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ON SOLVABILITY OF A BOUNDARY VALUE PROBLEM FOR A CLASS OF OPERATOR-DIFFERENTIAL EQUATIONS OF THIRD ORDER

Abstract

In the paper a boundary value problem for a class of operator-diffrential equations of third order is considered. The equations and boundary conditions are perturbed by some operators. Sufficient conditions on the coefficients of the equation and the operator participating in the boundary conditions, and that provide regular solvability of the problem under consideration are obtained.

Let H be a separable Hilbert space, A be a positive-definite self-adjoint operator in H, and $H_\gamma = D(A^\gamma)$ be a Hilbert space with the scalar product $(x, y)_\gamma = (A^\gamma x, A^\gamma y)$, $\gamma \geq 0$, $H_0 = H$. Denote by $L_2(R_+; H)$ Hilbert space of vector-functions with the values in H, measurable, quadratically integrable in the Bochner sense with the norm

$$\|f\|_{L_2(R_+;H)} = \left(\int_0^{+\infty} \|f(t)\|^2 \, dt \right)^{1/2} < \infty$$

Introduce the Hilbert space [1]

$$W_2^3(R_+; H) = \left\{ u : u''' \in L_2(R_+; H), \ A^3 u \in L_2(R_+; H) \right\}$$

with the norm

$$\|u\|_{W_2^3(R_+;H)} = \left(\|u'''\|^2_{L_2(R_+;H)} + \|A^3 u\|^2_{L_2(R_+;H)} \right)^{1/2}.$$

Let $L(X; Y)$ be the space of bounded operators acting from the space X to the space Y. Assume that $K \in L(W_2^3(R_+; H), H_{3/2})$ and define the subspace of the space $W_2^3(R_+; H)$

$$W_{2,K}^3(R_+; H) = \left\{ u : u \in W_2^3(R_+; H), u'(0) = 0, u''(0) = Ku \right\}.$$

Obviously $W_{2,K}^3(R_+; H)$ is a complete Hilebrt space. For $R = (-\infty, \infty)$ the spaces $L_2(R; H)$ and $W_2^3(R; H)$ are defined similarly.

Consider in space H the boundary value problem

$$P(d/dt)u(t) = u'' - A^3 u + \sum_{j=0}^{3} A_{3-j} u^{(j)} = f(t), \quad t \in R_+, \qquad (1)$$

$$u'(0)=0,\quad u''(0)=Ku, \tag{2}$$

where $f(t)$, $u(t)$ are the vector-functions defined in R_+, almost everywhere with the values in H, and the operator coefficients of boundary value problem (1)-(2) satisfy the conditions:

1) A is a positive-definite self-adjoint operator;

2) the operators $B_j=A_jA^{-j}$ $(j=\overline{0,3})$ are the bounded operators in H;

3) $K\in L(W_2^3(R_+;H),H_{1/2})$, moreover $\chi=\|K\|_{W_2^3(R_+;H)\to H_{1/2}}$.

Definition 1. *If for $f(t)\in L_2(R_+;H)$ there exists a vector-function $u(t)\in W_2^3(R_+;H)$ that satisfies equation (1) almost everywhere in R_+, then $u(t)$ is called a regular solution of equation (1).*

Definition 2. *If for any $f(t)\in L_2(R_+;H)$ there exists a regular solution of equation (1) that satisfies boundary conditions (2) in the sense of convergencce*

$$\lim_{t\to+0}\left\|u'(t)\right\|_{5/2}=0,\quad \lim_{t\to+0}\left\|u''(t)-Ku\right\|_{1/2}=0,$$

and the following estimation

$$\|u\|_{W_2^3(R_+;H)}\le const\,\|f\|_{L_2(R_+;H)},$$

we say that boundary value problem (1), (2) is regularly solvable.

In this paper we find the conditions on the coefficients of boundary value problem (1)-(2) that provide regular solvability of problem (1)-(2). The similar problems were studied in [2,3].

At first we investigate the regular solvability of the boundary value problem

$$P_0(d/dt)u(t)=u'''(t)-A^3u(t)=f(t),\quad t\in R_+, \tag{3}$$

$$u'(0)=0,\ u''(0)=Ku. \tag{4}$$

It holds

Theorem 1. *Let $\chi=\|K\|_{W_2^3(R_+;H)\to H_{1/2}}$. Then problem (3), (4) has a unique regular solution.*

Proof. Let $u_0(t)=e^{\omega_1tA}x_1+e^{\omega_2tA}x_2$ be a general solution of the equation $P_0(d/dt)u(t)=0$ from the space $W_2^3(R_+;H)$, where $x_1,\ x_2\in H_{5/2}$, $\omega_1=$ $=-\frac{1}{2}(1+\sqrt{3}i)$, $\omega_2=-\frac{1}{2}(1-\sqrt{3}i)$ and e^{ω_1tA}, e^{ω_2tA} are the semigroups of bounded operators generated by the operators ω_1A and ω_2A, respectively. Then from condition 3) it follows that

$$\omega_1Ax_1+\omega_2Ax_2=0,$$

$$\omega_1^2A^2x_1+\omega_2^2A^2x_2=K(e^{\omega_1tA}x_1+e^{\omega_2tA}x_2).$$

Hence we have $x_2=-\dfrac{\omega_1}{\omega_2}x_1$ and

$$\omega_1^2x_1-\omega_2\omega_1x_2=A^{-2}K\left(e^{\omega_1At}x_1-\frac{\omega_1}{\omega_2}e^{\omega_2tA}x_1\right).$$

Hence we get

$$\omega_1(\omega_2-\omega_1)x_1=-A^{-2}K(e^{\omega_1 tA}x_1-\frac{\omega_1}{\omega_2}e^{\omega_2 tA}x_1),$$

i.e.

$$x_1=-\frac{1}{\omega_1\sqrt{3}}A^{-2}K(e^{\omega_1 tA}x_1-\frac{\omega_1}{\omega_2}e^{\omega_2 tA}x_1).$$

Let

$$Qx=\frac{1}{\omega_1\sqrt{3}}A^{-2}K(e^{\omega_1 tA}x-\frac{\omega_1}{\omega_2}e^{\omega_2 tA}x),\quad x\in H_{5/2}. \tag{5}$$

Then we have

$$\|Qx\|_{5/2}=\frac{1}{\sqrt{3}}\left\|A^{1/2}K(e^{\omega_1 tA}x-\frac{\omega_1}{\omega_2}e^{\omega_2 tA}x)\right\|=$$

$$=\frac{1}{\sqrt{3}}\left\|K(e^{\omega_1 tA}x-\frac{\omega_1}{\omega_2}e^{\omega_2 tA}x)\right\|_{1/2}\leq$$

$$\leq\frac{k}{\sqrt{3}}\left\|e^{\omega_1 tA}x-\frac{\omega_1}{\omega_2}e^{\omega_2 tA}x\right\|_{W_2^3(R_+;H)}. \tag{6}$$

On the other hand, taking into account $\omega_1^3=\omega_2^3=1$, we get:

$$\left\|e^{\omega_1 tA}x-\frac{\omega_1}{\omega_2}e^{\omega_2 tA}x\right\|^2_{W_2^3(R_+;H)}=\left\|\omega_1^3A^3e^{\omega_1 tA}x-\omega_2^3\frac{\omega_1}{\omega_2}e^{\omega_2 tA}x\right\|^2_{L_2(R_+;H)}+$$

$$+\left\|A^3e^{\omega_1 tA}x-A^3\frac{\omega_1}{\omega_2}e^{\omega_2 tA}x\right\|^2_{L_2(R_+;H)}=$$

$$=2\left\|A^3e^{\omega_1 tA}x-A^3\frac{\omega_1}{\omega_2}e^{\omega_2 tA}x\right\|^2_{L_2(R_+;H)}. \tag{7}$$

Assuming $A^{5/2}x=z$, we have

$$\left\|A^3e^{\omega_1 tA}x-A^3\frac{\omega_1}{\omega_2}e^{\omega_2 tA}x\right\|^2_{L_2(R_+;H)}=$$

$$=\left\|A^{1/2}e^{\omega_1 tA}z\right\|^2_{L_2(R_+;H)}+\left\|A^{1/2}e^{\omega_2 tA}z\right\|^2_{L_2(R_+;H)}-$$

$$-2\,\mathrm{Re}\left(A^{1/2}e^{\omega_1 tA}z,A^{1/2}\frac{\omega_1}{\omega_2}e^{\omega_2 tA}z\right)_{L_2(R;H)}. \tag{8}$$

Using the spectral expansion for operator A, we get

$$\left\|A^{1/2}e^{\omega_1 tA}z\right\|^2_{L_2(R_+;H)}=\int_0^\infty\left(\int_{\mu_0}^\infty \mu e^{2\,\mathrm{Re}\,\omega_1 t\mu}(dE_\mu z,z)\right)dt=$$

$$=\int_{\mu_0}^\infty(\mu(dE_\mu z,z))\int_0^\infty e^{-tx}dt=\int_0^\infty((dE_\mu z,z))=$$

$$= \|z\|^2 = \left\| A^{5/2} x \right\|^2 = \|x\|_{5/2}^2 . \qquad (9)$$

Similarly we have

$$\left\| A^{1/2} e^{\omega_2 t A} x \right\|_{L_2(R_+;H)}^2 = \|x\|_{5/2}^2 . \qquad (10)$$

Using similar calculations, we get:

$$-2\,\mathrm{Re}(A^{1/2} e^{\omega_1 t A} z, \frac{\omega_1}{\omega_2} A^{1/2} e^{\omega_2 t A} z)_{L_2(R_x;H)} =$$

$$= -2\,\mathrm{Re}\,\frac{\omega_2}{\omega_1} (A e^{\omega_1 t A} z, z)_{L_2(R_x;H)} =$$

$$= -2\,\mathrm{Re}\,\frac{\omega_2}{\omega_1} \left(\int\limits_{\mu_0}^{\infty} \mu (dE_\mu z, z) \right) \int\limits_0^{\infty} e^{2\omega_1 t \mu} dt) =$$

$$= 2\,\mathrm{Re}\,\frac{\omega_2}{\omega_1} \frac{1}{2\omega_1} \|z\|^2 = \mathrm{Re}\,\frac{\omega_2}{\omega_1^2} \|z\|^2 =$$

$$= \mathrm{Re}\,\frac{\omega_2 \omega_1}{\omega_1^3} \|z\|^2 = \|z\|^2 = \|x\|_{5/2}^2 . \qquad (11)$$

Taking into account (9), (10) and (11) in (7), we get

$$\left\| e^{\omega_1 t A} x - \frac{\omega_1}{\omega_2} e^{\omega_2 t A} x \right\|_{L_2(R_x;H)}^2 = 3 \|x\|_{5/2}^2 .$$

Consequently, from inequality (6) it follows that $\|Q_x\|_{5/2} \leq \sqrt{2}\chi \|x\|_{5/2}$. Since $\chi < 1$, we have that the operator $E+Q$ is invertible in the space $H_{5/2}$. Consequently, from eqaution (5) it follows that $x_1 = 0$. Then $x_2 = 0$. Thus, $u_0(t) \equiv 0$.

Show that problem (3), (4) has a regular solution for any $f \in L_2(R_x; H)$. Obviously, the vector-function

$$u_1(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} (-i\xi^3 E - A^3)^{-1} \left(\int\limits_0^{\infty} f(s) e^{-i\xi(t-s)} ds \right) ds, \quad t \in R = (-\infty, \infty)$$

satisfies equation (3) almost everywhere for $t \in R_+$ and $u_1(t) \in W_2^3(R; H)$ [2].

Denote by $\eta(t)$ the contraction of the vector-function $u_1(t)$ on $[0, \infty)$. Then $\eta(t) \in W_2^3(R_+; H)$ and by the traces theorem $\eta(0) \in H_{5/2}$, $\eta'(0) \in H_{3/2}$, $\eta''(0) \in H_{1/2}$. We'll look for the solution of problem (3),(4) in the form $u(t) = \eta(t) + e^{\omega_1 t A} x_1 + e^{\omega_2 t A} x_2$. Using boundary conditions (4) we get

$$\omega_1 x_1 + \omega_2 x_2 = -A^{-1} \eta'(0)$$

and

$$\eta'(0) + \omega_1^2 x_1 + \omega_2^2 x_2 = A^{-2} K (e^{\omega_1 t A} x_1 + e^{\omega_2 t A} x_2 + \eta(t)).$$

Hence we have $x_2 = -\frac{\omega_1}{\omega_2}x_1 - \frac{i}{\omega_1}A^{-1}\eta'(0)$. Taking this into account in the second equality, we have: $x_1 + Qx_1 = \psi$, where

$$\psi = -\frac{1}{\sqrt{3}\omega_1}A^{-2}\eta'(0) + \frac{1}{\sqrt{3}\omega_1}A^{-2}K(\eta(t)) -$$

$$-\frac{1}{\sqrt{3}\omega_2}A^{-2}K(e^{\omega_2 tA}A^{-1}\eta'(0)) \in H_{5/2}.$$

Hence we find $x_1 = (E+Q)^{-1}\psi \in H_{5/2}$. Then

$$x_2 = -\frac{\omega_1}{\omega_2}(E+Q)^{-1}\psi - \frac{i}{\sqrt{3}\omega_1}A^{-1}\eta'(0) \in H_{5/2}.$$

Thus, $u(t)$ is a regular solution of problem (3), (4). Since

$$\left\| P_0(d(dt)u(t)) \right\|^2_{L_2(R_x;H)} \le 2\left\| u \right\|^2_{W_2^3(R_x;H)},$$

then from the Banach theorem on the inverse operator we get $\|u\|_{W_2^3(R_x;H)} \le const\,\|t\|_{L_2(R_x;H)}$. The theorem is prved.

It holds the following

Theorem 2. *Let conditions 1) and 3), $\chi < \frac{3^{1/4}}{2^{5/2}}$ be fulfilled. Then for any $u \in W^3_{2,K}(R_+;H)$ it holds the inequality*

$$\left\| A^{3-j}u^{(j)} \right\|_{L_2(R_+;H)} \le c_j(\chi)\left\| P_0 u \right\|_{L_2(R_+;H)},$$

where $c_0(\chi) = c_3(\chi) = \left(1 - \frac{2^{1/5}}{3^{1/4}}\chi\right)^{-1/2}$, $c_1(\chi) = c_2(\chi) = \frac{2^{1/3}}{3^{1/2}}\left(1 - \frac{2^{5/3}}{3^{1/4}}\chi\right)^{-1/2}$.

Proof. For $u \in W^3_{2,K}(R_+;H)$

$$\left\| P_0(d/dt)u \right\|^2_{L_2(R_x;H)} = \left\| u'' - A^3 u \right\|^2_{L_2(R_+;H)} =$$

$$= \|u\|^2_{W_2^3(R_+;H)} + 2\operatorname{Re}(A^{5/2}u''(0), A^{5/2}u(0)) \ge$$

$$\ge \|u\|^2_{W_2^3(R_+;H)} - 2\chi\|u\|_{W_2^3(R_x;H)} \cdot \|u(0)\|_{5/2} \tag{12}$$

Obviously,

$$\|u(0)\|^2_{5/2} = -2\operatorname{Re}(A^2u', A^3u)_{L_2(R_+;H)} \le$$

$$\le 2\left\| A^2u' \right\|_{L_2(R_+;H)} \cdot \left\| A^3 u \right\|_{L_2(R_+;H)} \tag{13}$$

On the other hand,

$$\left\| A^2u' \right\|^2_{L_2(R_+;H)} = (A^{5/2}u(0), A^{3/2}u'(0)) - (A^3u, Au'')_{L_2(R_+;H)} \le$$

$$\le \left\| A^3u \right\|_{L_2(R_+;H)} \cdot \left\| Au'' \right\|_{L_2(R_+;H)} \tag{14}$$

Similarly we have:

$$\left\|Au''\right\|^2_{L_2(R_+;H)} \le \left\|u'''\right\|_{L_2(R_+;H)} \cdot \left\|Au'\right\|_{L_2(R_+;H)}. \tag{15}$$

Taking into account (14) in (15), we get

$$\left\|Au''\right\|^2_{L_2(R_+;H)} \le \left\|u'''\right\|_{L_2(R_+;H)} \cdot \left\|A^3u\right\|^{1/2}_{L_2(R_+;H)} \cdot \left\|Au''\right\|^{1/2}_{L_2(R_+;H)},$$

or for $\varepsilon = \dfrac{1}{\sqrt[3]{2}}$ we have

$$\left\|Au''\right\|^2_{L_2(R_+;H)} \le \left\|u''\right\|^{\frac{4}{3}}_{L_2(R_+;H)} \cdot \left\|A^3u\right\|^{\frac{3}{2}}_{L_2(R_+;H)} =$$

$$= \left(\varepsilon \cdot \left\|u'''\right\|^2_{L_2(R_+;H)}\right)^{\frac{3}{2}} \cdot \left(\frac{1}{\varepsilon}\left\|A^3u\right\|^2_{L_2(R_+;H)}\right)^{\frac{1}{3}} \le$$

$$\le \frac{2}{3}\varepsilon \left\|u'''\right\|^2_{L_2(R_+;H)} + \frac{1}{3\varepsilon^2}\left\|A^3u\right\|^2_{L_2(R_+;H)} =$$

$$= \frac{2^{2/3}}{3}\left(\left\|A^3u\right\|^2_{L_2(R_+;H)} + \left\|u'''\right\|^2_{L_2(R_+;H)}\right) = \frac{2^{2/3}}{3}\left\|u\right\|^2_{W_2^3(R_+;H)},$$

i.e.

$$\left\|Au''\right\|_{L_2(R_+;H)} \le \frac{2^{2/3}}{3^{1/2}}\left\|u\right\|_{W_2^3(R_+;H)}. \tag{16}$$

Similarly we have

$$\left\|A'u\right\|_{L_2(R_+;H)} \le \frac{2^{1/3}}{3^{1/2}}\left\|u\right\|_{W_2^3(R_+;H)}. \tag{17}$$

Taking into account (16) and (17) in (13), we have:

$$\left\|u'(0)\right\|^2_{5/2} \le \left\|u(0)\right\|^2_{5/2} \le \frac{2^{4/3}}{3^{1/2}}\left\|u\right\|^2_{W_2^3(R_+;H)},$$

i.e.

$$\left\|u(0)\right\| \le \frac{2^{2/3}}{3^{1/4}}\left\|u\right\|_{W_2^3(R_+;H)}.$$

Then from inequality (12) it follows

$$\left\|P_0(d/dt)u\right\|^2_{W_2^3(R_+;H)} \ge \left(1 - \frac{2^{5/3}}{3^{1/4}}\chi\right)\left\|u\right\|^2_{W_2^3(R_+;H)}.$$

Hence we have

$$\left\|A^3u\right\|_{L_2(R_+;H)} \le \left(1 - \frac{2^{5/3}}{3^{1/4}}\chi\right)^{-1/2}\left\|P_0u\right\|_{L_2(R_+;H)}.$$

From inequality (15) and (16) it follows that

$$\left\|A^2u'\right\|_{L_2(R_+;H)} \le \frac{2^{1/3}}{3^{1/2}}\left(1 - \frac{2^{5/3}}{3^{1/4}}\chi\right)^{-1/2}\left\|P_0u\right\|_{L_2(R_+;H)},$$

$$\left\|Au''\right\|_{L_2(R_+;H)} \le \frac{2^{1/3}}{3^{1/2}}\left(1 - \frac{2^{5/3}}{3^{1/4}}\chi\right)^{-1/2}\left\|P_0u\right\|_{L_2(R_+;H)}.$$

The theorem is proved.

Now prove the main theorem.

Theorem 3. *Let conditions 1)-3) be fulfilled, and* $\alpha(\chi) = \sum_{j=0}^{3} c_j(\chi) \|B_{3-j}\| < 1$, *then problem (1), (2) is regularly solvable.*

Proof. Write problems (1), (2) in the form of the equation $Pu = P_0 u = P_1 u = f$, where $P_0 u = P_0(d/dt)u$, $P_1 u = P_1(d/dt)u = \sum_{j=0}^{3} A_{4-j} u^{(j)}$, $u \in W_{2,K}^3(R_+; H)$, $f \in L_2(R_+; H)$. Using theorem 1 we get that we can represent u in the form $u = P_0^{-1} v$. Hence with respect to v we get the equation $v+$ $+P_1 P_0^{-1} v = f$ in $L_2(R_+; H)$. Using theorem 2 we get that for all $v \in$ $\in L_2(R_+; H)$ the following ineqaulities hold

$$\left\| P_1 P_0^{-1} v \right\|_{L_2(R_+;H)} = \| P_1 u \|_{L_2(R_+;H)} \leq$$

$$\leq \sum_{j=0}^{3} \|B_{3-j}\| \left\| A^{3-j} u^{(j)} \right\|_{L_2(R_+;H)} \leq$$

$$\leq \sum_{j=0}^{3} \|B_{3-j}\| \, c_j(\chi) \, \|P_0 u\|_{L_2(R_+;H)} = \alpha(\chi) \, \|v\|_{L_2(R_+;H)} .$$

Since $\alpha(\chi) < 1$, then $v = (E + P_1 P_0^{-1})^{-1} f$ and $u \in P_0^{-1}(E + P_1 P_0^{-1})^{-1} f$.

Hence it follows that $\|u\|_{W_2^3(R_+;H)} \leq const \, \|f\|_{L_2(R_+;H)}$. The theorem is proved.

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