{for} \quad n \geq 3, \\ r_3 = \dfrac{2(n+2)}{n} \quad \text{for} \quad n \geq 1. \end{cases} \tag{11}$$

For studying the well-posedness of the statement of optimal control problem (1)-(7), we suppose that the following conditions are fulfilled:

1) there exist the functions $a(x,t) \in L_1(Q_T)$, $b(x) \in L_1(\Omega)$ and the constants M_3, $M_4 > 0$ such that for almost all $(x,t) \in Q_T$, $x \in \Omega$ and for any $u \in R$, $k \in U$ it holds

$$|F_l(x,t,u,k)| \leq a(x,t) + M_3 |u|^{r_1^*}, \ |G_l(x,u)| \leq b(x) + M_4 |u|^{r_2^*} \left(l = \overline{0, l_0 + l_1} \right),$$

where r_1^* and r_2^* are some numbers satisfying the conditions

$$r_1^* \in [2,\infty), \quad r_2^* \in [2,\infty), \quad for \quad n = 1 \ \ and \quad n = 2,$$

$$r_1^* \in [2, 2(n+2)/(n-2)), \quad r_2^* \in [2, 2n/(2n-2)), \quad for \quad n \geq 3;$$

2) the set of admissible controls satisfying restraints (6) is not empty, i.e.

$$W = \{\nu(x,t) \in V : J_l(\nu) \leq 0 \quad (l = \overline{1, l_0}),$$

$$J_l(\nu) \leq 0 \quad (l = \overline{l_0+1, l_0+l_1})\} \neq \varnothing$$

From conditions 1) and estimation (10) it follows that the operators generated by the functions $F_l(x,t,u(x,t,\nu),k(x,t))$, $G_l(x,u(x,T,\nu))$ $(l = \overline{0, l_0+l_1})$ act from $L_{r_1^*}(Q_T) \times K$, $L_{r_2^*}(\Omega)$ to $L_1(Q_T)$, $L_1(\Omega)$, respectively [22, p. 376], where $K = K_1 \times ... \times K_n$. Hence it follows that the functional $J_0(\nu)$ is determined in W and accepts finite values.

The following theorem is valid for well-posedness of problem (1)-(7).

Theorem 1. *Let the conditions in the statement of problem (1)-(7) and conditions 1), 2) be fulfilled. Then the set of optimal controls of problem (1)-(7)*

$$W_* = \{\nu_*(x,t) \in W : J_0(\nu_*) = J_{0*} \equiv \inf\{J_0(\nu) : \nu \in W\}\}$$

is not empty, W_* is weakly compact in B, and any minimizing sequence

$$\left\{\nu^{(m)}(x,t) = \left(k^{(m)}(x,t), q^{(m)}(x,t)\right)\right\} \subset W$$

of the functional $J_0(\nu)$ weakly converges in B to the set W_*.

Proof. Show that the functional $J_0(\nu)$ is weakly continuous in B in the set V. Let $\nu(x,t) = (k(x,t), q(x,t)) \in V$ be some element,

$$\left\{\nu^{(m)}(x,t)\right\} = \left\{\left(k^{(m)}(x,t), q^{(m)}(x,t)\right)\right\} \subset V$$

be an arbitrary sequence, weakly convergent in B to the element $\nu(x,t)$, i.e.

$$k_i^{(m)}(x,t) \to k_i(x,t) \quad weakly \quad W_{p_i}^{1,1}(Q_T) \ \left(i = \overline{1,n}\right), \tag{12}$$

$$q^{(m)}(x,t) \to q(x,t) \quad weakly \quad L_s(Q_T). \tag{13}$$

From the compactness of the embedding $W_{p_i}^{1,1}(Q_T) \to L_\infty(Q_T)$ for $p_i > n+1$ $\left(i = \overline{1,n}\right)$ [21, p. 78] and from (12) it follows that

$$k_i^{(m)}(x,t) \to k_i(x,t) \ \left(i = \overline{1,n}\right) \quad strongly\ in \ \ L_\infty(Q_T). \tag{14}$$

Assume $u^{(m)}(x,t) \to u\left(x,t,\nu^{(m)}\right)$. As it was noted in section 3, problem (1), (2) for $\nu = \nu^{(m)}$ has a unique solution from $W_{2,0}^{2,1}(Q_T)$ and it is valid the estimation

$$\left\|u^{(m)}\right\|_{2,Q_T}^{(2,1)} \leq M_7 \ (m = 1, 2, ...). \tag{15}$$

Then from the compactness of the imbedding $W_{2,0}^{2,1}(Q_T) \to L_{r_1^*}(Q_T)$ [23, p. 33, p. 39] and compactness of the mapping $u \to u|_{t=T}$ of the space $W_{2,0}^{2,1}$ in $L_{r_2^*}(\Omega)$ [24,

p. 160], [21, p. 78] it follows that from the sequence $\left\{u^{(m)}\right\}$ we can extract the sequence $\left\{u^{(m_k)}\right\}$ such that

$$u^{(m_k)}(x,t) \to u(x,t) \quad \textit{weakly in} \quad W_{2,0}^{2,1}(Q_T) \quad \textit{and strongly in} \quad L_{r_1^*}(Q_T), \tag{16}$$

$$u^{(m_k)}(x,t) \to u(x,T) \quad \textit{strongly in} \quad L_{r_2^*}(Q_T), \tag{17}$$

where the numbers r_1^* and r_2^* are determined in condition 1), and $u(x,t) \in W_{2,0}^{2,1}(Q_T)$ is some element.

Show that $u(x,t) \to u(x,t,\nu)$, i.e. $u(x,t)$ is the solution of problem (1), (2) corresponding to the control $\nu(z,t) \in V$. It is clear that the following identities

$$\int\limits_{Q_T} \left[-u^{(m_k)}\eta_t + \sum_{i=1}^{n} k_i^{(m_k)}(x,t)\, u_{x_i}^{(m_k)}\eta_{x_i} + q^{(m_k)}(x,t)\, u^{(m_k)}\eta \right] dxdt =$$

$$= \int\limits_{\Omega} \varphi(x)\,\eta(x,0)\,dx + \int\limits_{Q} f(x,t)\,\eta dxdt \tag{18}$$

are valid for all $\eta = \eta(x,t) \in W_{2,0}^{2,1}(Q_T)$ equal to zero $t = T$.

Using (14)-(16) and the Cauchy-Bunyakovsky inequality, we get

$$\left| \int\limits_{Q_T} \sum_{i=1}^{n} k_i^{(m_k)}(x,t)\, u_{x_i}^{(m_k)}\eta_{x_i} dxdt - \int\limits_{Q_T} \sum_{i=1}^{n} k_i(x,t)\, u_{x_i}\eta_{x_i} dxdt \right| \leq$$

$$\leq \left| \int\limits_{Q_T} \sum_{i=1}^{n} k_i(x,t) \left(u_{x_i}^{(m_k)} - u_{x_i} \right) \eta_{x_i} dxdt \right| +$$

$$+ \left| \int\limits_{Q_T} \sum_{i=1}^{n} \left(k_i^{(m_k)}(x,t) - k_i(x,t) \right) u_{x_i}^{(m_k)}\eta_{x_i} dxdt \right| \leq$$

$$\leq \left| \int\limits_{Q_T} \sum_{i=1}^{n} k_i(x,t) \left(u_{x_i}^{(m_k)} - u_{x_i} \right) \eta_{x_i} dxdt \right| +$$

$$+ \sum_{i=1}^{n} \left\| k_i^{(m_k)} - k_i \right\|_{\infty,Q_T} \left\| u_{x_i}^{(m_k)} \right\|_{2,Q_T} \left\| \eta_{x_i} \right\|_{2,Q_T} \to 0 \tag{19}$$

as $k \to \infty$.

Furthermore, it is obvious that

$$\int\limits_{Q_T} q^{(m_k)}(x,t)\, u^{(m_k)}\eta dxdt = \int\limits_{Q_T} q^{(m_k)}(x,t) \left(u^{(m_k)} - u \right) \eta dxdt +$$

$$+ \int\limits_{Q_T} q^{(m_k)}(x,t)\, u\eta dxdt. \tag{20}$$

Using the condition $\left|q^{(m_k)}(x,t)\right| \leq \rho$, almost everywhere on Q_T, the Cauchy-Bunyakovsky inequality and (16), we have

$$\left|\int\limits_{Q_T} q^{(m_k)}(x,t)\left(u^{(m_k)}-u\right)\eta dxdt\right| \leq$$

$$\leq \rho\left\|u^{(m_k)}-u\right\|_{2,Q_T}\|\eta\|_{2,Q_T} \to 0 \quad as \quad k\to\infty. \tag{21}$$

Furthermore, using the Holder inequality [21, p. 75], the conditions imposed on s, r_1^* and the boundedness of the imbedding $W_{2,0}^{2,1}(Q_T) \to L_{r_1^*}(Q_T)$, we get the inequalities

$$\|u\eta\|_{s/(s-1),Q_T} \leq \|u\|_{2s/(s-1),Q_T}\|\eta\|_{2,Q_T} \leq$$

$$\leq M_8\|u\|_{r_1^*,Q_T}\|\eta\|_{2,Q_T} \leq M_9\|u\|_{2,Q_T}^{(2,1)}\|u\|_{2,Q_T}.$$

Hence it follows that $u\eta \in L_{s/(s-1)}(Q_T)$. Then from (13) it follows that

$$\int\limits_{Q_T} q^{(m_k)}(x,t)\,u\eta dxdt \to \int\limits_{Q_T} q(x,t)\,u\eta dxdt \quad as \quad k\to\infty. \tag{22}$$

Now, passing to limit in (18) as $k\to\infty$, and taking into account (20), (22), we deduce that $u(x,t)$ satisfies identity (8), i.e. it is the solution of boundary value problem (1), (2) from $V_2^{1,1/2}(Q_T)$, corresponding to the control $\nu(x,t)\in V$. Hence and from $u(x,t)\in W_{2,0}^{2,1}(Q_T)$ it follows that $u(x,t)=u(x,t,\nu)$.

So, it is established that subject to relations (12), (13) from the sequence $\left\{u^{(m)}\right\}$ we can extract the subsequence $\left\{u^{(m_k)}\right\}$ for which relations (16), (17) in which $u(x,t)=u(x,t,\nu)$, are valid. Using the uniqueness of the solution of boundary value problem (1), (2) corresponding to the control $\nu(x,t)\in V$, we can show that relations (16), (17) are valid not only for the subsequence $\left\{u^{(m_k)}\right\}$, but also for all the sequence $\left\{u^{(m)}\right\}$, i.e.

$$u^{(m)}(x,t)\to u(x,t,\nu) \quad weakly \;\; in \quad W_{2,0}^{2,1}(Q_0)$$

$$and \;\; strongly \;\; in \;\; L_{r_1^*}(Q_T), \tag{23}$$

$$u^{(m)}(x,T)\to u(x,T,\nu) \quad strongly \;\; in \;\; L_{r_2^*}(\Omega). \tag{24}$$

Furthermore, from conditions 1) it follows that the operators generated by the functions $F_0(x,t,u(x,t,\nu),k(x,t))$, $G_0(x,u(x,T,\nu))$ continuously act from $L_{r_1^*}(Q_T)\times K$, $L_{r_2^*}(\Omega)$ to $L_1(Q_T)$, $L_1(\Omega)$, respectively [22, p. 376]. Then using (14), (23), (24), we get that $J_0\left(\nu^{(m)}\right)\to J_0(\nu)$ as $m\to\infty$. Thus, it is established that the functional $J_0(\nu)$ is weakly in B continuous on the set V and the more so on the set W.

Show that the set W is weakly compact in space B. Let

$$\left\{\nu^{(m)}\right\} = \left\{\left(k^{(m)},q^{(m)}\right)\right\} \subset W$$

be an arbitrary sequence of controls, i.e.

$$\nu^{(m)} = \left(k^{(m)}, q^{(m)}\right) \in V, J_l\left(\nu^{(m)}\right) \leq 0 \ \left(l = \overline{1, l_0}\right), J_l\left(\nu^{(m)}\right) = 0$$

$$\left(l = \overline{l_0 + 1, l_0 + l_1}\right) \ (m = 1, 2, ...).$$

The set V is convex, closed and bounded in the reflexive Banach space B and therefore weakly compact in B [25, p. 51]. Then from the sequence $\left\{\nu^{(m)}\right\} = \left\{\left(k^{(m)}, q^{(m)}\right)\right\}$ we can extract a subsequence $\left\{\nu^{(m_k)}\right\} = \left\{\left(k^{(m_k)}, q^{(m_k)}\right)\right\} \subset V$ such that

$$k_i^{(m_k)}(x,t) \to k_i(x,t) \quad weakly \quad in \ \ W_{p_i}^{1,1}(Q_T) \quad \left(i = \overline{1,n}\right),$$

$$q^{(m_k)}(x,t) \to q(x,t) \quad weakly \quad in \ \ L_s(Q_T)$$

as $k \to \infty$, where $\nu(x,t) = (k(x,t), q(x,t)) \in V$ is some element. Repeating the above arguments in proving weak continuity of the functional $J_0(\nu)$ and passing to limit in relations $J_l\left(\nu^{(m_k)}\right) \leq 0 \ \left(l = \overline{1, l_0}\right)$, $J_l\left(\nu^{(m_k)}\right) = 0 \ \left(l = \overline{l_0 + 1, l_0 + l_1}\right)$ as $k \to \infty$, we get $J_l(\nu) \leq 0 \ \left(l = \overline{1, l_0}\right)$, $J_l(\nu) = 0 \ \left(l = \overline{l_0 + 1, l_0 + l_1}\right)$. This means that $\nu \in W$ i.e. the set W is weakly compact in B. Then using the result from [25, p. 49], we get that problem (1), (7) is well-posed in weak topology of the space B, i.e. all the statements of theorem 2 are true. The theorem is proved.

Remark 1. From theorem 1 it follows the existence of the solution of problem (1), (7). However, as the following example shows, the solution of problem (1), (7) may be non-unique.

Example 1. Let in problem (1)-(7) $n = 1$, $T = 1$, $\Omega = \{x \in R : 0 < x < 1\}$, $\nu_1 = 1$, $\mu_1 = 2$, $d_1^{(1)} = 1$, $d_1 = 1$, $\rho = \pi^2 + 1$, $l_0 = 1$, $l_1 = 0$, $f(x,t) = 2\pi^2 \sin \pi x$, $\varphi(x) = \sin \pi x$, $F_0(x,t,u,k) = (u - \sin \pi x)^2$, $G_0(x,u) = 0$, $F_1(x,t,u,k) = u - 2/\pi$, $G_1(x,u) = 0$.

Then it is easy to verify that the minimal value of the functional $J_0(\nu)$ is attained on the controls $\nu_*^{(1)}(x,t) = \left(k_{1*}^{(1)}(x,t) = 1, q_*^{(1)}(x,t) = \pi^2\right)$, $\nu_*^{(2)}(x,t) = \left(k_{1*}^{(2)}(x,t) = 2, q_*^{(2)}(x,t) = 0\right)$ and $J_0\left(\nu_*^{(1)}\right) = J_0\left(\nu_*^{(2)}\right) = J_{0*} = 0$, $u\left(x,t,\nu^{(1)}\right) = u\left(x,t,\nu^{(2)}\right) = \sin \pi x$, $x \in \Omega$, $0 < t < T$, i.e. the solution of problem (1)-(7) is non-unique.

Remark 2. The statements of theorem 1 are valid also in the case when the set Q in problem (1)-(7) is of the form:

$$Q = \left\{q(x,t) \in L_s(Q_T) : \|q\|_{S,Q_T} \leq \rho\right\},$$

where $\rho > 0$ is some number.

4. Necessary optimality condition

In order to establish the necessary optimality condition in problem (1)-(7), we suppose that the following conditions are fulfilled:

3) the functions $F_l(x,t,u,k), G_l(x,u) \ \left(l = \overline{0, l_0 + l_1}\right)$ have partial derivatives F_{lu}, F_{lk_i}, $G_{lu} \ \left(l = \overline{0, l_0 + l_1};\ i = \overline{1,n}\right)$, measurable with respect to $(x,t) \in Q_T$,

$x \in \Omega$ and continuous with respect to $u \in R$, $k \in U$; the operators generated by the functions $F_{lu}(x,t,u(x,t),k(x,t))$, $F_{lk_i}(x,t,u(x,t),k(x,t))$, $G_{lu}(x,u(x))$ continuously act from to $L_{r_l}(Q_T) \times \prod_{i=1}^{n} W_{p_i}^{1,1}(Q_T)$, $L_{r_l}(Q_T) \times \prod_{i=1}^{n} W_{p_i}^{1,1}(Q_T)$, $\overset{\circ}{W}{}_2^1(\Omega)$ in $L_2(Q_T)$, $L_1(Q_T)$, $\overset{\circ}{W}{}_2^1(\Omega)$, respectively.

For problem (1)-(7) introduce the conjugate state $\psi_l = \psi_l(x,t,\nu)$ as the solution of the problem

$$\psi_{lt} + \sum_{i=1}^{n} \left(k_i(x,t)\psi_{lx_i}\right)_{x_i} - q(x,t)\psi_l =$$

$$= F_{lu}(x,t,u(x,t,\nu),k(x,t)), \quad (x,t) \in Q_T, \tag{25}$$

$$\psi_l|_{s_T} = 0, \quad \psi_l|_{t=T} = -G_{lu}(x,u(x,T,\nu)), \quad x \in \Omega\left(l = \overline{0, l_0 + l_1}\right). \tag{26}$$

Under the solution of problem (25), (26) corresponding to the control $\nu = \nu(x,t) \in V$, we'll understand the generalized solution from the space $V_2^{1,1/2}(Q_T)$, i.e. the function $\psi_l = \psi_l(x,t,\nu)$, $\overset{\circ}{V}{}_2^{1,1/2}(Q_T)$ satisfying the integral identity

$$\int_{Q_T} \left[\psi_l \eta_t + \sum_{i=1}^{n} k_i(x,t)\psi_{lx_i}\eta_{x_i} + q(x,t)\psi_l\eta\right] dxdt =$$

$$= -\int_{Q_T} F_{lu}(x,t,u(x,t),k(x,t))\eta dxdt -$$

$$= -\int_{Q_T} G_{lu}(x,u(x,T,\nu))\eta(x,T)dx \qquad \left(l = \overline{0, l_0 + l_1}\right) \tag{27}$$

for any function $\eta = \eta(x,t)$ from $\overset{\circ}{W}{}_2^{1,1}(Q_T)$ equal to zero for $t = 0$.

From the results [21, pp.181-189, pp. 203-212] it follows that under the made assumptions boundary value problem (25), (26) for the given $\nu(x,t) \in V$ has a unique generalized solution from $V_2^{1,1/2}(Q_T)$ and it belongs to the space $W_{2,0}^{2,1}(Q_T)$, satisfies equation (25) for almost all $(x,t) \in Q_T$ if its trace for $t = T$ coincides with the function $G_{lu}(x,u(x,T,\nu))$ and the following estimation is valid:

$$\|\psi_l\|_{2,Q_T}^{(2,1)} \le M_5\left[\|F_{lu}\|_{2,Q_T} + \|G_{lu}\|_{2,\Omega}^{(1)}\right] \quad \left(l = \overline{0, l_0 + l_1}\right). \tag{28}$$

Using estimation (28) and arguing as in [19], we can show that the following estimation is valid

$$\|\psi_l\|_{r_1,Q_T} + \|\psi_{lx}\|_{r_3,Q_T} + \|\psi_l(x,0,\nu)\|_{r_2,\Omega} \le$$

$$\le M_6\left[\|F_{lu}\|_{2,Q_T} + \|G_{lu}\|_{2,\Omega}^{(1)}\right] \quad \left(l = \overline{0, l_0 + l_1}\right) \tag{29}$$

where the numbers r_1, r_2, r_3 satisfy conditions (11).

The following theorem expresses the necessary optimality condition in problem (1)-(7).

Theorem 2. *Let the conditions of theorem 1 and conditions 3) be fulfilled,* $\nu_*(x,t) = (k_*(x,t), q_*(x,t)) \in W$ *be an optimal control in problem (1)-(7),* $u_*(x,t) = u(x,t,\nu_*)$, $\psi_{l^*}(x,t) = \psi_l(x,t,\nu_*)$ $\left(l = \overline{0, l_0 + l_1}\right)$ *be the solution of problems (1), (2) and (25), (26) corresponding to the control* $\nu_*(x,t)$. *Then there simultaneously exist the numbers* λ_l^* $\left(l = \overline{0, l_0 + l_1}\right)$, $\lambda_l^* \geq 0$ $\left(l = \overline{0, l_0}\right)$ *not equal to zero, such that for almost all* $(x,t) \in Q_T$ *and for any* $k(x,t) \in K$, $q = [-\rho, \rho]$ *the following inequality is valid:*

$$\sum_{l=1}^{l_0+l_1} \lambda_l^* \int\limits_{Q_T} \sum_{i=1}^{n} \left[u_{*x_i}\psi_{l*x_i} + F_{lk_i}(x,t,u_*(x,t),k_*(x,t))\right]\left[k_i(x,t) - k_{*i}(x,t)\right] dxdt +$$

$$+ \sum_{l=1}^{l_0+l_1} \lambda_l^* \left\{u_*(x,t)\,\psi_{l*}(x,t)\,(q - q_*(x,t))\right\} \geq 0. \tag{30}$$

Proof. For the control $k_*(x,t)$ determine the classical variation

$$k_\varepsilon(x,t) = k_*(x,t) + \varepsilon^{n+1}\left[k(x,t) - k_*(x,t)\right], \quad (x,t) \in Q_T, \tag{31}$$

where $\varepsilon \in (0,1)$ is an arbitrary number, $k(x,t) \in K$ is some element. Obviously, $k_\varepsilon(x,t) \in K$ for all $\varepsilon \in (0,1)$ and

$$k_{\varepsilon i}(x,t) \to k_{*i}(x,t) \quad \text{strongly in} \quad W_{p_i}^{1,1}(Q_T) \quad \left(i = \overline{1,n}\right) \quad \text{as} \quad \varepsilon \to 0. \tag{32}$$

For the control $q_*(x,t)$ we construct an impulse variation. Let m be an arbitrary natural number $\left(x^j, t^j\right) \in Q_T$ $\left(j = \overline{1,n}\right)$ be Lebesgue arbitrary points of all the considered functions, and $0 < t^1 < t^2 < ... < t^m < T$. Choose arbitrary natural numbers $\beta_j^k \geq 0$ $\left(j = \overline{1,m};\ k = \overline{1,k_j}\right)$, where k_j $\left(j = \overline{1,m}\right)$ are some natural numbers. Let $\varepsilon_0 \in (0,1)$ be a number that satisfies the inequalities $t^j - \varepsilon_0 \sum\limits_{s=1}^{k_j-1} \beta_j^s < t^{j+1}$ $\left(j = \overline{1,m}\right)$, where $t^{m+1} = T$. Define the parallelepipeds

$$\Pi_{jk}^\varepsilon = \Big\{(x,t) : x_i^j - \varepsilon/2 \leq x_i < x_i^j + \varepsilon/2 \left(i = \overline{1,n}\right),$$

$$t^j - \varepsilon \sum_{s=1}^{k_j} \beta_j^s \leq t < t^j - \varepsilon \sum_{s=1}^{k_j-1} \beta_j^s \Big\}$$

where $0 < \varepsilon < \varepsilon_0$ and ε_0 are so same that $\Pi_{jk}^\varepsilon \subset Q_T$. It is clear that for all rather small ε, the parallelepipeds Π_{jk}^ε are disjoint, and the volume of Π_{jk}^ε is $\left|\Pi_{jk}^\varepsilon\right| = \beta_j^k \varepsilon^{n+1}$.

Now for the function $q_*(x,t)$ we construct the impulse variation

$$q_\varepsilon(x,t) = \begin{cases} q_j^k \quad for \quad (x,t) \in \Pi_{jk}^\varepsilon \\ q_*(x,t) \quad for \quad (x,t) \in Q_T \setminus \bigcup\limits_{j,k} \Pi_{jk}^\varepsilon \end{cases} \tag{33}$$

where $\left|q_j^k\right| \leq \rho$ $\left(j = \overline{1,m};\ \ k = \overline{1,k_j}\right)$ are arbitrary numbers. Obviously, $q_\varepsilon(x,t) \in Q$ for all rather small $\varepsilon \in (0,1)$ and

$$q_\varepsilon(x,t) \to q_*(x,t) \quad \text{strongly in} \quad L_s(Q_T) \quad \text{as} \quad \varepsilon \to 0. \tag{34}$$

Introduce the denotation $\nu_\varepsilon(x,t) = (k_\varepsilon(x,t), q_\varepsilon(x,t))$, $u_\varepsilon(x,t) = u(x,t,\nu_\varepsilon)$, $\Delta u_\varepsilon(x,t) = u_\varepsilon(x,t) - u_*(x,t)$, $\Delta_\varepsilon k_i(x,t) = k_{\varepsilon i}(x,t) - k_{*i}(x,t)$ $\left(i = \overline{1,n}\right)$, $\Delta_\varepsilon q(x,t) = q_\varepsilon(x,t) - q_*(x,t)$. Then from conditions (1), (2) it follows that $\Delta u_\varepsilon = \Delta u_\varepsilon(x,t)$ is the solution from $W_{2,0}^{2,1}(Q_T)$ of the boundary value problem

$$\Delta u_{\varepsilon t} - \sum_{i=1}^{n} \left(k_{\varepsilon t}(x,t)\,\Delta u_{\varepsilon x_i}\right)_{x_i} + q_\varepsilon(x,t)\,\Delta u_\varepsilon =$$

$$= \sum_{i=1}^{n} \left(\Delta u_\varepsilon k_i u_{*x_i}\right)_{x_i} - \Delta_\varepsilon q u_*, \ (x,t) \in Q_T, \tag{35}$$

$$\Delta u_\varepsilon|_{s_T} = 0, \quad \Delta u_\varepsilon|_{t=0} = 0, \quad x \in \Omega. \tag{36}$$

For the function Δu_ε the following estimation is valid [21, p. 202-212]

$$\|\Delta u_\varepsilon\|_{2,Q_T}^{(2,1)} \leq$$

$$\leq M_{10}\left[\sum_{i=1}^{n}\left(\left\|\left(\Delta u_\varepsilon k_i\right)_{x_i} u_{*x_i}\right\|_{2,Q_T} + \|\Delta u_\varepsilon k_i u_{*x_i x_i}\|_{2,Q_T}\right) + \|\Delta_\varepsilon q u_*\|_{2,Q_T}\right].$$

Using this estimation and relations (32), (34), we can show that the solution of problem (35), (36) possesses the property of stability with respect to the parameter ε, i.e.

$$\|\Delta u_\varepsilon\|_{2,Q_T}^{(2,1)} = \|u_\varepsilon - u_*\|_{2,Q_T}^{2,1} \to 0 \quad as \quad \varepsilon \to 0. \tag{37}$$

Calculate the first variations of the functionals $J_l(\nu)$ $\left(l = \overline{0, l_0 + l_1}\right)$. Using (32), (34) and condition 3), we can represent the increment of the functional $J_l(\nu)$ at the point $\nu_*(x,t) \in V$ in the form

$$\Delta_\varepsilon J_l(\nu_*) = J_l(\nu_\varepsilon) - J_l(\nu_*) = \int\limits_{Q_T} F_{lu}(\eta_{\varepsilon l})\,\Delta u_\varepsilon dxdt +$$

$$+\varepsilon^{n+1}\int\limits_{Q_T}\sum_{i=1}^{n} F_{lk_i}(\mu_{\varepsilon l})\left[k_i(x,t) - k_{*i}(x,t)\right]dxdt +$$

$$+\int\limits_{\Omega} G_{lu}(\nu_{\varepsilon l})\,\Delta u_\varepsilon(x,T)\,dx\left(l = \overline{0, l_0 + l_1}\right), \tag{38}$$

where

$$\eta_{\varepsilon l} = (x,t,u_* + \theta_{1l}\Delta u_\varepsilon, k_\varepsilon), \mu_{\varepsilon l} = (x,t,u_*, k_* + \theta_{2l}\Delta u_\varepsilon, k),$$

$$\nu_{\varepsilon l} = (x, u_*(x,T) + \theta_{3l}\Delta u_\varepsilon(x,T)), \theta_{il} \in [0,1]\left(i = \overline{1;3};\ l = \overline{0, l_0 + l_1}\right).$$

Let $\psi_{l\varepsilon}(x,t) = \psi_l(x,t,\nu_\varepsilon)$ be the solution from $W_{2,0}^{2,1}(Q_T)$ of the boundary value problem

$$\psi_{l\varepsilon t} + \sum_{i=1}^{n}\left(k_{\varepsilon t}(x,t)\,\psi_{t\varepsilon x_i}\right)_{x_i} - q_\varepsilon(x,t)\,\psi_{l\varepsilon} = F_{lu}(\eta_{\varepsilon t}), \quad (x,t) \in Q_T, \tag{39}$$

$$\psi_{l\varepsilon}|_{s_T}=0,\quad \psi_{l\varepsilon}|_{t=T}=-G_{lu}\left(\nu_{\varepsilon l}\right),\quad x\in\Omega,\ \left(l=\overline{0,l_0+l_1}\right) \tag{40}$$

Using (32), (34), (37) and condition 3), we can show that the solution of problem (39), (40) possesses the property of stability with respect to the parameter ε, i.e.

$$\|\Delta\psi_{l\varepsilon}\|_{s_T}^{(2,1)}=\|\psi_{l\varepsilon}-\psi_{l*}\|_{2,Q_T}^{(2,1)}\to 0\ \left(l=\overline{0,l_0+l_1}\right)\quad \text{as}\quad \varepsilon\to 0. \tag{41}$$

Using the solutions of boundary value problems (35), (36) and (39), (40) we can represent equality (38) in the form

$$\Delta_\varepsilon J_l\left(\nu_*\right)=\Delta_\varepsilon^{(1)}J_l\left(\nu_*\right)+\Delta_\varepsilon^{(2)}J_l\left(\nu_*\right)\left(l=\overline{0,l_0+l_1}\right),$$

where

$$\Delta_\varepsilon^{(1)}J_l\left(\nu_*\right)=\varepsilon^{n+1}\int\limits_{Q_T}\sum_{i=1}^{n}\left[u_{*x_i}\psi_{l\varepsilon x_i}+F_{lk_i}\left(\mu_{\varepsilon l}\right)\right]\left[k_i\left(x,t\right)-k_{*i}\left(x,t\right)\right]dxdt, \tag{42}$$

$$\Delta_\varepsilon^{(2)}J_l\left(\nu_*\right)=\sum_{j=1}^{m}\sum_{k=1}^{k_j}\int\limits_{\Pi_{jk}^{\varepsilon}}u_*\psi_{l\varepsilon}\left[q_j^k-q_*\left(x,t\right)\right]dxdt\quad \left(l=\overline{0,l_0+l_1}\right). \tag{43}$$

The first variation of the functional $J_l\left(\nu\right)$ on the element ν_* is defined as

$$\delta J_l\left(\nu_*\right)=\lim_{\varepsilon\to 0}\frac{\Delta_\varepsilon^{(1)}J_l\left(\nu_*\right)}{\varepsilon^{n+1}}+\lim_{\varepsilon\to 0}\frac{\Delta_\varepsilon^{(2)}J_l\left(\nu_*\right)}{\varepsilon^{n+1}}=$$

$$=\delta^{(1)}J_l\left(\nu_*\right)+\delta^{(2)}J_l\left(\nu_*\right)\ \left(l=\overline{0,l_0+l_1}\right). \tag{44}$$

Show that

$$\delta^{(1)}J_l\left(\nu_*\right)=\int\limits_{Q_T}\sum_{i=1}^{n}\left[u_{*x_i}\psi_{l*x_i}+F_{lk_i}\left(x,t,u_*\left(x,t\right),k_*\left(x,t\right)\right)\right]\times$$

$$\times\left[k_i\left(x,t\right)-k_{*i}\left(x,t\right)\right]dxdt, \tag{45}$$

$$\delta^{(2)}J_l\left(\nu_*\right)=\sum_{j=1}^{m}\sum_{k=1}^{k_j}\beta_j^k u_*\left(x^j,t^j\right)\psi_{l*}\left(x^j,t^j\right)\left[q_j^k-q_*\left(x^j,t^j\right)\right]\quad \left(l=\overline{0,l_0+l_1}\right). \tag{46}$$

Using (42), (45), the Cauchy-Bunyakovsky inequality (32), and condition 3), we get

$$\left|\frac{\Delta_\varepsilon^{(1)}J_l\left(\nu_*\right)}{\varepsilon^{n+1}}-\delta^{(1)}J_l\left(\nu_*\right)\right|=$$

$$=\left|\int\limits_{Q_T}\sum_{i=1}^{n}\left[u_{*x_i}\left(\psi_{l\varepsilon x_i}-\psi_{l_*x_i}\right)+F_{lk_i}\left(\mu_{\varepsilon l}\right)-F_{lk_i}\left(x,t,u_*\left(x,t\right),k_*\left(x,t\right)\right)\right]\times\right.$$

$$\left.\times\left[k_i\left(x,t\right)-k_{*i}\left(x,t\right)\right]dxdt\right|\leq$$

$$\leq \sum_{i=1}^{n} \Big[\|\Delta u_{*x_i}\|_{2,Q_T} \left\|\psi_{l\varepsilon x_i} - \psi_{l_* x_i}\right\|_{2,Q_T} +$$

$$+ \|F_{lk_i}(\mu_{\varepsilon l}) - F_{lk_i}(x,t,u_*(x,t),k_*(x,t))\|_{1,Q_T} \Big] \times$$

$$\times \|k_i - k_{*i}\|_{\infty,Q_T} \to 0 \qquad \left(l = \overline{0, l_0 + l_1}\right)$$

as $\varepsilon \to 0$. Hence the validity of formula (45) follows.

Now, show the validity of formula (46). For that we establish for instance, that

$$\int\limits_{Q_T} \left| \frac{\Delta_\varepsilon^{(2)} J_l(\nu_*)}{\varepsilon^{n+1}} - \delta^{(2)} J_l(\nu_*) \right| dQ \to 0 \quad \left(l = \overline{0, l_0 + l_1}\right) \tag{47}$$

as $\varepsilon \to 0$, where $Q_j \subset Q_T$ is some vicinity of the Lebesgue point $\left(x^j, t^j\right) \in Q_T$. Using the definition of the Lebesgue point, we have

$$\int\limits_{Q_T} \left| \frac{\Delta_\varepsilon^{(2)} J_l(\nu_*)}{\varepsilon^{n+1}} - \delta^{(2)} J_l(\nu_*) \right| dQ = \int\limits_{Q_T} \left| \frac{1}{\varepsilon^{n+1}} \sum_{j=1}^{m} \sum_{k=1}^{k_j} \int\limits_{\Pi_{jk}^\varepsilon} u_* \psi_{l\varepsilon} \left[q_j^k - q_*(x,t) \right] d\Pi - \right.$$

$$\left. - \sum_{j=1}^{m} \sum_{k=1}^{k_j} \beta_j^k u_*\left(x^j, t^j\right) \psi_{l*}\left(x^j, t^j\right) \left[q_j^k - q_*\left(x^j, t^j\right) \right] \right| dQ \leq$$

$$\leq \int\limits_{Q_j} \left| \frac{1}{\varepsilon^{n+1}} \sum_{j=1}^{m} \sum_{k=1}^{k_j} \int\limits_{\Pi_{jk}^\varepsilon} u_* \psi_{l*} \left[q_j^k - q_*(x,t) \right] d\Pi - \right.$$

$$\left. - \sum_{j=1}^{m} \sum_{k=1}^{k_j} \beta_j^k u_*\left(x^j, t^j\right) \psi_{l*}\left(x^j, t^j\right) \left[q_j^k - q_*\left(x^j, t^j\right) \right] \right| dQ \leq$$

$$\leq \int\limits_{Q_j} \left| \frac{1}{\varepsilon^{n+1}} \sum_{j=1}^{m} \sum_{k=1}^{k_j} \int\limits_{\Pi_{jk}^\varepsilon} u_* \Delta\psi_{l\varepsilon} \left[q_j^k - q_*(x,t) \right] d\Pi \right| dQ =$$

$$= \frac{o\left(\varepsilon^{n+1}\right)}{\varepsilon^{n+1}} mes Q_j + \int\limits_{Q_j} \left| \frac{1}{\varepsilon^{n+1}} \sum_{j=1}^{m} \sum_{k=1}^{k_j} \int\limits_{\Pi_{jk}^\varepsilon} u_* \Delta\psi_{l\varepsilon} \left[q_j^k - q_*(x,t) \right] d\Pi \right| dQ \tag{48}$$

$\left(l = \overline{0, l_0 + l_1}\right)$.

Make a change of variables $x = x^j + \xi$, $t = t^j + \tau$ and denote by

$$\widetilde{\Pi}_{jk}^\varepsilon = \left\{ (\xi, \tau) \in E_{n+1} : -\varepsilon/2 \leq \xi_i < \varepsilon/2 \left(i = \overline{1,n}\right) \ , \ \varepsilon \sum_{s=1}^{k} \beta_j^s \leq \tau < -\varepsilon \sum_{s=1}^{k-1} \beta_j^s \right\}$$

a shifted parallelepiped. Then using $|q_*(x,t)| \leq \rho$ almost everywhere on Q_T, $\left|q_j^k\right| \leq \left(j=\overline{1,m};\ k=\overline{1,k_j}\right)$, the Cauchy-Bunyakovsky inequality, estimation (9) and relation (41), we get

$$\int\limits_{Q_T}\left|\frac{1}{\varepsilon^{n+1}}\sum_{j=1}^{m}\sum_{k=1}^{k_j}\int\limits_{\Pi_{jk}^{\varepsilon}} u_*\Delta\psi_{l\varepsilon}\left[q_j^k-q_*(x,t)\right]d\Pi\right|dQ \leq$$

$$\leq \frac{1}{\varepsilon^{n+1}}\sum_{j=1}^{m}\sum_{k=1}^{k_j}\int\limits_{\widetilde{\Pi}_{jk}^{\varepsilon}}\left\{\left|u_*\left(x^i+\xi,t^j+\tau\right)\Delta\psi_{l\varepsilon}\left(x^i+\xi,t^j+\tau\right)\times\right.\right.$$

$$\left.\left.\times\left[q_j^k-q_*\left(x^j+\xi,t^i+\tau\right)\right]\right|\right\}d\widetilde{\Pi} \leq$$

$$\leq \frac{2\rho}{\varepsilon^{n+1}}\sum_{j=1}^{m}\sum_{k=1}^{k_j}\int\limits_{\widetilde{\Pi}_{jk}^{\varepsilon}}\left\{\left|u_*\left(x^i+\xi,t^j+\tau\right)\right|^2dQ\right\}^{1/2}\times$$

$$\times\left\{\int\limits_{Q_T}\left|\Delta\psi_{l\varepsilon}\left(x^i+\xi,t^j+\tau\right)\right|^2dQ\right\}^{1/2}d\widetilde{\Pi}\to 0\quad\left(l=\overline{0,l_0+l_1}\right)$$

as $\varepsilon\to 0$. Hence and from (48) it follows the validity of (47). This means that formula (46) is valid. The cylinder Q_T may be covered by the vicinities of Q_j. Therefore, we can affirm that for all $\left(x^j,t^j\right)\in Q_T$ formula (46) is valid.

Denote $\delta k(x,t)=k(x,t)-k_*(x,t)$. According to formulae (44)-(46), to each collection of parameters $\mu=\left\{\beta_j^k,\delta_k,q_j^k,\left(x^j,t^j\right)\right\}$ there responds the collection of variation of functionals $A_\mu=\{\delta J_0,\delta J_1,...,\delta J_{l_0},\delta J_{l_0+1},...,\delta J_{l_0+l_1}\}$ that may be considered as a vector of the space $E_{l_0+l_1+1}$ emanating from the point $(J_{0*},0,...,0,0,...,0)\in E_{l_0+l_1+1}$, where J_{0^*} is the minimal value of the functional J_0 on the set W. Without loss of generality, we can assume that $J_{0^*}=0$. If it is not so, then by means of shift along the axis J_0 we pass to such a space where the vector A_μ emanates from the origin of coordinates.

To the possible collection μ there responds the set $P\subset E_{l_0+l_1+1}$ of the vector A_μ of the first variations of the functionals $J_l(\nu)$ $\left(l=\overline{0,l_0+l_1}\right)$. Using the linearity of the first variations of the functional $J_l(\nu)$ $\left(l=\overline{0,l_0+l_1}\right)$ with respect to parameters P, we can show that δ_k, β_j^k is a convex cone in $E_{l_0+l_1+1}$.

Let $\nu_*(x,t)=(k_*(x,t),q_*(x,t))\in W$ be an optimal control in problem (1)-(7). Then we can show that [12] the cone P constructed for the optimal control $\nu_*(x,t)$, and the set $L=\{A\in E_{l_0+l_1+1}:A=(a_0,a_1,...,a_{l_0},a_{l_0+1},...,a_{l_0+l_1}),$ $a_l\leq 0\left(l=\overline{0,l_0}\right),a_l=0\,(l=l_0+1,l_0+l_1)\}$ are separated in $E_{l_0+l_1+1}$ by the nontrivial functional $\lambda^*=\left(\lambda_0^*,\lambda_1^*,...,\lambda_{l_0}^*,\lambda_{l_0+1}^*,...,\lambda_{l_0+l_1}^*\right)\in\left(E_{l_0+l_1+1}\right)^*=E_{l_0+l_1+l}$, where $\lambda_l^*\geq 0$ $\left(l=\overline{0,l_0}\right)$. Hence and from the form of the set P it follows that

$$\sum_{l=0}^{l_0+l_1}\lambda_l^*\delta J_l(\nu_*)\geq 0. \tag{49}$$

Assuming $m = 1$, $\left(x^j, t^j\right) = (x, t)$, $k_1 = \beta_1^1 = 1$, $q_1^1 = q$ and substituting expressions (44)-(46) in (49), we get the validity of inequality (17). The theorem is proved

5. Conclusion

Theorem 1 shows that problem (1)-(7) was well-posed in weak topology of the control space B, i.e., $W_* \neq \varnothing$ and any minimizing sequence of the functional $J_0(\nu)$ weakly in B converges to the set W_* [25,p.163]. Example 1 shows that the optimal control in problem (1)-(7) generally speaking, is non-unique. If instead of the functional (7) we consider a problem on minimization of the functional

$$J_\varepsilon(\nu) = J_0(\nu) + \varepsilon \|\nu - \omega\|_B^2 \tag{50}$$

on the set V at conditions (1), (2), (6) where $\varepsilon > 0$ is the given number, $\omega \in B$ is the goven element, we can show that there exists a dense subset B_0 of the space B such that for any $\omega \in B_0$ this problem has a unique solution [26;27, p.373].

Note that problem (1)-(7) is ill-posed in the metric of the space B [6,8] i.e. there may exists minimizing sequences of the functional $J_0(\nu)$ not converging in the norm of the space B to the set of optimal controls W_*. For construction of the minimizing sequence of the functional $J_0(\nu)$ convergeng in the norm of the space B to the space W_* we consider a problem on minimization of the Tichonov functional

$$T_k(\nu) = J_0(\nu) + \alpha_k \Omega(\nu)$$

on the set V at conditions (1), (2), (6), where $\{\alpha_k\}$ is a positive sequence converging to zero, $\Omega(\nu)$ is any stabilizer of problem (1)-(7) [23, p. 165].

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Rafiq K. Tagiyev, Sadiq A. Hashimov
Baku State University,
23, B. Vahabzade str., AZ 1148, Baku, Azerbaijan
Tel.: (99412) 539 47 20 (off.).

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