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ON SPECTRAL PROPERTIES OF A DIFFERENTIAL BUNDLE ON THE AXIS

Abstract

Spectral proprieties for a fourth order polynomial differential operator equation with four-fold charactreistical root of the principal characteristical polynomial on the axis are studied . A formula of spectral expansion of rather smooth finite functions in principal functions of the continuous spectrum and in eigen functions responding to eigen values of the bundle are studied.

Consider a bundle L_λ generated in space $L_2(-\infty;\infty)$ by the differential expression

$$l_\lambda \equiv y^{IV} - 4i\lambda y''' + \left(4i\lambda^3 + P_{30}(x)\right) y' + \left(\lambda^4 + P_{41}(x)\lambda + P_{40}(x)\right) y =$$

$$= l_\lambda^0(y) + P_{30}(x) y' + \left(P_{41}(x)\lambda + P_{40}(x)\right) y, \ -\infty < x < \infty \tag{1}$$

where λ is a spectral parameter, $P_{30}(x), P_{41}(x), P_{40}(x)$ are complex- valued functions, $P_{30}(x) = P_{40}(x)\, i$, and the following conditions are fulfilled:

$$\int_{-\infty}^{\infty} |x|^4 \left| P_{ks}^{(j)}(x) \right| dx < \infty, \quad \overline{j = 0, k+s}.$$

The basic spectral properties of bundles with such a differential expression on a finite interval were studied in detail in [1,2]. But spectral aspects for singular differential bundles with four-fold characteristical root of the principal characteristical polynominal in Brikhoff sense were not considered. Here, enlisting double asymptotics of fundamental systems of the solution of the equation $l_\lambda(y) = 0$, we rescarch spectral properties of the bundle, and derive a formula for expanding rather smooth finite functions in eigen functions of continuous spectrum.

Fundamental systems of solutions $l_\lambda(y) = 0$, at each of half- planes $\pm Jm\lambda \geq 0$ as $|\lambda| \to \infty$ are easily obtained form formula (10) of the paper [2], for them and their derivatives the following asymptotic formulas are valid

$$y_i^{(v)}(x,\lambda) = \left[g_{iv}^{(0)}(x) + \frac{1}{\lambda} g_{iv}^{(1)}(x) + \frac{1}{\lambda^2} g_{iv}^{(2)}(x) + \right.$$

$$\left. + \frac{1}{\lambda^3} g_{iv}^{(3)}(x) + \frac{E_{iv}(x,\lambda)}{\lambda^4} \right] e^{i\lambda x}, \ v = 0, 3, \tag{2}$$

where $g_{iv}^{(0)}(x), \ i = \overline{0,4}$ are fundamental systems of solutions of special homogeneous equations of fourth order, $g_{iv}^{(1)}(x), g_{iv}^{(2)}(x), g_{iv}^{(3)}(x)$ are particular solutions of inhomogeneous differential equations of fourth order constructed by the scheme of the paper [2].

Fundamental systems of solutions having asymptotics as $|x| \to \infty$ are found by using the method of [3], and these solutions have the following representations

$$y_i(x,\lambda) = x^{m-1} e^{i\lambda x} + 0(1),\ |x| \to \pm\infty,\ \pm Jm\lambda \geq 0 \tag{3}$$

It is easily established that no nontrivial linear combination of four solutions $y_i(x,\lambda),\ i = \overline{1,4}$ and $\lambda \neq 0$ and $y_i^{(0)}(x),\ i = \overline{1,4}$ for $\lambda = 0$ of the equation,$l_\lambda^{(0)}(y) = 0$, belong to the space $L_2(-\infty;\infty)$. Hence it follows that an unperturbed bundle on a real axis has no eigen values.

Let $Jm\lambda > 0$. For $f(x) \in L_2(-\infty;\infty)$ finding the solutions of the eqation $l_\lambda(y) = f$ belonging to the space $L_2(-\infty;\infty)$ in the form

$$y(x,\lambda) = \sum_{v=1}^{4} c_v(x,\lambda)\, y_v(x,\lambda) \tag{4}$$

and applying the method of variation of constants, we get that for finite functions $f(x) \in L_2(-\infty;\infty)$ it holds the equality

$$y(x,\lambda) = \int_{-\infty}^{x} \frac{\sum\limits_{k=1}^{4} (-1)^k W_{4k}(\xi,\lambda)\, y_k(x,\lambda)}{W(\xi,\lambda)} f(\xi)\, d\xi, \tag{5}$$

where $W(\xi,\lambda) = 12e^{4i\lambda}\left[1 + O\left(\frac{1}{\lambda}\right)\right]$ is a Wronskian determinant from $y_i(\xi,\lambda), i = \overline{1,4}, W_{4k}(\xi,\lambda), ik = \overline{1,4}$ is the cofactor of the elements from the 4-th row of $W(\xi,\lambda)$. Exactly in the same way, for $Jm\lambda < 0$ we have

$$y(x,\lambda) = \int_{x}^{\infty} \frac{\sum\limits_{k=1}^{4} (-1)^k y_k(x,\lambda)\, W_{4k}(\xi,\lambda)}{W(\xi,\lambda)} f(\xi)\, d\xi, \tag{6}$$

Relations (5),(6) determine some bounded operator in $L_2(-\infty;\infty)$ with the kernels $R^+(x,\xi,\lambda),\ R^-(x,\xi,\lambda)$ for $Jm\lambda > 0$ and $Jm\lambda < 0$, respectively,

$$R^-(x,\xi,\lambda) = \left\{ \left| \begin{array}{c} \frac{\sum\limits_{k=1}^{4} (-1)^k y_k(x,\lambda) W_{4k}(\xi,\lambda)}{W(\xi,\lambda)} \xi \leq x \\ 0, \xi > x \end{array} \right| \right.$$

$$R^+(x,\xi,\lambda) = \left\{ \left| \begin{array}{c} \frac{\sum\limits_{k=1}^{4} (-1)^k y_k(x,\lambda) W_{4k}(\xi,\lambda)}{W(\xi,\lambda)} \xi \geq x \\ 0, \xi > x \end{array} \right| \right. \tag{7}$$

The kenels $R^{\pm}(x,\xi,\lambda)$at the points λ at which $W(\xi,\lambda)$ vanishes, may be regularized, i.e.$R^{\pm}(x,\xi,\lambda)$ will not have poles at the zeros of the denominator.

Theorem 1. *All the numbers λ of which $Jm\lambda \neq 0$ belong to the resolvent set of the unperturbed bundle L_λ^0. The resolvent $R_\lambda^0 = L_\lambda^{-1}$ is a bounded integral operator in $L_2(-\infty;\infty)$ with the kernel*

$$R(x,\xi,\lambda) = \begin{cases} R^+(x,\xi,\lambda), & Jm\lambda > 0 \\ \\ R^-(x,\xi,\lambda), & Jm\lambda < 0 \end{cases}, \tag{8}$$

s*atisfying the Carleman type conditions for which the estimation $R(x,\xi,\lambda) = 0(1)$ uniform with respect to x,ξ at each finite interval is valid for rather large $|\lambda|$.*

Proof. The validity of the following two equalities are directly verified

$$L_{\lambda_0}^0 R_{\lambda_0} f(x) = f(x), \forall f(x) \in L_2(-\infty;\infty),$$

$$Jm\lambda_0 \neq 0; R_{\lambda_0} L_{\lambda_0}^0 f(x) = f(x), \forall f(x) \in D(L_\lambda).$$

It follows from the first equality that the range of values of the operator L_{λ_0} coincides with all the space $L_2(-\infty;\infty)$ and from the second one it follows that $\ker L_{\lambda_0} = 0$.This means that the bundle L_{λ_0} has no eigen functions. Consequently, the operator L_{λ_0} has an inverse determined on the space, $L_2(-\infty;\infty)$ and $\{L_{\lambda_0}\cdot\}^{-1} = R$

Using the formula $y_i(x,\lambda_0), ik = \overline{1,4}$, it is easy to show that in fact the kernel $R(x,\xi,\lambda_0)$ defines a bounded integral operator in the space $L_2(-\infty;\infty)$ Fulfilment of the kernel $R(x,\xi,\lambda)$ the Carleman type condition is verified in ordinary way. The theorem is proved.

Theorem 2. *At each sector $\pi_\pm : Jm\lambda \geq 0$ the bundle L_λ^0 has a pure continuous spectrum. All the points λ of the real axis are the points of the continuous spectrum.*

Proof. For definiteness let $Jm\lambda \geq 0$. Considering a rectangle of a complex λ-plane defined by the inequalities $Q = \{\alpha \leq \operatorname{Re}\lambda \leq \beta,\ 0 \leq Jm\lambda \leq \beta\}$ by means of the operator

$$Af(x) = ae^{-\alpha\beta z} \int_{-\infty}^{x} e^{-\alpha\beta y} f(y)\, dy,\ \alpha >, \beta > 0$$

defining in the space $L_2(-\infty;\infty)$ a linear bounded operator, and it is valid the inequality $\|A\| \leq \frac{1}{\beta}$ following from general theorems for integral operators (see [4, p.61] theorem 2.10.3]) and from the estimation $|R^+(x,\xi,\lambda)|$, it is easy to show that it holds the estimation of the norm of the resolvent $\|R_\lambda\| \geq \frac{C}{q}$ for $\lambda \in Q$, where C is a constant, $q = Jm\lambda > 0$.

From this estimation it follows that by approaching λ to the point $\lambda_0 \in I = (-\infty,\infty)$ the norm of the resolvent unboundedly increases. Consequently, such a point should belong to the spectrum of the bundle L_λ^0. Show that at $\lambda_0 \in I$ the range of the bundle L_λ^0 is dense in $L_2(-\infty;\infty)$. Assume the contrary, then there exists the function not equal to zero and such that $\varphi \in L_2(-\infty;\infty)$ i.e. $\left(y, L_\lambda^{0*}, \varphi\right) = 0$

for $\left(y, L_\lambda^0, \varphi\right) = 0$ ally $y(x, y) = D\left(L_\lambda^0\right)$. Hence it follows that for $\lambda \in I, \varphi$ is a non-trivial solution of the adjoint equation $L_\lambda^{0*}\varphi = 0$ for $\lambda = \overline{\lambda}$, but then λ will be an eigen value of the bundle L_λ^0 that is impossible. The theorem is proved.

Theorem 3. *For finite functions continuously differentiable to 8-th order in the vicinity of* $\pm\infty$ *it holds an expansion uniformly convergent for all* $x \in (-\infty, \infty)$ *in principal functions of the continuous spectrum of the bundle* L_λ^0*:*

$$f(x) = \frac{1}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} \lambda^3 \left[R_{\lambda+i0}^0 - R_{\lambda-i0}^0\right] f d\lambda, \tag{9}$$

where

$$\left[R_{\lambda+i0}^0 - R_{\lambda-i0}^0\right] f = \int_{-\infty}^{\infty} \left[R^+(x, \xi, \lambda + i0) - R^-(x, \xi, \lambda - i0)\right] f(\xi)\, d\xi.$$

Proof. Having taken the circle $\Gamma_N = \{\lambda : |\lambda| = N\}$ of large radius N centered at the orgin, assuming $\Gamma_N = \Gamma'_N + \Gamma''_N$, where Γ'_N is a closed contour in the half-plane π_+ generated by the semi - circle Γ_N and the straight line $Jm\lambda = +\varepsilon$, and Γ''_N is a closed contour generated by the semi - cricle Γ_N in the half-plane π and the straight line $Jm\lambda = -\varepsilon$, from the equality $l_\lambda^0 \left(R^\pm (x, \xi, \lambda)\right)_\varepsilon = \delta(x - \xi)$ (where $\delta(x - \xi)$ is Dirac's function) expressing $R^\pm(x, \xi, \lambda)$ by the remaining addends, taking into account that the integral

$R_\lambda f(x) = = \int_{-\infty}^{\infty} R(x, \xi, \lambda) f\xi d\xi$ is integrable by parts, then taking from $y = (x, \lambda)$ the integral along Γ'_N and Γ''_N ,summing the contours taken along the sections, compressing to the real axis, then passing to limit as $N \to \infty$, we arrive at the representation $f(x) = \frac{1}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} \lambda^3 \left[R_{\lambda+i0} - R_{\lambda-i0}\right] f d\lambda$.

The theorem is proved.

The both $l_\lambda y = 0$ perturbed and $l_\lambda^0 y = 0$ unperturbed equations have square integrable four solutions in $L_2(-\infty; \infty)$ and $L_2(0; \infty)$. The main factors of the expressions in the exponent's index and principal terms of expansion in perimeter of the coefficient at the, exponent of these solutions are equal, the principal terms of the asymptotics as $x \to \pm\infty$ are also equal. Therefore such theorems 3,4 from the paper [3] hold also at the considered cases with appropriate changes in formulations.

Theorem 4. *All the numbers* λ *of which* $\operatorname{Im}\lambda \neq 0$, $\lambda \neq 0$, $W(\xi, \lambda) \neq 0$ *belong to the resolvent set of the bundle* L_λ*. The resolvent* $R_\lambda = L_\lambda^{-1}$ *is a bounded integral operator in* $L_2(-\infty; \infty)$ *with the kernel,* $K(x, \xi, \lambda) = \begin{cases} K^+(x, \xi, \lambda), \ \operatorname{Im}\lambda > 0, \\ K^-(x, \xi, \lambda), \ \operatorname{Im}\lambda < 0, \end{cases}$

The kernel satisfies the Carleman type conditions for which for rather large $|\lambda|$ *the estimation* $K(x, \xi, \lambda) = O(1)$ *uniform with respect to* x, ξ *at each finite interval is valid.*

Theorem 5. *The spectrum of the bundle L_λ consists of eigen values being the non real zeros of $W^{\pm}(\xi,\lambda)$, forming a bounded finite or denumerable set in λ plane whose limiting points may be found only on a real axis and continuous spectrum coinciding with the real axis.*

Theorem 6. *Let the bundle L_λ have a finitely many eigen values and have no spectral properties. Then any functions $f(x)$ possessing a continuos derivative up to the 8-th order inclusively and finite in the vicinity of $\pm\infty$ has a uniformly convergent for all $x \in (-\infty,+\infty)$ spectral expansion*

$$f(x) = \frac{1}{2\pi\sqrt{-1}} \int\limits_{-\infty}^{\infty} \lambda^3 \left[R^+_{\lambda+i0} - R^-_{\lambda-i0}\right] f d\lambda +$$

$$+\sum_{j=1}^{p_1} \underset{\lambda=\lambda_j}{res} \left[\lambda^3 R^+_\lambda f\right] + \sum_{j=1}^{p_2} \underset{\lambda=\lambda_j}{res} \left[\lambda^3 R^-_\lambda f\right], \tag{10}$$

where

$$\left[R^+_{\lambda+i0} - R^-_{\lambda-i0}\right] f = \int\limits_{-\infty}^{\infty} \left[K^+(x,\xi,\lambda+i0) - K^-(x,\xi,\lambda-i0)\right] f\xi d\xi$$

and is a combination of eigen functions of the continuous spectrum of the operators L_λ and L^*_λ, respectively, and p_1, p_2 are the numbers of eigen values at upper and lower half-planes.

References

[1]. Pechentsev A.S. *Boundary value problems for parameter containing differential equations with multiple roots of characteristic equation* // Differen Uran. 1984, No 2, pp.263-273 (Russian)

[2]. Namazova N.M. *On some boundary value problems for a fourth order differential equation with multiple characteristics* // Transaction of NAS of Azerbaijan, 2012, vol. XXXII, No 4, pp.77-86.

[3]. Mirzoyev S.S., Orujov E.G. Aliyev A.R. *Spectral analysis of a fourth order differential equation on the axis.*// Doklady RAN, 2012, vol. 442, No 3, pp.312-314. (Russian)

[4]. Birman M.Sh., Solomyak M.Z. *Spectral theory of self – adjoint operators in Hilbert space* , LGU Leningrad 1980, 264 p. (Russian)

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