Abulfaz M.MAMEDOV

ON REGULAR SOLVABILITY OF A CLASS OF OPERATOR-DIFFRENTIAL EQUATIONS OF FIFTH ORDER WITH A CONTINUOUS COEFFICIENT ON THE AXIS

Abstract

In the paper, we obtain regular solvability conditions of a class of operator-differential equations of fifth order with a discontinuous coefficient on the axis, and the principal part of operator-differential equations contain a normal operator. The found conditions are expressed by the properties of the coefficients of the operator-differential equation.

Let H be a separable Hilbert space, the operator A, A_j $\left(j=\overline{0,4}\right)$ be linear operators in H and satisfy the following conditions:

1) A is a normal invertible operator whose spectrum is contained in the angular sector

$$S_\varepsilon = \left\{\lambda : |\arg\lambda| \le \varepsilon,\ 0 \le \varepsilon < \frac{\pi}{10}\right\};$$

2) The operators $B_j = A_j A^{-j}$ $\left(j=\overline{1,5}\right)$ are continuous in H.

Consider the operator-differential equation

$$\frac{d^5 u(t)}{dt^5} - \rho(t) A^5 u(t) + \sum_{j=0}^{4} A_{5-j} u^{(j)}(t) = f(t),\ t \in R = (-\infty, \infty), \tag{1}$$

where $u(t)$ and $f(t)$ are vector-functions determined almost everywhere in R with the values in H, and

$$\rho(t) = \begin{cases} \alpha^5, & t \in R_- = (-\infty, 0), \\ \beta^5, & t \in R_+ = (0, +\infty), \end{cases}$$

and $\alpha > 0,\ \beta > 0,\ \alpha \ne \beta$.

Denote by $L_2(R;H)$ the space of vector-functions $f(t)$ determined almost everywhere in R with the values in H, summable in the square over R, i.e. summable functions for which

$$\|f\|_{L_2(R;H)} = \left(\int_{-\infty}^{+\infty} \|f(t)\|_H^2\, dt\right)^{1/2} < \infty. \tag{2}$$

It is known that $L_2(R;H)$ is a Hilbert space with the scalar product

$$(f,g)_{L_2(R;H)} = \int_{-\infty}^{+\infty} (f(t), g(t))_H dt$$

generating the norm (2).

It is known well that [1]

$$W_2^5(R;H)=\left\{u:\frac{d^5u}{dt^5},A^5u\in L_2(R;H)\right\}$$

is a Hilbert space with the scalar product

$$(u,v)_{W_2^5(R;H)}=\left(\frac{d^5u}{dt^5},\frac{d^5v}{dt^5}\right)_{L_{2(R;H)}}+\left(A^5u,A^5v\right)_{L_2(R;H)}$$

and here the norm of the element $u\in W_2^5(R;H)$ is given by the formula

$$\|u\|_{W_2^5(R;H)}=\left(\left\|\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}+\left\|A^5u\right\|^2_{L_2(R;H)}\right)^{1/2}. \tag{3}$$

Note that here the derivatives are understood in the sense of theory of generalized functions [1].

Subject to condition 1), the operator A is represented in the form $A=UC$, where U is a unitary, C is a positive –definite operator in H, and for $x\in D(A)$ it holds the equality $\|Ax\|_H=\|A^*x\|_H=\|Cx\|_H$ and $UCx=CUx$, here A^* is an operator conjugate to the operator A.

Denote by $\{H_\gamma\}$ $(0<\gamma<\infty)$ the Hilbert scale of spaces generated by the operator A, i.e. $H_\gamma=D(C^\gamma)$, $(x,y)_\gamma=(C^\gamma x,C^\gamma y)$, $x,y\in D(C^\gamma)$.

Definition. *If for any $f(t)\in L_2(R;H)$ there exists $u(t)\in W_2^5(R;H)$, that satisfies equation (1) almost everywhere in R and it holds the inequality*

$$\|u\|_{W_2^5(R;H)}\le const\,\|f\|_{L_2(R;H)},$$

then equation (1) is called regularly solvable.

Define the following operators:

$$P_0u=\frac{d^5u}{dt^5}-\rho(t)A^5u,\qquad P_1u=\sum_{j=0}^{4}A_{5-j}u^{(j)},$$

where $u\in W_2^5(R;H)$. Then we can write equation (1) in the form

$$Pu=P_0u+P_1u=f,$$

where $f\in L_2(R;H)$, $u\in W_2^5(R;H)$.

In the present paper, under some restrictions on the coefficients, we'll prove a theorem on regular solvability of equation (1).

At first prove the following theorem.

Theorem 1. *Let condition 1) be fulfilled.*

Then equation

$$P_0u\equiv\frac{d^5u}{dt^5}-\rho(t)A^5u=f(t) \tag{4}$$

is regularly solvable.

Proof. Denote by

$$u_1(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(i\xi^5E-\alpha^5A^5\right)^{-1}\left(\int\limits_{-\infty}^{+\infty}f(s)e^{i(t-s)\xi}ds\right)d\xi \tag{5}$$

and

$$u_2(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(i\xi^5E-\beta^5A^5\right)^{-1}\left(\int\limits_{-\infty}^{+\infty}f(s)e^{i(t-s)\xi}ds\right)d\xi, \tag{6}$$

here E is a unit operator in H.

It is obvious that $u_1(t)$ and $u_2(t)$ satisfy the equations $\frac{d^5u}{dt^5}-\alpha^5A^5u=f$ and $\frac{d^5u}{dt^5}-\beta^5A^5u=f$, respectively, in R almost everywhere. Show that $u_1(t),u_2(t)\in$ $\in W_2^5(R;H)$.

Obviously, the Fourier transformation of the vector-function $u_1(t)$ is of the form:

$$\widehat{u}_1(\xi)=\left(i\xi^5E-\alpha^5A^5\right)^{-1}\widehat{f}(\xi), \tag{7}$$

where $\widehat{f}(\xi)$ is the Fourier transformation of the vector-function $f(t)$. Then by the Plancherel theorem

$$\|u_1\|^2_{W_2^5(R;H)}=\left\|\frac{d^5u_1}{dt^5}\right\|^5_{L_2(R;H)}+\left\|A^5u_1\right\|^2_{L_2(R;H)}=$$

$$=\left\|\xi^5\widehat{u}_1(\xi)\right\|^2_{L_2(R;H)}+\left\|A^5\widehat{u}_1(\xi)\right\|^2_{L_2(R;H)}. \tag{8}$$

From the spectral expansion of the operator A it follows that for any $\xi\in R$ it holds the estimation:

$$\left\|A^5(i\xi^5E-\alpha^5A^5)^{-1}\right\|=\sup_{\lambda\in\sigma(A)}\left|\lambda^5(i\xi^5-\alpha^5\lambda^5)^{-1}\right|\le$$

$$\le\sup_{\substack{\mu>0\\|\varphi|\le\varepsilon}}\left|\mu^5(i\xi^5-\alpha^5\mu^5e^{5i\varphi})^{-1}\right|\le\sup_{\substack{\mu>0\\|\varphi|\le\varepsilon}}\mu^5\left|\xi^{10}+\alpha^{10}\mu^{10}-2\alpha^5\mu^5|\xi|^5\sin 5\varphi\right|^{-1/2}\le$$

$$\le\sup_{\substack{\mu>0\\|\varphi|\le\varepsilon}}\mu^5\left|\xi^{10}+\alpha^{10}\mu^{10}-(\xi^{10}+\alpha^{10}\mu^{10}\sin^2 5\varphi)\right|^{-1/2}\le\frac{1}{\alpha^5\cos 5\varphi}. \tag{9}$$

Allowing for (7) and (9), we get

$$\left\|A^5\widehat{u}_1(\xi)\right\|_{L_2(R;H)}=\left\|A^5(i\xi^5E-\alpha^5A^5)^{-1}\widehat{f}(\xi)\right\|_{L_2(R;H)}\le$$

$$\le\left\|A^5(i\xi^5E-\alpha^5A^5)^{-1}\right\|\cdot\left\|\widehat{f}(\xi)\right\|_{L_2(R;H)}\le\frac{1}{\alpha^5\cos 5\varphi}\left\|\widehat{f}(\xi)\right\|_{L_2(R;H)}.$$

Then it is clear that $A^5u_1(t)\in L_2(R;H)$.

It is similarly proved that $\frac{d^5u_1}{dt^5}\in L_2(R;H)$. Then we get that $u_1(t)\in W_2^5(R;H)$. $u_2(t)\in W_2^5(R;H)$ is proved in the same way.

Denote by $\psi_1(t)$ and $\psi_2(t)$ the contractions of the vector-functions $u_1(t)$ and $u_2(t)$ on R_- and R_+, respectively. Then $\psi_1(t) \in W_2^5(R_-;H)$, $\psi_2(t) \in W_2^5(R_+;H)$ and by the traces theorem [1] $\psi_i^{(j)}(0) \in H_{5-j-\frac{1}{2}}$, $i=1,2;$ $j=\overline{0,4}$.

Construct the vector-function

$$u(t) = \begin{cases} \theta_1(t) = \psi_1(t) + e^{\alpha\lambda_1 tA}\varphi_1 + e^{\alpha\lambda_2 tA}\varphi_2 + e^{\alpha\lambda_5 tA}\varphi_5, & t \in R_-, \\ \theta_2(t) = \psi_2(t) + e^{\alpha\lambda_3 tA}\varphi_3 + e^{\alpha\lambda_4 tA}\varphi_4, & t \in R_+, \end{cases}$$

where $\lambda_k = \cos\dfrac{2\pi(k-1)}{5} + i\sin\dfrac{2\pi(k-1)}{5}$, $k=\overline{1,5}$ are the roots of the equation $\lambda^5=1$, and φ_k $(k=\overline{1,5})$ are still unknown vectors from $H_{9/2}$. It is obvious that the vector-function $u(t)$ is a general solution of equation (4). Choose φ_k $(k=\overline{1,5})$ so that $u(t) \in W_2^5(R;H)$.

For that there should be $\theta_1^{(j)}(0) = \theta_2^{(j)}(0)$, $j=\overline{0,4}$. Hence we get the system of equations with respect to φ_k, $k=\overline{1,5}$:

$$\begin{cases} \varphi_1+\varphi_2-\varphi_3-\varphi_4+\varphi_5 = \psi_2(0)-\psi_1(0), \\ \alpha\lambda_1\varphi_1+\alpha\lambda_2\varphi_2-\beta\lambda_3\varphi_3-\beta\lambda_4\varphi_4+\alpha\lambda_5\varphi_5 = A^{-1}(\psi_2'(0)-\psi_1'(0)), \\ \alpha^2\lambda_1^2\varphi_1+\alpha^2\lambda_2^2\varphi_2-\beta^2\lambda_3^2\varphi_3-\beta^2\lambda_4^2\varphi_4+\alpha^2\lambda_5^2\varphi_5 = A^{-2}(\psi_2''(0)-\psi_1''(0)), \\ \alpha^3\lambda_1^3\varphi_1+\alpha^3\lambda_2^3\varphi_2-\beta^3\lambda_3^3\varphi_3-\beta^3\lambda_4^3\varphi_4+\alpha^3\lambda_5^3\varphi_5 = A^{-3}(\psi_2'''(0)-\psi_1'''(0)), \\ \alpha^4\lambda_1^4\varphi_1+\alpha^4\lambda_2^4\varphi_2-\beta^4\lambda_3^4\varphi_3-\beta^4\lambda_4^4\varphi_4+\alpha^4\lambda_5^4\varphi_5 = A^{-4}(\psi_2^{IV}(0)-\psi_1^{IV}(0)). \end{cases} \quad (10)$$

From the traces theorem it follows that the vectors at the right side of equations in (10) belong to the space $H_{9/2}$, and we can show that the principal operator-matrix in (10)

$$\Delta_0 = \begin{bmatrix} E & E & -E & -E & E \\ \alpha\lambda_1 E & \alpha\lambda_2 E & -\beta\lambda_3 E & -\beta\lambda_4 E & \alpha\lambda_5 E \\ \alpha^2\lambda_1^2 E & \alpha^2\lambda_2^2 E & -\beta^2\lambda_3^2 E & -\beta^2\lambda_4^2 E & \alpha^2\lambda_5^2 E \\ \alpha^3\lambda_1^3 E & \alpha^3\lambda_2^3 E & -\beta^3\lambda_3^3 E & -\beta^3\lambda_4^3 E & \alpha^3\lambda_5^3 E \\ \alpha^4\lambda_1^4 E & \alpha^4\lambda_2^4 E & -\beta^4\lambda_3^4 E & -\beta^4\lambda_4^4 E & \alpha^4\lambda_5^4 E \end{bmatrix}$$

is invertible in $H_{9/2}^5 = H_{9/2}\times H_{9/2}\times H_{9/2}\times H_{9/2}\times H_{9/2}$. Then the unknown vectors φ_k $(k=\overline{1,5})$ are uniquely determined and belong to the space $H_{9/2}$.

From these arguments it follows that the homogeneous equation $P_0 u \equiv \dfrac{d^5u}{dt^5} - \rho(t)A^5u = 0$ has only a zero solution $u_0(t)\equiv 0$. Therefore, the operator P_0 isomorphically maps the space $W_2^5(R;H)$ onto $L_2(R;H)$.

Since

$$\|f\|_{L_2(R;H)} = \|P_0u\|_{L_2(R;H)} \le const\,\|u\|_{W_2^5(R;H)},$$

then applying the Banach theorem on the inverse operator, we get that the solution of the equation $u(t) \in W_2^5(R;H)$ satisfies the inequality

$$\|u\|_{W_2^5(R;H)} \le const\,\|f\|_{L_2(R;H)}.$$

The theorem is proved.

In order to prove regular solvability of equation (1), in some conditions on the coefficients, it is necessary to estimate the norms of the operators of intermediate derivatives with the norm of the principal part of equation (1).

Prove the following theorem.

Theorem 2. *Let A be a normal invertible operator whose spectrum is contained in the angular sector*

$$S_\varepsilon = \left\{ \lambda : |\arg \lambda| \le \varepsilon, 0 \le \varepsilon < \frac{\pi}{10} \right\}.$$

Then the following estimations hold:

$$\left\| A^{5-j} u^{(j)} \right\|_{L_2(R;H)} \le C_j(\varepsilon;\alpha;\beta) \left\| P_0 u \right\|_{L_2(R;H)},$$

where the coefficients $C_j(\varepsilon;\alpha;\beta)$ $(j = \overline{0,4})$ are determined as follows:

$$C_0(\varepsilon;\alpha;\beta) = \frac{1}{\min(\alpha^5, \beta^5)} \cdot \frac{1}{\cos 5\varepsilon},$$

$$C_1(\varepsilon;\alpha;\beta) = \frac{16^{1/5}}{5^{1/2}} \cdot \frac{\max\left(\alpha^{1/2};\beta^{1/2}\right)}{\min\left(\alpha^{9/2};\beta^{9/2}\right)} \cdot (1 - \sin 5\varepsilon)^{-1/2},$$

$$C_2(\varepsilon;\alpha;\beta) = \frac{2^{1/5} 3^{3/10}}{5^{1/2}} \cdot \frac{\max(\alpha;\beta)}{\min\left(\alpha^4;\beta^4\right)} \cdot (1 - \sin 5\varepsilon)^{-1/2},$$

$$C_3(\varepsilon;\alpha;\beta) = \frac{2^{1/5} 3^{3/10}}{5^{1/2}} \cdot \frac{\max\left(\alpha^{3/2};\beta^{3/2}\right)}{\min\left(\alpha^{7/2};\beta^{7/2}\right)} \cdot (1 - \sin 5\varepsilon)^{-1/2},$$

$$C_4(\varepsilon;\alpha;\beta) = \frac{16^{1/5}}{5^{1/2}} \cdot \frac{\max\left(\alpha^2;\beta^2\right)}{\min\left(\alpha^3;\beta^3\right)} \cdot (1 - \sin 5\varepsilon)^{-1/2}.$$

Proof. At first that for $u(t) \in W_2^5(R;H)$ it holds the following inequality:

$$\left\| \rho^{-1/2} P_0 u \right\|^2_{L_2(R;H)} \ge \left\| \rho^{-1/2} \frac{d^5 u}{dt^5} \right\|^2_{L_2(R;H)} + \left\| \rho^{1/2} A^5 u \right\|^2_{L_2(R;H)} -$$

$$-2 \sin 5\varepsilon \left\| \rho^{-1/2} \frac{d^5 u}{dt^5} \right\|_{L_2(R;H)} \cdot \left\| \rho^{1/2} A^5 u \right\|_{L_2(R;H)}. \tag{11}$$

To prove inequality (11), multiply the both sides of the equality

$$P_0 u \equiv \frac{d^5 u}{dt^5} - \rho(t) A^5 u$$

scalarly the function $\rho^{-1/2}$. Then we get

$$\left\| \rho^{-1/2} \frac{d^5 u}{dt^5} - \rho^{1/2} A^5 u \right\|^2_{L_2(R;H)} = \left\| \rho^{1/2} P_0 u \right\|^2_{L_2(R;H)}.$$

Hence we have

$$\left\|\rho^{1/2}P_0u\right\|^2_{L_2(R;H)}=\left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}+$$

$$+\left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)}-2\operatorname{Re}\left(\frac{d^5u}{dt^5},A^5u\right)_{L_2(R;H)}. \qquad (12)$$

After integration by parts, we get

$$\left(\frac{d^5u}{dt^5},A^5u\right)_{L_2(R;H)}=\int_{-\infty}^{+\infty}\left(\frac{d^5u}{dt^5},A^5u\right)_H dt=$$

$$=-\int_{-\infty}^{+\infty}\left(A^{*5}u,\frac{d^5u}{dt^5}\right)_H dt=-\left(A^{*5}u,\frac{d^5u}{dt^5}\right)_{L_2(R;H)}.$$

Then

$$\left|2\operatorname{Re}\left(\frac{d^5u}{dt^5},A^5u\right)_{L_2(R;H)}\right|=\left|\left(\frac{d^5u}{dt^5},A^5u\right)_{L_2(R;H)}+\left(A^5u,\frac{d^5u}{dt^5}\right)_{L_2(R;H)}\right|=$$

$$=\left|\left(A^5u,\frac{d^5u}{dt^5}\right)_{L_2(R;H)}-\left(A^{*5}u,\frac{d^5u}{dt^5}\right)_{L_2(R;H)}\right|=$$

$$=\left|\left(\left(A^5-A^{*5}\right)u,\frac{d^5u}{dt^5}\right)_{L_2(R;H)}\right|=\left|\left(\left(E-A^{*5}A^{-5}\right)A^5u,\frac{d^5u}{dt^5}\right)_{L_2(R;H)}\right|\leq$$

$$\leq\left|\left(\left(E-A^{*5}A^{-5}\right)\rho^{1/2}A^5u,\rho^{-1/2}\frac{d^5u}{dt^5}\right)_{L_2(R;H)}\right|\leq$$

$$\leq\left\|E-(A^*A^{-1})^5\right\|\cdot\left\|\rho^{1/2}A^5u\right\|_{L_2(R;H)}\cdot\left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|_{L_2(R;H)}.$$

From the spectral expansion of the operator A it follows that

$$\left\|E-(A^*A^{-1})^5\right\|\leq\underset{\lambda\in\sigma(A)}{Sup}\left|1-\left(\frac{\overline{\lambda}}{\lambda}\right)^5\right|\leq 2\sin 5\varepsilon.$$

Then

$$2\operatorname{Re}\left(\frac{d^5u}{dt^5},A^5u\right)_{L_2(R;H)}\leq$$

$$\leq 2\sin 5\varepsilon\left\|\rho^{1/2}A^5u\right\|_{L_2(R;H)}\cdot\left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|_{L_2(R;H)}. \qquad (13)$$

Taking this inequality into account in (12), we get validity of (11). Then using (11), we get

$$\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(R;H)}\geq\left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)}+\left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|_{L_2(R;H)}-$$

$$-\left(\sin^2 5\varepsilon \left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)} + \left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}\right) =$$

$$= \cos^2 5\varepsilon \left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)}. \tag{14}$$

Using inequality (13), we get

$$\left\|A^5u\right\|^2_{L_2(R;H)} = \left\|\rho^{-1/2}(\rho^{1/2}A^5u)\right\|^2_{L_2(R;H)} \le \max \rho^{-1}(t)\left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)} \le$$

$$\le \frac{1}{\min\left(\alpha^5 \cdot \beta^5\right)} \cdot \frac{1}{\cos^2 5\varepsilon}\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(R;H)} \le$$

$$\le \frac{1}{\min\left(\alpha^{10}, \beta^{10}\right)} \cdot \frac{1}{\cos^2 5\varepsilon}\left\|P_0u\right\|^2_{L_2(R;H)}.$$

Hence we get

$$\left\|A^5u\right\|_{L_2(R;H)} \le \frac{1}{\min\left(\alpha^5; \beta^5\right)} \cdot \frac{1}{\cos 5\varepsilon}\left\|P_0u\right\|_{L_2(R;H)} = C_0(\varepsilon;\alpha;\beta)\left\|P_0u\right\|_{L_2(R;H)},$$

where $C_0(\varepsilon;\alpha;\beta) = \dfrac{1}{\min\left(\alpha^5; \beta^5\right)} \cdot \dfrac{1}{\cos 5\varepsilon}$.

From inequality (11), we easily get

$$\left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)} + \left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)} \le (1-\sin 5\varepsilon)^{-1}\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(R;H)}. \tag{15}$$

Obviously,

$$\left\|A\frac{d^4u}{dt^4}\right\|^2_{L_2(R;H)} = \int_{-\infty}^{+\infty}\left(A\frac{d^4u}{dt^4}, A\frac{d^4u}{dt^4}\right)_H dt = \int_{-\infty}^{+\infty}\left(C\frac{d^4u}{dt^4}, C\frac{d^4u}{dt^4}\right)_H dt =$$

$$= -\int_{-\infty}^{+\infty}\left(C^2\frac{d^3u}{dt^3}, \frac{d^5u}{dt^5}\right)_H dt \le \left\|C^2\frac{d^3u}{dt^3}\right\|_{L_2(R;H)} \cdot \left\|\frac{d^5u}{dt^5}\right\|_{L_2(R;H)} =$$

$$= \left\|A^2\frac{d^3u}{dt^3}\right\|_{L_2(R;H)} \cdot \left\|\frac{d^5u}{dt^5}\right\|_{L_2(R;H)}. \tag{16}$$

In the same way we get:

$$\left\|A^2\frac{d^3u}{dt^3}\right\|^2_{L_2(R;H)} \le \left\|A^4\frac{du}{dt}\right\|_{L_2(R;H)} \cdot \left\|\frac{d^5u}{dt^5}\right\|_{L_2(R;H)}, \tag{17}$$

$$\left\|A^3\frac{d^2u}{dt^2}\right\|^2_{L_2(R;H)} \le \left\|A^5u\right\|_{L_2(R;H)} \cdot \left\|A\frac{d^4u}{dt^4}\right\|_{L_2(R;H)}, \tag{18}$$

$$\left\|A^4\frac{du}{dt}\right\|^2_{L_2(R;H)} \le \left\|A^5u\right\|_{L_2(R;H)} \cdot \left\|A^3\frac{d^2u}{dt^2}\right\|_{L_2(R;H)}. \tag{19}$$

Taking into account (16)-(19) and applying inequality (15), we get the remaining required estimations in the following way:

1.

$$\left\| A^4 \frac{du}{dt} \right\|^2_{L_2(R;H)} \leq \left\| A^5 u \right\|_{L_2(R;H)} \times$$

$$\times \left\| A^3 \frac{d^2u}{dt^2} \right\|_{L_2(R;H)} \leq \left\| A^5 u \right\|^{3/2}_{L_2(R;H)} \cdot \left\| A \frac{d^4u}{dt^4} \right\|^{1/2}_{L_2(R;H)} \leq$$

$$\leq \left\| A^5 u \right\|^{3/2}_{L_2(R;H)} \left\| \frac{d^5u}{dt^5} \right\|^{1/4}_{L_2(R;H)} \cdot \left\| A^2 \frac{d^3u}{dt^3} \right\|^{1/4}_{L_2(R;H)} \leq$$

$$\leq \left\| A^5 u \right\|^{3/2}_{L_2(R;H)} \cdot \left\| \frac{d^5u}{dt^5} \right\|^{3/8}_{L_2(R;H)} \cdot \left\| A^4 \frac{du}{dt} \right\|^{1/8}_{L_2(R;H)}.$$

Hence we get that for any $\delta > 0$

$$\left\| A^4 \frac{du}{dt} \right\|^2_{L_2(R;H)} \leq \left(\left\| A^5 u \right\|^2_{L_2(R;H)} \right)^{4/5} \cdot \left(\left\| \frac{d^5u}{dt^5} \right\|^2_{L_2(R;H)} \right)^{1/5} \leq$$

$$\leq \frac{\max \rho^{1/5}(t)}{\min \rho^{4/5}(t)} \left(\left\| \rho^{1/2} A^5 u \right\|^2_{L_2(R;H)} \right)^{4/5} \times$$

$$\times \left(\left\| \rho^{-1/2} \frac{d^5u}{dt^5} \right\|^2_{L_2(R;H)} \right)^{1/5} = \frac{\max(\alpha;\beta)}{\min(\alpha^4;\beta^4)} \times$$

$$\times \left(\delta \left\| \rho^{1/2} A^5 u \right\|^2_{L_2(R;H)} \right)^{4/5} \left(\frac{1}{\delta^4} \left\| \rho^{-1/2} \frac{d^5u}{dt^5} \right\|^2_{L_2(R;H)} \right)^{1/5}.$$

Applying the Young inequality, we have

$$\left\| A^4 \frac{du}{dt} \right\|^2_{L_2(R;H)} \leq \frac{\max(\alpha;\beta)}{\min(\alpha^4;\beta^4)} \left(\frac{4\delta}{5} \left\| \rho^{1/2} A^5 u \right\|^2_{L_2(R;H)} + \frac{1}{5\delta^4} \left\| \rho^{-1/2} \frac{d^5u}{dt^5} \right\|^2_{L_2(R;H)} \right).$$

Now choose $\delta > 0$ such that $\frac{4\delta}{5} = \frac{1}{5\delta^4}$, i.e. $\delta = 4^{-1/5}$. Then taking into account inequality (15), we get

$$\left\| A^4 \frac{du}{dt} \right\|^2_{L_2(R;H)} \leq \frac{4^{4/5}}{5} \cdot \frac{\max(\alpha;\beta)}{\min(\alpha^4;\beta^4)} \cdot (1 - \sin 5\varepsilon)^{-1} \left\| \rho^{-1/2} P_0 u \right\|^2_{L_2(R;H)} \leq$$

$$\leq \frac{4^{4/5}}{5} \cdot \frac{\max(\alpha;\beta)}{\min(\alpha^9;\beta^9)} \cdot (1 - \sin 5\varepsilon)^{-1} \left\| P_0 u \right\|^2_{L_2(R;H)}.$$

Hence we get

$$\left\| A^4 \frac{du}{dt} \right\|_{L_2(R;H)} \leq \frac{16^{1/5}}{5^{1/2}} \cdot \frac{\max(\alpha^{1/2};\beta^{1/2})}{\min(\alpha^{9/2};\beta^{9/2})} \times$$

$$\times (1 - \sin 5\varepsilon)^{-1/2} \left\| P_0 u \right\|_{L_2(R;H)} = C_1(\varepsilon;\alpha;\beta) \left\| P_0 u \right\|_{L_2(R;H)},$$

where $C_1(\varepsilon;\alpha;\beta)=\dfrac{16^{1/5}}{5^{1/2}}\cdot\dfrac{\max(\alpha^{1/2};\beta^{1/2})}{\min(\alpha^{9/2};\beta^{9/2})}\cdot(1-\sin 5\varepsilon)^{-1/2}$.

2.

$$\left\|A^3\frac{d^2u}{dt^2}\right\|^2_{L_2(R;H)}\le\left\|A^4\frac{du}{dt}\right\|_{L_2(R;H)}\cdot\left\|\frac{d^5u}{dt^5}\right\|_{L_2(R;H)}\le$$

$$\le\left\|A^5u\right\|_{L_2(R;H)}\cdot\left\|\frac{d^5u}{dt^5}\right\|^{1/2}_{L_2(R;H)}\cdot\left\|A^2\frac{d^3u}{dt^3}\right\|^{1/2}_{L_2(R;H)}\le$$

$$\le\left\|A^5u\right\|_{L_2(R;H)}\cdot\left\|\frac{d^5u}{dt^5}\right\|^{3/4}_{L_2(R;H)}\cdot\left\|A^4\frac{du}{dt}\right\|^{1/4}_{L_2(R;H)}\le$$

$$\le\left\|A^5u\right\|^{9/8}_{L_2(R;H)}\cdot\left\|\frac{d^5u}{dt^5}\right\|^{3/4}_{L_2(R;H)}\cdot\left\|A^3\frac{d^2u}{dt^2}\right\|^{1/8}_{L_2(R;H)}.$$

Hence we have

$$\left\|A^3\frac{d^2u}{dt^2}\right\|^8_{L_2(R;H)}\le\left(\left\|A^5u\right\|^2_{L_2(R;H)}\right)^{3/5}\left(\left\|\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}\right)^{2/5}\le$$

$$\le\frac{\max(\alpha^2;\beta^2)}{\min(\alpha^3;\beta^3)}\cdot\left(\left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)}\right)^{3/5}\left(\left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}\right)^{2/5}.$$

Then applying the Young inequality, for $\delta>0$ we get

$$\left\|A^3\frac{d^2u}{dt^2}\right\|^2_{L_2(R;H)}\le\frac{\max(\alpha^2;\beta^2)}{\min(\alpha^3;\beta^3)}\times$$

$$\times\left(\frac{1}{\delta^2}\left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)}\right)^{3/5}\left(\delta^3\left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}\right)^{2/5}\le$$

$$\le\frac{\max(\alpha^2;\beta^2)}{\min(\alpha^3;\beta^3)}\cdot\left(\frac{3}{5\delta^2}\left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)}+\frac{2\delta^3}{5}\left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}\right).$$

For $\dfrac{3}{5\delta^2}=\dfrac{2\delta^3}{5}$, i.e. $\delta=\dfrac{3^{1/5}}{2^{1/5}}$ allowing for (15) we get

$$\left\|A^3\frac{d^2u}{dt^2}\right\|_{L_2(R;H)}\le\frac{2^{1/5}3^{3/10}}{5^{1/2}}\cdot\frac{\max(\alpha;\beta)}{\min(\alpha^4;\beta^4)}\times$$

$$\times(1-\sin 5\varepsilon)^{-1/2}\left\|P_0u\right\|_{L_2(R;H)}=C_2(\varepsilon;\alpha;\beta)\left\|P_0u\right\|_{L_2(R;H)},$$

where $C_2(\varepsilon;\alpha;\beta)=\dfrac{2^{1/5}\cdot 3^{3/10}}{5^{1/2}}\cdot\dfrac{\max(\alpha;\beta)}{\min(\alpha^4;\beta^4)}\cdot(1-\sin 5\varepsilon)^{-1/2}$

3.

$$\left\|A^2\frac{d^3u}{dt^3}\right\|^2_{L_2(R;H)}\le\left\|A^4\frac{dt}{dt}\right\|_{L_2(R;H)}\cdot\left\|\frac{d^5u}{dt^5}\right\|_{L_2(R;H)}\le$$

$$\leq \left\|A^5u\right\|^{1/2}_{L_2(R;H)}\cdot\left\|A^3\frac{d^2u}{dt^2}\right\|^{1/2}_{L_2(R;H)}\cdot\left\|\frac{d^5u}{dt^5}\right\|_{L_2(R;H)}\leq$$

$$\leq \left\|A^5u\right\|^{3/4}_{L_2(R;H)}\cdot\left\|A\frac{d^4u}{dt^4}\right\|^{1/4}_{L_2(R;H)}\cdot\left\|\frac{d^5u}{dt^5}\right\|_{L_2(R;H)}\leq$$

$$\leq \left\|A^5u\right\|^{3/4}_{L_2(R;H)}\cdot\left\|\frac{d^5u}{dt^5}\right\|^{9/8}_{L_2(R;H)}\cdot\left\|A^2\frac{d^3u}{dt^3}\right\|^{1/8}_{L_2(R;H)}.$$

Applying the Young inequality, we have that for any $\delta>0$

$$\left\|A^2\frac{d^3u}{dt^3}\right\|^2_{L_2(R;H)}\leq\left(\left\|A^5u\right\|^2_{L_2(R;H)}\right)^{2/5}\left(\left\|\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}\right)^{3/5}\leq$$

$$\leq\frac{\max(\alpha^3;\beta^3)}{\min(\alpha^2;\beta^2)}\cdot\left(\frac{1}{\delta^3}\left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)}\right)^{2/5}\left(\delta^2\left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}\right)^{3/5}\leq$$

$$\leq\frac{\max(\alpha^3;\beta^3)}{\min(\alpha^2;\beta^2)}\cdot\left(\frac{1}{5\delta^3}\left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)}+\frac{3\delta^2}{5}\left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}\right).$$

For $\frac{3}{5\delta^3}=\frac{3\delta^2}{5}$, i.e. $\delta=\frac{2^{1/5}}{3^{1/5}}$, allowing for (15) we get

$$\left\|A^2\frac{d^3u}{dt^3}\right\|_{L_2(R;H)}\leq\frac{2^{1/5}3^{3/10}}{5^{1/2}}\cdot\frac{\max(\alpha^{3/2};\beta^{3/2})}{\min(\alpha^{7/2};\beta^{7/2})}\cdot(1-\sin 5\varepsilon)^{-1/2}\left\|P_0u\right\|_{L_2(R;H)},$$

where $C_3(\varepsilon;\alpha;\beta)=\frac{2^{1/5}\cdot 3^{3/10}}{5^{1/2}}\cdot\frac{\max(\alpha^{3/2};\beta^{3/2})}{\min(\alpha^{7/2};\beta^{7/2})}\cdot(1-\sin 5\varepsilon)^{-1/2}$.

4. Estimate the norm $A\frac{d^4u}{dt^4}$.

$$\left\|A\frac{d^4u}{dt^4}\right\|^2_{L_2(R;H)}\leq\left\|A^2\frac{d^3u}{dt^3}\right\|_{L_2(R;H)}\cdot\left\|\frac{d^5u}{dt^5}\right\|_{L_2(R;H)}\leq$$

$$\leq\left\|A^4\frac{du}{dt}\right\|^{1/2}_{L_2(R;H)}\cdot\left\|\frac{d^5u}{dt^5}\right\|^{3/2}_{L_2(R;H)}\leq\left\|A^5u\right\|^{1/4}_{L_2(R;H)}\cdot\left\|\frac{d^5u}{dt^5}\right\|^{3/2}_{L_2(R;H)}\times$$

$$\times\left\|A^3\frac{d^2u}{dt^2}\right\|^{1/4}_{L_2(R;H)}\leq\left\|A^5u\right\|^{3/8}_{L_2(R;H)}\cdot\left\|\frac{d^5u}{dt^5}\right\|^{3/2}_{L_2(R;H)}\cdot\left\|A\frac{d^4u}{dt^4}\right\|^{1/8}_{L_2(R;H)}.$$

Hence we have

$$\left\|A\frac{d^4u}{dt^4}\right\|^2_{L_2(R;H)}\leq\left(\left\|A^5u\right\|^2_{L_2(R;H)}\right)^{1/5}\cdot\left(\left\|\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}\right)^{4/5}.$$

For $\delta>0$, applying the Young ineqiality,we get:

$$\left\|A\frac{d^4u}{dt^4}\right\|^2_{L_2(R;H)}\leq\frac{\max(\alpha^4;\beta^4)}{\min(\alpha;\beta)}\times$$

$$\times\left(\frac{1}{\delta^4}\left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)}\right)^{1/5}\cdot\left(\delta\left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}\right)^{4/5}\leq$$

$$\leq\frac{\max(\alpha^4;\beta^4)}{\min(\alpha;\beta)}\cdot\left(\frac{1}{5\delta^4}\left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)}+\frac{4\delta}{5}\left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}\right).$$

For $\frac{1}{5\delta^4}=\frac{4\delta}{5}$, i.e. $\delta=4^{-1/5}$, allowing for (15) we get

$$\left\|A\frac{d^4u}{dt^4}\right\|^2_{L_2(R;H)}\leq\frac{4^{4/5}}{5}\cdot\frac{\max(\alpha^4;\beta^4)}{\min(\alpha;\beta)}\times$$

$$\times\left(\left\|\rho^{1/2}A^5u\right\|^2_{L_2(R;H)}+\left\|\rho^{-1/2}\frac{d^5u}{dt^5}\right\|^2_{L_2(R;H)}\right)\leq$$

$$\leq\frac{16^{2/5}}{5}\cdot\frac{\max(\alpha^4;\beta^4)}{\min(\alpha;\beta)}(1-\sin 5\varepsilon)^{-1}\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(R;H)}.$$

Hence we have

$$\left\|A\frac{d^4u}{dt^4}\right\|_{L_2(R;H)}\leq\frac{16^{1/5}}{5^{1/2}}\cdot\frac{\max(\alpha^2;\beta^2)}{\min(\alpha^3;\beta^3)}\times$$

$$\times(1-\sin 5\varepsilon)^{-1/2}\left\|P_0u\right\|_{L_2(R;H)}=C_4\left(\varepsilon;\alpha;\beta\right)\left\|P_0u\right\|_{L_2(R;H)},$$

here $C_4\left(\varepsilon;\alpha;\beta\right)=\frac{16^{1/5}}{5^{1/2}}\cdot\frac{\max(\alpha^2;\beta^2)}{\min(\alpha^3;\beta^3)}\cdot(1-\sin 5\varepsilon)^{-1/2}$.

The theorem is completely proved.

Now we can prove the main theorem on regular solvability of equation (1).

Theorem 3. *Let conditions 1), 2) be fulfilled, and the following inequality hold:*

$$K\left(\varepsilon;\alpha;\beta\right)=\sum_{j=0}^{4}C_j\left(\varepsilon;\alpha;\beta\right)\left\|B_{5-j}\right\|<1,$$

where the constant numbers $C_j\left(\varepsilon;\alpha;\beta\right)$ $(j=\overline{0,4})$ are determined from theorem 2. Then equation (1) is regularly solvable.

Proof. After substitution of $P_0u=v$ we can write equation (1), i.e.

$$Pu=P_0u+P_1u$$

in the form $v+P_1P_0^{-1}v=f$ or in the form $(E+P_1P_0^{-1})v=f$, where $v\in L_2(R;H)$, $f\in L_2(R;H)$.

On the other hand, for any $v\in L_2(R;H)$ we have:

$$\left\|P_1P_0^{-1}v\right\|_{L_2(R;H)}=\left\|P_1u\right\|_{L_2(R;H)}=$$

$$=\left\|\sum_{j=0}^{4}A_{5-j}u^{(j)}\right\|_{L_2(R;H)}\leq\sum_{j=0}^{4}\left\|A_{5-j}u^{(j)}\right\|_{L_2(R;H)}=$$

$$= \sum_{j=0}^{4} \left\| A_{5-j} A^{-(5-j)} A^{5-j} u^{(j)} \right\|_{L_2(R;H)} \leq$$

$$\leq \sum_{j=0}^{4} \left\| A_{5-j} A^{-(5-j)} \right\| \cdot \left\| A^{5-j} u^{(j)} \right\|_{L_2(R;H)} \leq$$

$$\leq \sum_{j=0}^{4} \| B_{5-j} \| \, C_j(\varepsilon;\alpha;\beta) \, \| P_0 u \|_{L_2(R;H)} = K(\varepsilon;\alpha;\beta) \, \| v \|_{L_2(R;H)} ,$$

where the constant numbers $C_j(\varepsilon;\alpha;\beta)$ $(j = \overline{0,4})$ are determined from theorem 2.

Since $K(\varepsilon;\alpha;\beta) < 1$, the operator $E + P_1 P_0^{-1}$ is invertible in $L_2(R;H)$, therefore $v = P_0 u = (E + P_1 P_0^{-1})^{-1} f$, i.e. $u = P_0^{-1}(E + P_1 P_0^{-1})^{-1} f$.

From the last one we get

$$\| u \|_{W_2^5(R;H)} \leq const \, \| f \|_{L_2(R;H)} .$$

The theorem is proved.

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Abulfaz M.Mamedov
Nakhchivan State University,
University campus, AZ 7000, Nakhchivan, Azerbaijan
Tel.: (99412) 539 47 20 (off.).

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