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EXISTENCE OF GLOBAL MINIMAL ATTRACTOR FOR A SYSTEMS THEORY THERMOELASTICITY

Abstract

In the present paper we investigate the existence of a global minimal attractor of mixed problems for one one-dimensional semi-linear hyperbolic parabolic system when the principal linear part of the system under consideration doesn't decompose into hyperbolic and parabolic operators

1. Introduction. Behavior of the solutions of mixed problems for semi-linear and parabolic equations has been studied well ([see e.i. [1-6]), but we can't say it about the system of semi-linear hyperbolic-parabolic equations. Existence of a global minimal attractor for hyperbolic-parabolic systems when the linear part of the appropriate system decomposes into hyperbolic and parabolic operators, was investigated in the papers [7,8]. In the present paper we investigate the existence of gloval minimal attractor of mixed problems for one one-dimensional semi-linear hyperbolic parabolic system when the principal linear part of the system under consideration doesn't decompose into hyperbolic and parabolic operators

2. Problem Statement. In the domain $Q = [0, \infty) \times (0, 1)$ consider the mixed problem

$$\left.\begin{array}{l} u_{tt} - u_{xx} + \theta_x + u_t + |u|^{p-1} u + f_1(u) = g_1(x) \\ \\ \theta_t - \theta_{xx} + u_{tx} + f_2(u) \end{array}\right\} \qquad (1)$$

with boundary conditions

$$u(t, 0) = u(t, 1) = 0, \quad t > 0, \qquad (2)$$

$$\theta(t, 0) = \theta(t, 1) = 0, \quad t > 0, \qquad (3)$$

and initial conditions

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad \theta(0, x) = \theta_0(x) \quad x \in (0, 1), \qquad (4)$$

where $p > 1$, $f_1(u)$, $f_2(u)$, $g_1(x)$ and $g_2(x)$ are the given functions to be revised below.

In the paper we investigate the existence and uniqueness of global solutions of problem (1)-(4) and also the behavior of the solutions as $t \to +\infty$.

3. Existence and uniqueness of global solutions. In Hilbert space $H = \overset{0}{W}{}_2^1(0, 1) \times L_2(0, 1) \times L_2(0, 1)$ introduce the scalar product

$$\langle w, z \rangle = \int_0^1 v_{1x}\overline{z_{1x}}dx + \int_0^1 v_{2x}\overline{z_{2x}}dx + \int_0^1 v_{3x}\overline{z_{3x}}dx,$$

where $w = (v_1, v_2, v_3)\,,\ z = (z_1, z_2, z_3)$.

By substituting $v_1 = u,\ v_2 = u,\ v_3 = u$ we can reduce problem (1)-(3) to the Cauchy problem

$$w = Aw + F(w)\,, \tag{5}$$

$$w(0) = w_0 \tag{6}$$

in Hilbert space $H = \overset{0}{W}{}_2^1(0,1) \times L_2(0,1) \times L_2(0,1)\,,$ where $w = (u, u_t, \theta)$

$$A = \begin{pmatrix} 0 & I & 0 \\ \dfrac{\partial}{\partial x^2} & -I & -\dfrac{\partial}{\partial x} \\ 0 & -\dfrac{\partial}{\partial x} & \dfrac{\partial^2}{\partial x^2} \end{pmatrix},$$

$$D(A) = \left[W_2^2(0,1) \cap \overset{\circ}{W}{}_2^2(0,1)\right] \times \overset{\circ}{W}{}_2^1(0,1) \times \left[W_2^2(0,1) \cap \overset{\circ}{W}{}_2^1(0,1)\right],$$

$$F(x) = \begin{pmatrix} 0 \\ g_1(x) - f_1(u) \\ g_2(x) - f_2(u) \end{pmatrix}.$$

Assume that the following conditions are fulfilled:

$$f_1(u) \in C^1(R)\,, \quad i = 1, 2 \tag{7}$$

$$|f_1(u)| \in c\left(1 + |u|^{\rho_1}\right), \quad \textit{where} \quad \rho_1 < p, \quad c > 0, \tag{8}$$

$$|f_2(u)| \in c\left(1 + |u|^{\rho_2}\right), \quad \textit{where} \quad \rho_2 < \frac{p+1}{2}, \quad c > 0. \tag{9}$$

Lemma 1. *A is a dissipative operator in H.*

Proof. Let $w = (v_1, v_2 v_3) \in H$. Then

$$\operatorname{Re}\langle Aw, w\rangle = \operatorname{Re}\left[\int_0^1 v_{2x}\overline{v}_{1x}dx + \int_0^1 (v_{1xx} - v_2)\,\overline{v}_2 dx + \int_0^1 (v_{2x} - v_{3xx})\,\overline{v}_3 dx\right] -$$

$$-\int_0^1 |v_2|^2\, dx - \int_0^1 |v_{3x}|^2\, dx \le 0$$

Lemma 2. *A is an invertible operator in H and A^{-1} is a compact operator in H.*

Poof. Let $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in H$. Consider the equation

$$Aw = \varphi$$

Taking into account the expression of the operator A, we have

$$\begin{aligned} v_2 &= \varphi_1, \\ v_{1xx} - v_{3x} &= \varphi_2 + \varphi_1, \\ v_{3xx} &= \varphi_2 + \varphi_{1x}, \end{aligned} \tag{10}$$

In $L_2(0,1)$ consider the operator $G = -\frac{d^2}{dx^2}$ with domain of definition $W_2^2(0,1) \cap \overset{\circ}{W}(0,1)$. It is known [10] that G^{-1} is a bounded operator from $L_2(0,1)$ to $W_2^2(0,1) \cap \overset{\circ}{W}{}_2^1(0,1)$. Form (10) we get

$$v_1 = G^{-1}\left[\left(G^{-1}(\varphi_3 + \varphi_{1x})\right)_x + \varphi_2 + \varphi_1\right]$$

$$v_2 = \varphi_1,$$

$$v_3 = G^{-1}[\varphi_3 + \varphi_{1x}].$$

Hence it is easy to see that A has a bounded inverse operator from H to $D(A)$. On the other hand, $D(A)$ is compactly imbedded in H, therefore A^{-1} ia a compact operator acting in H.

From [10] and lemmas 1,2 we gat the following statement.

Lemma 3. *A is a maximal dissipative operator in H.*

Lemma 4. *The operator A has no imaginary eigen numbers.*

Proof. Let $\lambda \in R$.

Consider the equation

$$Aw = i\lambda w, \tag{11}$$

where $w = (v_1, v_2, v_3) \in D(A)$, $\lambda \in R$, $\lambda \neq 0$. Then prove that $w = 0$. Write (11) with respect to coordinates and get

$$\begin{aligned} v_2 &= i\lambda v_1, \\ v_{1xx} - v_2 - v_{3x} &= i\lambda v_2, \\ -v_{2x} + v_{3xx} &= i\lambda v_3. \end{aligned} \tag{12}$$

Hence we have

$$\langle v_{2x}, \overline{v_{1x}} \rangle = i\lambda |v_{1x}|^2,$$

$$\langle \overline{v_{1xx}} - v_2 - \overline{v_{3x}}, v_2 \rangle = i\lambda |v_2|^2,$$

$$\langle -v_{2x} + v_{3xx}, \overline{v_3} \rangle = i\lambda |v_3|.$$

Summing up these equalities, we gat the equality

$$-|v_{3x}|^2 - |v_2|^2 = i\lambda\left[|v_{1x}|^2 - |v_2|^2 + |v_3|^2\right],$$

Hence it follows, that

$$v_{3x} = 0, \ v_2 = 0.$$

Then from (11) we get that $v_3 = 0$, and for v_1 we get the following problem

$$v_{1xx} = 0, \quad v_1(0) = v_1(1) = 0.$$

from which it follows that

$$v_1 = 0.$$

Based on Lumer-Fillips (see [10]) and Lax (see [9]) theorems, from lemmas 1-3 we get the following statement.

Theorem 1. *In space H the operator A generates a strongly continuous contracting descending semi-group.*

Lemma 5. *In space H, the nonlinear operator $F(w)$ satisfies the local Lipschits condition, i.e.*

$$\left\| F(w^1) - F(w^2) \right\|_H \le c\left(\left\| w^1 \right\|_H, \left\| w^2 \right\|_H \right)\left(\left\| w^1 - w^2 \right\|_H \right).$$

From theorem 1, lemmas 3, 5 and also from a priori estimates obtained below, we get that for any $w_0 \in H$ problem (5),(6) has a unique solution (u, θ), where

$$u \in C^1([0,\infty); L_2(0,1)) \cap C\left([0,\infty); \overset{\circ}{W}{}_2^1(0,1) \right), \quad \theta \in C([0,\infty); L_2(0,1)).$$

Therewith, if $w_0 \in D(A)$, then

$$u \in C^2([0,\infty); L_2(0,1)) \cap C^1\left([0,\infty); \overset{\circ}{W}{}_2^1(0,1) \right) \cap$$

$$\cap C\left([0,\infty); W_2^2(0,1) \cap \overset{\circ}{W}{}_2^1(0,1) \right),$$

$$u \in C^1([0,\infty); L_2(0,1)) \cap C\left([0,\infty); W_2^2(0,1) \cap \overset{\circ}{W}{}_2^1(0,1) \right).$$

Thus, problem (5),(6) generates a nonlinear semi-group. We'll denote this semi-group by $W(t)$.

4. A priori estimates and existence of absorbing set. In domain $Q = [0,\infty) \times (0,1)$ consider the linear operators

$$L_1(u) \equiv u_{tt} - u_{xx} + \theta_x + u_t$$

$$L_2(\theta) \equiv \theta_t - \theta_{xx} + u_{tx}$$

with boundary conditions (3).

Let

$$u \in C^2([0,\infty); L_2(0,1)) \cap C^1\left([0,\infty); \overset{\circ}{W}{}_2^1(0,1) \right) \cap$$

$$\cap C\left([0,\infty); W_2^2(0,1) \cap \overset{\circ}{W}{}_2^1(0,1) \right),$$

$$\theta \in C^1\left([0,\infty);L_2(0,1)\right)\cap C\left([0,\infty);W_2^2(0,1)\cap \overset{\circ}{W}{}_2^1(0,1)\right).$$

Multiply $L_1(u)$ by $\overline{u_t}+\eta\overline{u}$ and integrate with respect to domain $(0,1)$, where $\eta>0$

$$\int_0^1 L_1(u(t,x))\left(\overline{u_t}(t,x)+\eta\overline{u}(t,x)\right)dx=\int_0^1 u_{tt}(t,x)\cdot\overline{u_t}(t,x)\,dx+$$

$$+\eta\int_0^1 u_{tt}(t,x)\cdot\overline{u}(t,x)\,dx-\int_0^1 u_{xx}(t,x)\cdot\overline{u_t}(t,x)\,dx-$$

$$-\eta\int_0^1 u_{xx}(t,x)\cdot\overline{u}(t,x)\,dx+\int_0^1 \theta_x(t,x)\cdot\overline{u_t}(t,x)\,dx+$$

$$+\eta\int_0^1 \theta_x(t,x)\cdot\overline{u}(t,x)\,dx+\int_0^1 |u_t(t,x)|^2\,dx+\eta\int_0^1 u_t(t,x)\cdot\overline{u}(t,x)\,dx,$$

$$\int_0^1 L_2(\theta(t,x))\,\theta(t,x)\,dx=\int_0^1 \theta_t(t,x)\cdot\theta(t,x)\,dx-$$

$$-\int_0^1 \theta_{xx}(t,x)\,\theta(t,x)\,dx-\int_0^1 u_{tx}(t,x)\cdot\theta(t,x)\,dx$$

Transform each addend integrating by parts. Then using the Poincare inequality, we get

$$\int_0^1 L_1(u(t,x))\left(u_t(t,x)+\eta u(t,x)\right)dx+\int_0^1 L_2(\theta(t,x))\,\theta(t,x)\,dx\geq$$

$$\geq\frac{d}{dt}\left[\frac{1}{2}\|u_t(t,\cdot)\|^2_{L_2(G)}+\eta\langle u_t(t,\cdot),u(t,\cdot)\rangle+\frac{1}{2}\|u_x(t,\cdot)\|^2_{L_2(G)}+\right.$$

$$\left.+\frac{\eta}{2}\|u(t,\cdot)\|^2_{L_2(G)}+\frac{1}{2}\|\theta(t,\cdot)\|^2_{L_2(G)}\right]+$$

$$+\|u_t(t,\cdot)\|^2_{L_2(G)}+\eta\|u_x(t,\cdot)\|^2_{L_2(G)}+c_0\|\theta(t,\cdot)\|^2_{L_2(G)}, \qquad (13)$$

where c_0 is determined from the Poincare inequality

$$c_0\|\theta(t,\cdot)\|^2_{L_2(G)}\leq\|\theta(t,\cdot)\|^2_{L_2(G)}$$

Using (1) and (8),(9), from (13) we get

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t(t,\cdot)\|^2_{L_2(G)}+\eta\langle u_t(t,\cdot),u(t,\cdot)\rangle+\frac{1}{2}\|u_x(t,\cdot)\|^2_{L_2(G)}+\right.$$

$$+\frac{\eta}{2}\left\|u\left(t,\cdot\right)\right\|^2_{L_2(G)}+\frac{1}{2}\left\|\theta\left(t,\cdot\right)\right\|^2_{L_2(G)}\Bigg]+$$

$$+\left\|u_t\left(t,\cdot\right)\right\|^2_{L_2(G)}+\eta\left\|u_x\left(t,\cdot\right)\right\|^2_{L_2(G)}+\left\|\theta_x\left(t,\cdot\right)\right\|^2_{L_2(G)}=$$

$$=\left\langle g_1\left(x\right)-\left|u\right|^{p-1}u-f_1\left(u\right),u_t\left(t,\cdot\right)+\eta u\left(t,\cdot\right)\right\rangle+\left\langle g_2\left(x\right)-f_2\left(u\right),\theta\left(t,\cdot\right)\right\rangle.$$

Hence, for homogeneous problem (2),(3)

$$L_1\left(u\right)=0,\quad L_2\left(u\right)=0$$

we have

$$\frac{d}{dt}E_1\left(t\right)+E_2\left(t\right)\leq 0, \tag{14}$$

where

$$E_1\left(t\right)=\frac{1}{2}\left\|u_t\left(t,\cdot\right)\right\|^2_{L_2(G)}+\eta\left\langle u_t\left(t,\cdot\right),u\left(t,\cdot\right)\right\rangle+$$

$$+\frac{1}{2}\left\|u_x\left(t,\cdot\right)\right\|^2_{L_2(G)}+\frac{1}{2}\left\|u\left(t,\cdot\right)\right\|^2_{L_2(G)}+\frac{1}{2}\left\|\theta\left(t,\cdot\right)\right\|^2_{L_2(G)}$$

$$E_2\left(t\right)=\left\|u_t\left(t,\cdot\right)\right\|^2_{L_2(G)}+\eta\left\|u_x\left(t,\cdot\right)\right\|^2_{L_2(G)}+\left\|\theta\left(t,\cdot\right)\right\|^2_{L_2(G)}.$$

Taking into account Poincare and Hölder inequalities, we have the following inequality

$$mE\left(t\right)\leq E_i\left(t\right)\leq ME\left(t\right),\quad i=1,2 \tag{15}$$

where $0<m<M$ don't depend on $t>0$, and

$$E\left(t\right)=\left\|u_t\left(t,\cdot\right)\right\|^2_{L_2(G)}+\left\|u_x\left(t,\cdot\right)\right\|^2_{L_2(G)}+\left\|\theta\left(t,\cdot\right)\right\|^2_{L_2(G)}.$$

In view of (14),(15) there exist such $C>0$ and $\omega>0$ that

$$E\left(t\right)\leq CE\left(0\right)\exp\left(-\omega t\right),$$

i.e. the following statement is true.

Theorem 2. *The operator A generates an exponentially decreasing semi-group $\exp\left(tA\right),\ t>0$ i.e.*

$$\left\|\exp\left(tA\right)\right\|_{H\to H}\leq C\exp\left(-\omega t\right),\ t>0.$$

Using conditions (8),(9), we get

$$\left\langle g_1\left(x\right)-\left|u\right|^{p-1}u-f_1\left(u\right),u_t\left(t,\cdot\right)+\eta u\left(t,\cdot\right)\right\rangle+\left\langle g_2\left(x\right)-f_2\left(u\right),\theta\left(t,\cdot\right)\right\rangle\leq$$

$$\leq-\frac{1}{p+1}\frac{d}{dt}\left|u\right|^{p+1}-\eta\int_0^1\left|u\right|^{p+1}dx-\frac{d}{dt}\int_0^1F_1\left(u\right)dx-\eta\int_0^1f_1\left(u\right)uds+$$

$$+c\int_0^1\left(1+\left|g_1\left(x\right)\right|\right)\left(\left|u_t\left(t,\cdot\right)\right|+\eta\left|u\left(t,\cdot\right)\right|\right)dx+$$

$$+c\int_0^1 \left(1+|g_2(x)|+|u(t,x)|^{\rho_2}\right)|\theta(t,x)|\,dx \tag{16}$$

where $F_1(s)=\int_0^s f_1(\tau)\,d\tau$.

Applying the Holder and Young inequalities, we have

$$c\int_0^1 (1+|g_1(x)|)\left(|u_t(t,\cdot)|+\eta\,|u(t,\cdot)|\right)dx \le \frac{c}{\varepsilon}\left(1+\int_0^1 |g_1(x)|^2\,dx\right)+$$

$$\le c\varepsilon\int_0^1 \left(|u_t(t,\cdot)|^2+\eta\,|u(t,\cdot)|^2\right)dx. \tag{17}$$

Similarly,

$$c\int_0^1 \left(1+|g_2(x)|+|u(t,\cdot)|^{\rho_2}\right)|\theta(t,x)|\,dx \le \frac{c}{\varepsilon}\left(1+\int_0^1 |g_2(x)|^2\,dx\right)+$$

$$\le c\varepsilon\int_0^1 |u(t,x)|^{p+1}\,dx + c\varepsilon\int_0^1 |\theta(t,x)|^2\,dx+\frac{c}{\varepsilon^3}. \tag{18}$$

Chosing ε rather small, and denoting from (15)-(18) we get

$$\frac{d}{dt}\overline{E}(t)+C_{1\varepsilon}\overline{E}(t)\le C_{2\varepsilon}, \tag{19}$$

where $\overline{E}(t)=E_1(t)+\int_0^1 |u|^{p+1}\,dx+\int_0^1 F_1(u)\,dx$.

Then using the Young and Poincare inequality from (7)-(9),(15) and (19) for some positive l_1,l_2 we have

$$E(t)\le l_1E(0)\,.\quad \exp\left(-C_{1\varepsilon}t\right)+l_2,\quad t>0.$$

Let $R=2l_2E(0)\le r$. Then for any $t_r\ge \frac{1}{C_{1\varepsilon}}\ln\frac{l_1r}{l_2}$ the inequality $E(t)\le R$ is fulfilled.

Thus, the ball

$$B_R=\left\{w=(v_1,v_2,v_3):\|v_{1x}(t,\cdot)\|^2_{L_2(G)}+\|v_{2x}(t,\cdot)\|^2_{L_2(G)}+\|v_{3x}(t,\cdot)\|^2_{L_2(G)}\le R\right\}$$

is an absorbing set for the nonlinear semi-group $W(t)$ corresponding to problem (5),(6), i.e. the following theorem is valid.

Theorem 3. *The semi-group $W(t)$ has an absorbing set in space H.*

5. Asymptotic compactness and existence of global minimal attractor. Represent the nonlinear semi-group $W(t)$ as follows

$$W(t) = \exp(tA) + V(t),$$

where $\varphi(t) = V(t)\,w_0$ is the solution of the following problem

$$\left.\begin{array}{l} \dfrac{d}{dt}\varphi(t) = A\varphi(t) + F(w), \\ \\ \varphi(0) = 0 \end{array}\right\}, \tag{19}$$

where $F(w) = \begin{pmatrix} 0 \\ \\ g_1(x) - f_1(u) \\ \\ g_1(x) - f_2(u) \end{pmatrix}$, $w = w(t)$ is the solution of problem (5),(6).

Theorem 4. *Let conditions (7)-(9) be fulfilled. Then for any bounded set $B \subset H$ the set $\bigcup\limits_{t\geq 0} V(t)\,B$ is precompact in H.*

Proof. Denoting $y = \dfrac{d}{dt}\varphi(t)$ from (19), we get

$$\left.\begin{array}{l} \dfrac{d}{dt}y(t) = Ay(t) + \Phi(w(t)), \\ \\ y(0) = y_0 \end{array}\right\}, \tag{20}$$

where $\Phi(w) = -\begin{pmatrix} 0 \\ \\ f_{1u}(u)\,u_t \\ \\ f_{2u}(u)\,u_t \end{pmatrix}$, $\quad y_0 = \begin{pmatrix} 0 \\ \\ g_1(x) - f_1(u_0) \\ \\ g_2(x) - f_2(u_0) \end{pmatrix}$.

From (19) we get

$$\|y(t)\|_H \leq C\exp(-\omega t)\,\|y_0\|_H + C\int\limits_0^t \exp(-\omega(t-s))\,\|\Phi(w(s))\|_H\,ds. \tag{21}$$

Let $w_0 \in B_r \subset H$, then from theorem 3 it follows that

$$\|y_0\|_H \leq \psi(r), \quad \|\Phi(w(s))\|_H \leq \psi(r), \quad s \geq 0. \tag{22}$$

where $\psi(.)$ is some continuous function. From (21),(22) it follows that

$$\|y(t)\|_H \leq \Psi(r), \quad r \geq 0.$$

Contequently

$$\|AV(t)\,w_0\|_H \leq \Psi(r), \quad t \geq 0.$$

Since A^{-1} is a compact operator, the set $\underset{t\geq 0}{\cup} V(t)\,B_r$ is precompact in H.

Theorems 3 and 4 yield.

Theorem 5. *Let conditions (7),(9) be fulfilled. Then the* $W(t)$, $t \geq 0$ *semigroup generated by problem (5),(6) is asymptotically compact in space* H.

From the results of [6] and theorem 2-4 we get the following main statement on the existence of a global minimal attractor.

Theorem 6. *Let conditions (8)-(9) be fulfilled. Then the* $W(t)$, $t \geq 0$ *semigroup generated by problem (5),(6) has a compact global attractor in space* H. *This attractor is invariant and bound.*

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