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# HARDY-LITTLEWOOD-STEIN-WEISS INEQUALITY IN THE VARIABLE EXPONENT MORREY SPACES

**Abstract**

*We prove the boundedness of the weighted Hardy-Littlewood maximal operator and the singular integral operator on variable Morrey spaces* $\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}(\Omega)$ *over a bounded open set* $\Omega \subset \mathbb{R}^n$ *and a Hardy-Littlewood-Stein-Weiss type* $\mathcal{L}^{p(\cdot),\lambda(\cdot),|x-x_0|^\gamma}(\Omega)$ *to* $\mathcal{L}^{q(\cdot),\lambda(\cdot),|x-x_0|^\mu}(\Omega)$ *-theorem for the potential operators* $I^{\alpha(\cdot)}$ , $x_0 \in \overline{\Omega}$ *, also of variable order. In the case of constant* $\alpha$ *, the limiting case is also studied when the potential operator* $I^\alpha$ *acts into* $BMO_{|\cdot|^\gamma}$ *.*

## 1. Introduction

In the theory of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $\mathcal{L}^{p,\lambda}(\Omega)$ play an important role. Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [38]), they are defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}} := \sup_{x,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))},$$

where $0 \le \lambda < n$, $1 \le p < \infty$. Later, Morrey spaces found important applications to Navier-Stokes ([37], [57]) and Schrödinger ([40], [42], [43], [55], [56]) equations, elliptic problems with discontinuous coefficients ([9], [17]), and potential theory ([1], [2]). An exposition of the Morrey spaces can be found in the books [19] and [36].

As is known, last two decades there is an increasing interest to the study of variable exponent spaces and operators with variable parameters in such spaces, we refer for instance to the surveying papers [15], [23], [32], [51], on the progress in this field, including topics of Harmonic Analysis and Operator Theory, see also references therein.

Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ , were introduced and studied in [3] and [39] in the Euclidean setting and in [24] in the setting of metric measure spaces, in case of bounded sets. In [3] there was proved the boundedness of the maximal operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the log-condition on $p(\cdot)$ . For potential operators there was proved a Sobolev-Spanne type $\mathcal{L}^{p(\cdot),\lambda(\cdot)} \to \mathcal{L}^{q(\cdot),\lambda(\cdot)}$ –theorem, under the same log-condition on $p(\cdot)$ and $\lambda(\cdot)$ and the assumptions $\inf_{x\in\Omega} \alpha(x) > 0$ , $\sup_{x\in\Omega}[\lambda(x) + \alpha(x)p(x)] < n$. In the case of constant $\alpha$ , there was also proved a boundedness theorem in the limiting case $p(x) = \frac{n-\lambda(x)}{\alpha}$, when the potential operator $I^\alpha$ acts from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ into $BMO$ . In [39] the maximal operator and potential operators were considered in a somewhat more general space, but under more restrictive conditions on $p(x)$ . P. Hästö in [22] used his new "local-to-global" approach to extend the result of [3] on the maximal operator to the case of the whole space $\mathbb{R}^n$ .

In [24] there was proved the boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces. In [20] there was proved the boundedness of the maximal operator, singular integral operator and the potential operators in generalized variable exponent Morrey spaces $\mathcal{M}^{p(\cdot),\omega}(\Omega)$ .

We introduce the variable exponent weighted Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^{\gamma}}(\Omega)$ over an open set $\Omega \subseteq \mathbb{R}^n$ . Weighted Morrey spaces of such a kind in the case of constant $p$ and $\lambda$ were studied in [16] there was proved the boundedness of the fractional integral operator in weighted Morrey spaces $\mathcal{L}^{p,\lambda,|\cdot|^{\gamma}}$ in the general setting of metric measure spaces. Within the frameworks of the spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ , over bounded sets $\Omega \subseteq \mathbb{R}^n$ , $x_0 \in \overline{\Omega}$ we consider the weighted Hardy-Littlewood maximal operator

$$M_{\beta}f(x) = |x - x_0|^{\beta} \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} \frac{|f(y)|}{|y - x_0|^{\beta}} dy \qquad (1)$$

potential type operators

$$I^{\alpha(x)}f(x) = \int_{\Omega} |x - y|^{\alpha(x)-n} f(y)dy, \ 0 < \alpha(x) < n,$$

the fractional maximal operator

$$M^{\alpha(x)}f(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha(x)}{n}-1} \int_{\widetilde{B}(x,r)} |f(y)|dy, \ 0 \le \alpha(x) < n$$

of variable order $\alpha(x)$ and Calderon-Zygmund type singular operator

$$Tf(x) = \int_{\Omega} K(x,y)f(y)dy$$

where $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \le C|x - y|^{-n} \ \text{ for all } \ x \neq y,$$

$$|K(x,y) - K(x,z)| \le C\frac{|y - z|^{\sigma}}{|x - y|^{n+\sigma}}, \ \ \sigma > 0, \ \text{ if } \ |x - y| > 2|y - z|,$$

$$|K(x,y) - K(\xi,y)| \le C\frac{|x - \xi|^{\sigma}}{|x - y|^{n+\sigma}}, \ \ \sigma > 0, \ \text{ if } \ |x - y| > 2|x - \xi|.$$

**Notation:**
$\mathbb{R}^n$ is the $n$-dimensional Euclidean space,
$\Omega \subseteq \mathbb{R}^n$ is an open set, $\ell =$ diam $\Omega$ ;
$0 \in \overline{\Omega}$ ;
$\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$ ;
$B(x,r) = \{y \in \mathbb{R}^n : |x - y| < r\})$, $\widetilde{B}(x,r) = B(x,r) \cap \Omega$ ;
by $c$ , $C, c_1, c_2$ etc, we denote various absolute positive constants, which may have different values even in the same line.

## 2. Preliminaries on variable exponent Lebesgue and Morrey spaces

Let $p(\cdot)$ be a measurable function on $\Omega$ with values in $[1,\infty)$. An open set $\Omega$ is assumed to be bounded throughout the whole paper. We suppose that

$$1 < p_- \le p(x) \le p_+ < \infty, \tag{2}$$

where $p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) > 1, \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \infty.$

By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on $\Omega$ such that

$$I_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 :\ I_{p(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent. The Hölder inequality is valid in the form

$$\int_\Omega |f(x)||g(x)|dx \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

The space $L^{p(\cdot)}$ coincides with the space

$$\left\{f(x) : \left|\int_\Omega f(y)g(y)dy\right| < \infty \ \ for\ all\ \ g \in L^{p'(\cdot)}(\Omega)\right\} \tag{3}$$

up to the equivalence of the norms

$$\|f\|_{L^{p(\cdot)}} \sim \sup_{\|g\|_{L^{p'(\cdot)}} \le 1} \left|\int_\Omega f(y)g(y)dy\right| \tag{4}$$

see [35], Theorem 2.3 or [50], Theorem 3.5.

For the basics on variable exponent Lebesgue spaces we refer to [54], [35].

The weighted Lebesgue space $L^{p(\cdot),\omega}(\Omega)$ is defined as the set of all measurable functions for which

$$\|f\|_{L^{p(\cdot),\omega}(\Omega)} = \inf\left\{\eta > 0 :\ \int_\Omega \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \omega(x)dx \le 1\right\}.$$

**Definition 2.1.** *By $WL(\Omega)$ (weak Lipschitz) we denote the class of functions defined on $\Omega$ satisfying the log-condition*

$$|p(x) - p(y)| \le \frac{A}{-\ln|x-y|},\ \ |x-y| \le \frac{1}{2}\ \ x, y \in \overline{\Omega}, \tag{5}$$

*where $A = A(p) > 0$ does not depend on $x, y$.*

**Theorem 2.2** *([13]). Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $p \in WL(\Omega)$ satisfy condition (2). Then the maximal operator $M$ is bounded in $L^{p(\cdot)}(\Omega)$.*

**Theorem 2.3** *([30]). Let* $\Omega$ *be bounded and* $p \in WL(\Omega)$ *satisfy condition* (2), (13), $x_0 \in \overline{\Omega}$ *and let*

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{p'(x_0)}. \tag{6}$$

*Then the weighted maximal operator* $M_\beta$ *is bounded in* $L^{p(\cdot)}(\Omega)$ .

The following theorem for bounded sets $\Omega$ , but for variable $\alpha(x)$ , was proved in [51].

**Theorem 2.4.** *Let* $\Omega \subset \mathbb{R}^n$ *be bounded,* $p, \alpha \in WL(\Omega)$ *satisfy assumption* (2), $x_0 \in \overline{\Omega}$ *and the conditions*

$$\inf_{x\in\Omega} \alpha(x) > 0,\ \sup_{x\in\Omega} \alpha(x)p(x) < n, \tag{7}$$

$$\alpha(x_0)p(x_0) - n < \gamma < n(p(x_0) - 1), \tag{8}$$

$$\mu = \frac{q(x_0)\gamma}{p(x_0)}. \tag{9}$$

*Then the operator* $I^{\alpha(\cdot)}$ *is bounded from* $L^{p(\cdot),|x-x_0|^\gamma}(\Omega)$ *to* $L^{q(\cdot),|x-x_0|^\mu}(\Omega)$ *with*

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}. \tag{10}$$

Singular operators within the framework of the spaces with variable exponents were studied in [14]. From Theorem 4.8 and Remark 4.6 of [14] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded $\Omega$ , but valid for an arbitrary open set $\Omega$ under the corresponding condition in $p(x)$ at infinity.

**Theorem 2.5** *([14]). Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set and* $p \in WL(\mathbb{R}^n)$ *satisfy condition* (2). *Then the singular integral operator* $T$ *is bounded in* $L^{p(\cdot)}(\Omega)$ .

We will also make use of the estimate provided by the following lemma ( see [47], Corollary to Lemma 3.22).

**Lemma 2.6.** *Let* $\Omega$ *be a bounded domain and* $p$ *satisfy the assumption* $1 \le p_- \le p(x) \le p_+ < \infty$ *and condition* (5). *Let also* $\beta \in WL(\Omega)$ *and* $\sup_{x\in\Omega}[n + \nu(x)p(x)] < 0$ , $\sup_{x\in\Omega}[n + \nu(x)p(x) + \beta(x)] < 0$ . *Then*

$$\| |x-y|^{\nu(x)} \chi_{B(x,r)}(y) \|_{L^{p(\cdot),|\cdot|^{\beta(x)}}} \le C r^{\nu(x)+\frac{n}{p(x)}} (r + |x|)^{\frac{\beta(x)}{p(x)}},$$

$$x \in \Omega, 0 < r < \ell = diam\, \Omega, \tag{11}$$

*where* $C$ *does not depend on* $x$ *and* $r$ .

**Remark 2.7.** It may be shown that the constant $C$ in () may be estimated as $C = C_0 \ell^{n\ \frac{1}{p_-} - \frac{1}{p_+}}$ , where $C_0$ does not depend on $\Omega$.

Let $\lambda(x)$ be a measurable function on $\Omega$ with values in $[0, n]$ . The variable Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ and variable weighted Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}(\Omega)$ is defined as the set of integrable functions $f$ on $\Omega$ with the finite norms

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{-\frac{\lambda(x)}{p(x)}} \|f \chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}(\Omega)}=\sup_{x\in\Omega,\,t>0} t^{-\frac{\lambda(x)}{p(x)}}\||\cdot|^{\frac{\gamma}{p(\cdot)}}f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}.$$

**Lemma 2.8** *([3]). Let* $\Omega$ *be bounded,* $0\le\lambda\le n$ *and* $0\le\mu\le n$. *If* $p,q\in WL(\Omega)$ *satisfy condition* (2), $p(x)\le q(x)$ *and*

$$\frac{n-\lambda(x)}{p(x)}\ge\frac{n-\mu(x)}{q(x)}, \tag{12}$$

*then*

$$\mathcal{L}^{q(\cdot),\mu(\cdot)}(\Omega)\hookrightarrow\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega).$$

If $\gamma>0$, then

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}(\Omega)}=\sup_{x\in\Omega,\,t>0} t^{-\frac{\lambda(x)}{p(x)}}\||\cdot|^{\frac{\gamma}{p(\cdot)}}f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}\le$$
$$\le C\sup_{x\in\Omega,\,t>0} t^{-\frac{\lambda(x)}{p(x)}}\|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}=C\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)}$$

or $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)\hookrightarrow\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}(\Omega)$.

To make if $\gamma\ge\beta\ge 0$, $0\le\lambda\le n$ and $0\le\mu\le n$. If $p,q\in WL(\Omega)$ satisfy condition (2), (12), $p(x)\le q(x)$, then

$$\mathcal{L}^{q(\cdot),\mu(\cdot),|\cdot|^\gamma}(\Omega)\hookrightarrow\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\beta}(\Omega).$$

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x)=\sup_{r>0}|B(x,r)|^{-1}\int_{\widetilde{B}(x,r)}|f(y)-f_{\widetilde{B}(x,r)}|dy,$$

where $f_{\widetilde{B}(x,t)}(x)=|\widetilde{B}(x,t)|^{-1}\int_{\widetilde{B}(x,t)}f(y)dy$.

**Definition 2.9.** *We define the* $BMO_{|\cdot|^\beta}(\Omega)$ *space as the set of all locally integrable functions* $f$ *with finite norm*

$$\|f\|_{BMO_{|\cdot|^\beta}}=\sup_{x\in\Omega}|x|^\beta M^\sharp f(x)=\|M^\sharp f\|_{L_{\infty,|\cdot|^\beta}}.$$

The following statements are known.

**Theorem 2.10** *([3]). Let* $\Omega$ *be bounded and* $p\in WL(\Omega)$ *satisfy condition* (2) *and let a measurable function* $\lambda$ *satisfy the conditions*

$$0\le\lambda(x),\qquad \sup_{x\in\Omega}\lambda(x)<n. \tag{13}$$

*Then the maximal operator* $M$ *is bounded in* $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$.

Theorem 2.10 was extended to unbounded domains in [22].

Note that the boundedness of the maximal operator in Morrey spaces with variable $p(x)$ was studied in [24] in the more general setting of quasimetric measure spaces.

**Theorem 2.11** *([3]). Let* $\Omega$ *be bounded,* $p,\alpha,\lambda\in WL(\Omega)$ *and* $p$ *satisfy condition* (2). *Let also* $\lambda(x)\ge 0$ *and the conditions*

$$\inf_{x\in\Omega}\alpha(x)>0,\ \sup_{x\in\Omega}[\lambda(x)+\alpha(x)p(x)]<n \tag{14}$$

*hold. Then the operator* $I^{\alpha(\cdot)}$ *is bounded from* $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ *to* $\mathcal{L}^{q(\cdot),\lambda(\cdot)}(\Omega)$ *, where*

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n-\lambda(x)}. \tag{15}$$

**Theorem 2.12** *([3]). Let* $\Omega$ *be bounded and* $p,\alpha,\lambda \in WL(\Omega)$ *satisfy conditions* (2) *and the conditions*

$$\inf_{x\in\Omega} \alpha(x) > 0,\ \lambda(x) + \alpha(x)p(x) = n \tag{16}$$

*hold. Then the operator* $M^{\alpha(\cdot)}$ *is bounded from* $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ *to* $L^{\infty}(\Omega)$ *.*

**Theorem 2.13** *([3]). Let* $\Omega$ *be bounded and* $p,\lambda \in WL(\Omega)$ *satisfy conditions* (2) *and let* $0<\alpha<n$ *,* $0 \le \lambda(x)$ *,* $\sup \lambda(x) < n-\alpha$ *,*

$$p(x) = \frac{n-\lambda(x)}{\alpha}.$$

*Then the operator* $I^{\alpha}$ *is bounded from* $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ *to* $BMO(\Omega)$ *.*

## 3. The weighted maximal operator in the spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$

**Theorem 3.1** *[30]. Let* $\Omega$ *be bounded and* $p,\lambda \in WL(\Omega)$ *satisfy condition* (2), (13), $x_0 \in \Omega$ *. If* $0 \le \beta < \frac{n}{p'(x_0)}$ *Then*

$$\left(M^{\beta}f(y)\right)^{p(y)} \le C\left(M[|f(y)|^{p(y)}]+1\right) \tag{17}$$

*for all* $f \in \mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ *such that* $\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}} \le 1$ *, where* $C = C(p,\beta)$ *is a constant not depending on* $x$ *and* $x_0$ *.*

**Theorem 3.2** *Let* $\Omega$ *be bounded and* $p,\lambda \in WL(\Omega)$ *satisfy condition* (2), (13)*. The weighted maximal operator* $M^{\beta}$ *with* $x_0 \in \Omega$ *is bounded in* $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ *if and only if*

$$-\frac{n-\lambda(x_0)}{p(x_0)} < \beta < \frac{n}{p'(x_0)}. \tag{18}$$

**Proof** *Suffciency.* We have to show that

$$I_{p(\cdot),\lambda(\cdot)}(M^{\beta}f) \le C, \text{ for all } f \text{ with } \|f\|_{p(\cdot),\lambda(\cdot)} \le c,$$

where $c>0$ and $C=C(c)$ does not depend on $f$ .

**1. The case** $0 \le \beta < \frac{n}{p'(x_0)}$ **.** We continue the function $f$ by zero beyond the set $\Omega$ whenever necessary, and obtain

$$\int_{\widetilde{B}(x,t)} \left(M^{\beta}f(y)\right)^{p(y)} dy = \int_{\mathbb{R}^n} \left(\left(M^{\beta}f(y)\right)^{\frac{p(y)}{\theta}}\right)^{\theta} \chi_{\widetilde{B}(x,t)}(y)dy,$$

where $1<\theta<p^-$ .

The following estimate is known

$$\left(M^{\beta}f(y)\right)^{\frac{p(y)}{\theta}} \le C\left(M[|f(y)|^{\frac{p(y)}{\theta}}]+1\right).$$

Then the pointwise estimate (17) is applicable, which yields

$$\int_{\widetilde{B}(x,t)} \left(M^\beta f(y)\right)^{p(y)} dy \le C \left( \int_\Omega \left(M|f(y)|^{\frac{p(y)}{\theta}}\right)^\theta \chi_{\widetilde{B}(x,t)}(y)dy + \int_\Omega \chi_{\widetilde{B}(x,t)}(y)dy \right).$$

By the Fefferman-Stein inequality

$$\int\limits_{\mathbb{R}^n} (Mf)^p g dy \le \int\limits_{\mathbb{R}^n} f^p Mg dy, \quad p = const,$$

for all non-negative functions $f, g \in L^p$ , $1 < p < \infty$ (see [18]), we get

$$\int_{\widetilde{B}(x,t)} (Mf(y))^{p(y)} dy \le C \left( \int_\Omega |f(y)|^{p(y)} M\chi_{\widetilde{B}(x,t)}(y)dy + t^n \right).$$

As is known (see, [5], Lemma 2, p. 160), for all $r > 0$ and $x, y \in \mathbb{R}^n$

$$\frac{t^n}{|x-y|+t} \le M\chi_{\widetilde{B}(x,t)}(y) \le \frac{4^n t^n}{|x-y|+t}.$$

Therefore, we have the following inequalities

$$\int_{\widetilde{B}(x,t)} \left(M^\beta f(y)\right)^{p(y)} dy \le C \left( \int_\Omega \left(|f(y)|^{\frac{p(y)}{\theta}}\right)^\theta M\chi_{\widetilde{B}(x,t)}(y)dy + t^n \right) \le$$

$$\le C \left( t^{\lambda(x)} + \sum_{j=1}^\infty 2^{-jn} (2^j t)^{\lambda(x)} + t^n \right) \le C \left( ct^{\lambda(x)} + t^n \right) \le Ct^{\lambda(x)}.$$

2. **The case** $-\frac{n-\lambda(x)}{p(x)} < \beta < 0$ **.** Estimate (17) with $\beta = 0$ says that

$$(M\varphi(y))^{\frac{p(y)}{p^-}} \le C \left( M|\varphi(y)|^{\frac{p(y)}{p^-}} + 1 \right) \tag{19}$$

for all $\varphi \in \mathcal{L}^{s(\cdot),\lambda(\cdot)}(\Omega)$ with $\|\varphi\|_{s(\cdot),\lambda(\cdot)} \le 1$ . For $\varphi(x) = \frac{f(x)}{|x-x_0|^\beta}$ , $s(x) = \frac{p(x)}{p^-}$ we have

$$\|\varphi\chi_{\widetilde{B}(x,r)}\|_{s(\cdot)} = \left\| \frac{f\chi_{\widetilde{B}(x,r)}}{|x-x_0|^\beta} \right\|_{s(\cdot)} \le C\|f\chi_{\widetilde{B}(x,r)}\|_{s(\cdot)}.$$

Then, by the Lemma 2.8, we have

$$\|\varphi\|_{s(\cdot),\lambda(\cdot)} \le C\|f\|_{s(\cdot),\lambda(\cdot)} \le C\|f\|_{p(\cdot),\lambda(\cdot)}.$$

Then $\|\varphi\|_{s(\cdot),\lambda(\cdot)} \le 1$ . From (19) we now get

$$\int_{\widetilde{B}(x,t)} \left(M^\beta f(y)\right)^{p(y)} dy \le C \int_\Omega \left( |y-x_0|^{\beta s(x_0)} M \left| \frac{f(y)}{|y-x_0|^\beta} \right|^{s(y)} \right)^{p^-} \chi_{\widetilde{B}(x,t)}(y)dy \le$$

$$\le C \int_\Omega |y-x_0|^{\beta p(x_0)} \left( M \left| \frac{f(y)}{|y-x_0|^\beta} \right|^{s(y)} \right)^{p^-} \chi_{\widetilde{B}(x,t)}(y)dy + \int_{\widetilde{B}(x,t)} |y-x_0|^{\beta p(x_0)} dy.$$

By the Fefferman-Stein inequality, we have

$$\int_{\widetilde{B}(x,t)} \left(M^\beta f(y)\right)^{p(y)} dy \le C \int_\Omega \frac{|f(y)|^{p(y)}}{|y-x_0|^{\beta p(x_0)}} M(|\cdot - x_0|^{\beta p(x_0)} \chi_{\widetilde{B}(x,t)})(y) dy +$$
$$+ C t^{n+\beta p(x_0)} \le C \int_{\widetilde{B}(x,t)} \frac{|f(y)|^{p(y)}}{|y-x_0|^{\beta p(x_0)}} M(|\cdot - x_0|^{\beta p(x_0)} \chi_{\widetilde{B}(x,t)})(y) dy +$$
$$+ \int_{\mathbb{R}^n \setminus \widetilde{B}(x,t)} \frac{|f(y)|^{p(y)}}{|y-x_0|^{\beta p(x_0)}} M(|\cdot - x_0|^{\beta p(x_0)} \chi_{\widetilde{B}(x,t)})(y) dy + C t^{n+\beta p(x_0)} \le$$
$$\le C \int_{\widetilde{B}(x,t)} \frac{|f(y)|^{p(y)}}{|y-x_0|^{\beta p(x_0)}} |y-x_0|^{\beta p(x_0)} dy +$$
$$+ \int_{\mathbb{R}^n \setminus \widetilde{B}(x,t)} \frac{|f(y)|^{p(y)}}{|y-x_0|^{\beta p(x_0)}} M(|\cdot - x_0|^{\beta p(x_0)} \chi_{\widetilde{B}(x,t)})(y) dy + C t^{n+\beta p(x_0)}.$$

Now estimate $M(|\cdot - x_0|^{\beta p(x_0)} \chi_{\widetilde{B}(x,t)})(y)$ :

$$M(|\cdot - x_0|^{\beta p(x_0)} \chi_{\widetilde{B}(x,r)})(y) = \sup_{t>0} \frac{1}{|B(y,t)|} \int_{\widetilde{B}(y,t) \cap \widetilde{B}(x,r)} |z-x_0|^{\beta p(x_0)} dz,$$

where $|x-y| > r$ .

1) If $t > |x-y| > r$ . Then

$$M(|\cdot - x_0|^{\beta p(x_0)} \chi_{\widetilde{B}(x,r)})(y) \le \frac{1}{|x-y|^n} \int_{\widetilde{B}(x,r)} |z-x_0|^{\beta p(x_0)} dz \le$$
$$\le \frac{1}{(|x-y|-r)^n} \int_{\widetilde{B}(x,r)} |z-x_0|^{\beta p(x_0)} dz \le C \frac{r^{n+\beta p(x_0)}}{(|x-y|-r)^n}.$$

2) If $0 < t \le |x-y| - r$ . Then

$$M(|\cdot - x_0|^{\beta p(x_0)} \chi_{\widetilde{B}(x,r)})(y) = 0.$$

3) If $|x-y| - r < t < |x-y|$ . Then

$$M(|\cdot - x_0|^{\beta p(x_0)} \chi_{\widetilde{B}(x,r)})(y) \le \frac{1}{(|x-y|-r)^n} \int_{\widetilde{B}(x,r)} |z-x_0|^{\beta p(x_0)} dz \le C \frac{r^{n+\beta p(x_0)}}{(|x-y|-r)^n}.$$

Therefore, we have the following inequalities

$$\int_{\widetilde{B}(x,t)} \left(M^\beta f(y)\right)^{p(y)} dy \le$$
$$\le C \left( t^{\lambda(x)} + \sum_{j=1}^{\infty} \int_{\widetilde{B}(x,2^{j+1}t) \setminus \widetilde{B}(x,2^j t)} \frac{|f(y)|^{p(y)}}{|y-x_0|^{\beta p(x_0)}} \frac{r^{n+\beta p(x_0)}}{(|x-y|-r)^n} dy + t^{n+\beta p(x_0)} \right) \le$$
$$\le C \left( t^{\lambda(x)} + \sum_{j=1}^{\infty} \frac{(2^{j+1}t)^{\lambda(x)}}{(2^j)^{\beta p(x_0)}} \frac{1}{(2^j-1)^n} dy + t^{n+\beta p(x_0)} \right) \le C t^{\lambda(x)}.$$

*Necessity.* We take a function $f(y) = \chi_{\widetilde{B}(x,t)}(y)$ . Then we have

$$\int_{\widetilde{B}(x,r)} |f(y)|dy = \int_{\widetilde{B}(x,r)} |y-x_0|^{\beta p(x_0)} \chi_{\widetilde{B}(x,t)}(y)dy \leq$$

$$\leq C \int_{\widetilde{B}(x,r)} |y-x_0|^{\beta p(x_0)} dy \leq C r^{\beta p(x_0)+n} \leq C r^{\lambda(x_0)}.$$

From last inequality implies that $\beta > -\frac{n-\lambda(x)}{p(x)}$ .

We can be check that $I_{p,\lambda}(Mf(x)) \leq C$ .

1) Let now $f(x) = |x-x_0|^{-n}$ and $\beta > \frac{n}{p'(x_0)}$ .

Then easy can proved that $I_{p,\lambda}(f(x)) \leq C$ , and $Mf$ just does not exist (a.e. $Mf(x) = \infty$ ).

2) Let now $f(x) = |x-x_0|^{-n} \left(\ln \frac{1}{|x-x_0|}\right)^{\gamma}$ , $|x-x_0| \leq \frac{1}{2}$ , $-1 < \gamma < -\frac{1}{p(x_0)}$ and $\beta = \frac{n}{p'(x_0)}$ .

Then $I_{p,\lambda}(f(x)) \leq C$ , $Mf$ just does not exist.

Thus $\beta < \frac{n}{p'(x_0)}$ .

## 4. Hardy-Littlewood-Stein-Weiss type theorem in the variable exponent weighted Morrey spaces

**Theorem 4.1.** *Let* $\Omega$ *be bounded,* $x_0 \in \overline{\Omega}$ , $p, \alpha, \lambda \in WL(\Omega)$ *and* $p$ *satisfy condition* (2)*. Let also* $\lambda(x) \geq 0$ , (9),

$$0 \leq \gamma < n(p(x_0)-1) \tag{20}$$

*and the conditions* (14), (15) *hold. Then the operator* $I^{\alpha(\cdot)}$ *is bounded from* $\mathcal{L}^{p(\cdot),\lambda(\cdot),|x-x_0|^{\gamma}}(\Omega)$ *to* $\mathcal{L}^{q(\cdot),\lambda(\cdot),|x-x_0|^{\mu}}(\Omega)$ .

**Proof.** Since $\gamma \geq 0$ . Let $x_0 \in \overline{\Omega}$ , $\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|x-x_0|^{\gamma}}} \leq 1$ . For given $t > 0$ we denote

$$f_1(x) = f(x)\chi_{\widetilde{B}(x,2t)}(x), \quad f_2(x) = f(x) - f_1(x). \tag{21}$$

As always, we continue the function $f$ by zero beyond the set

$$I^{\alpha(x)}f(x) = I^{\alpha(x)}f_1(x) + I^{\alpha(x)}f_2(x).$$

The pointwise estimate

$$|I^{\alpha(x)}f_1(x)| \leq C t^{\alpha(x)} Mf(x), \tag{22}$$

with a constant $C > 0$ not depending on $f$ and $x$ is well known in the case of constant $\alpha$ and is also valid for variable $\alpha(\cdot)$ , under the condition $\inf\limits_{x\in\Omega} \alpha(x) > 0$ (see [48], formula (56)). We assume for simplicity that $x_0 = 0$ . For $I^{\alpha(x)}f_2(x)$ , by the Hölder inequality and Lemma ?? we have

$$\left|I^{\alpha(x)}f_2(x)\right| \le \int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{\alpha(x)-n}|f(y)|dy =$$

$$= \sum_{j=0}^{\infty}\int_{\widetilde{B}(x,2^{j+1}t)\setminus\widetilde{B}(x,2^jt)} |x-y|^{\alpha(x)-n}|f(y)|dy \le$$

$$\le C\sum_{j=0}^{\infty}\int_{\widetilde{B}(x,2^{j+1}t)\setminus\widetilde{B}(x,2^jt)} |x-y|^{\alpha(x)-n+\frac{\lambda(x)}{p(x)}}(2^jt)^{\frac{\lambda(x)}{p(y)}}|f(y)|dy \le$$

$$\le C\sum_{j=0}^{\infty}\left\| |x-y|^{\alpha(x)-n+\frac{\lambda(x)}{p(x)}}\right\|_{L^{p'(\cdot),|\cdot|^{\gamma/(1-p(\cdot))}}(\widetilde{B}(x,2^{j+1}t)\setminus\widetilde{B}(x,2^jt))}\times$$

$$\times\left\|(2^jt)^{\frac{\lambda(x)}{p(\cdot)}}f\right\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,2^{j+1}t)\setminus\widetilde{B}(x,2^jt))} \le$$

$$\le C\sum_{j=0}^{\infty}(2^{j+1}t)^{\alpha(x)-\frac{n-\lambda(x)}{p(x)}}(2^{j+1}t+|x|)^{-\frac{\gamma}{p(x)}} \le Ct^{\alpha(x)-\frac{n-\lambda(x)}{p(x)}}|x|^{-\frac{\gamma}{p(x)}}.$$

Then

$$\left|I^{\alpha(x)}f_2(x)\right| \le Ct^{\alpha(x)-\frac{n-\lambda(x)}{p(x)}}|x|^{-\frac{\gamma}{p(x)}}. \tag{23}$$

Thus, from (22) and (23) we have

$$\left|I^{\alpha(x)}f(x)\right| \le Ct^{\alpha(x)}Mf(x) + Ct^{\alpha(x)-\frac{n-\lambda(x)}{p(x)}}|x|^{-\frac{\gamma}{p(x)}}.$$

Minimizing with respect to $t$, at $t=(Mf(x))^{-\frac{p(x)}{n-\lambda(x)}}|x|^{-\frac{\gamma}{n-\lambda(x)}}$ we arrive at

$$\left|I^{\alpha(x)}f(x)\right| \le C|x|^{-\frac{\gamma\alpha(x)}{n-\lambda(x)}}(Mf(x))^{\frac{p(x)}{q(x)}}.$$

Since satisfy conditions $\alpha, p \in WL(\Omega)$, we see that

$$c|x|^{\gamma} \le |x|^{\mu-\frac{\gamma\alpha(x)q(x)}{n-\lambda(x)}} \le C|x|^{\gamma}.$$

Hence, by Theorem 3.2, we have

$$\int_{\widetilde{B}(x,t)}|y|^{\mu}\left|I^{\alpha(x)}f(y)\right|^{q(y)}dy \le$$

$$\le C\int_{\widetilde{B}(x,t)}|y|^{\gamma}(Mf(y))^{p(y)}dy = C\int_{\widetilde{B}(x,t)}\left(M_{\beta}\overline{f}(y)\right)^{p(y)}dy \le Ct^{\lambda(x)},$$

where $\beta=\frac{\gamma}{p(0)}$, $\overline{f}(y)=\frac{f(y)}{|y|^{\frac{\gamma}{p(y)}}}$ and $M_{\beta}$ is the weighted maximal operator (1).

Thus the proof of the theorem is completed.

**Theorem 4.2.** *Let* $\Omega$ *be bounded and* $p,\alpha,\lambda \in WL(\Omega)$, $p$ *satisfy conditions* (2) *and let* $x_0 \in \overline{\Omega}$, $\mu=\frac{\gamma}{p(x_0)}$, (6),(16). *Then the operator* $M^{\alpha(\cdot)}$ *is bounded from* $\mathcal{L}^{p(\cdot),\lambda(\cdot),|x-x_0|^{\gamma}}(\Omega)$ *to* $L^{\infty,|x-x_0|^{\mu}}(\Omega)$.

**Proof.** Let $f\in\mathcal{L}^{p(\cdot),\lambda(\cdot),|x-x_0|^\gamma}(\Omega)$ , $\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|x-x_0|^\gamma}}\leq 1$ . We assume for simplicity that $x_0=0$ . Then applying Hölder's inequality we have

$$|B(x,r)|^{\frac{\alpha(x)}{n}-1}\int_{\widetilde{B}(x,r)}|f(y)|dy=|B(x,r)|^{\frac{\alpha(x)}{n}-1}\int_{\widetilde{B}(x,r)}r^{\frac{\lambda}{p(y)}}r^{-\frac{\lambda}{p(y)}}|f(y)|dy\leq$$

$$\leq|B(x,r)|^{\frac{\alpha(x)}{n}-1}r^{\frac{\lambda}{p(x)}}\int_{\widetilde{B}(x,r)}r^{-\frac{\lambda}{p(y)}}|f(y)|dy\leq r^{\alpha(x)-n+\frac{\lambda}{p(x)}}\int_{\widetilde{B}(x,r)}r^{-\frac{\lambda}{p(y)}}|f(y)|dy\leq$$

$$\leq r^{\alpha(x)-n+\frac{\lambda}{p(x)}}\left\|\chi_{\widetilde{B}(x,r)}\right\|_{L^{p'(\cdot),|\cdot|^{\gamma/(1-p(\cdot))}}(\Omega)}\left\|r^{\frac{\lambda(x)}{p(\cdot)}}f\right\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,r))}\leq r^{\alpha(x)-\frac{n-\lambda}{p(x)}}|x|^{-\frac{\gamma}{p(x)}}.$$

We again refer to the logarithmic condition for $p(x)$ which provides the equivalence

$$|x|^{\frac{\gamma}{p(x)}}\sim|x|^{\frac{\gamma}{p(0)}}.$$

**Theorem 4.3.** *Let* $\Omega$ *be bounded and* $p,\lambda\in WL(\Omega)$ *satisfy conditions* (2) *and let* (6), $x_0\in\overline{\Omega}$ , $\mu=\frac{\gamma}{p(x_0)}$ , $0<\alpha<n$ , $0\leq\lambda(x)$ , $\sup\lambda(x)<n-\alpha$ ,

$$p(x)=\frac{n-\lambda(x)}{\alpha}.$$

*Then the operator* $I^\alpha$ *is bounded from* $\mathcal{L}^{p(\cdot),\lambda(\cdot),|x-x_0|^\gamma}(\Omega)$ *to* $BMO_{|x-x_0|^\mu}(\Omega)$ .

**Proof.** Let $f\in\mathcal{L}^{p(\cdot),\lambda(\cdot),|x-x_0|^\gamma}(\Omega)$ . In [1] was proved

$$M^\sharp(I^\alpha f)(x)\leq CM^\alpha f(x). \tag{24}$$

The proof Theorem 4.3, by the Theorem 4.2 and inequality (24).

## 5. Singular operators in the variable exponent weighted Morrey spaces

Let

$$T^*f(x)=\sup_{\varepsilon>0}|T_\varepsilon f(x)|$$

be the maximal singular operator, where $T_\varepsilon f(x)$ is the usual truncation

$$T_\varepsilon f(x)=\int_{|x-y|\geq\varepsilon}K(x,y)f(y)dy.$$

The following statements are known.

**Theorem 5.1** *[27]. Let* $\Omega$ *be bounded,* $p\in WL(\Omega)$ *and* $p$ *satisfy condition* (2) *and* (6). *Then the operators* $T$ *and* $T^*$ *are bounded in the space* $L^{p(\cdot),|\cdot|^\gamma}(\Omega)$ .

**Theorem 5.2.** *Let* $\Omega$ *be bounded,* $p,\lambda\in WL(\Omega)$ *and* $p$ *satisfy condition* (2), (13), (6). *Then the operators* $T$ *and* $T^*$ *are bounded in the space* $\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}(\Omega)$ .

**Proof.** We represent function $f$ as in (21) and have

$$\|Tf\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))}\leq\|Tf_1\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))}+\|Tf_2\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))}.$$

By the Theorem 5.1 we obtain

$$\|Tf_1\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))}\leq\|Tf_1\|_{L^{p(\cdot),|\cdot|^\gamma}(\Omega)}\leq C\|f_1\|_{L^{p(\cdot),|\cdot|^\gamma}(\Omega)},$$

so that

$$\|Tf_1\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}} \le C\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}}. \tag{25}$$

To estimate $\|Tf_2\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))}$ , we observe that

$$|Tf_2(z)| \le C \int\limits_{\Omega\backslash B(x,2t)} \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \le t$ , $|z-y| \ge 2t$ imply $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$ , and therefore

$$|Tf_2(z)| \le C \int_{\Omega\backslash \widetilde{B}(x,2t)} |x-y|^{-n}|f(y)|dy.$$

By the Hölder inequality and Lemma **??** we have

$$|Tf_2(z)| \le \int_{\Omega\backslash \widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy = \sum_{j=0}^{\infty} \int_{\widetilde{B}(x,2^{j+1}t)\backslash \widetilde{B}(x,2^jt)} |x-y|^{-n}|f(y)|dy \le$$

$$\le C\sum_{j=0}^{\infty} \int_{\widetilde{B}(x,2^{j+1}t)\backslash \widetilde{B}(x,2^jt)} |x-y|^{-n+\frac{\lambda(x)}{p(x)}} (2^jt)^{\frac{\lambda(x)}{p(y)}} |f(y)|dy \le$$

$$\le C\sum_{j=0}^{\infty} \left\| |x-y|^{-n+\frac{\lambda(x)}{p(x)}} \right\|_{L^{p'(\cdot),|\cdot|^{\gamma/(1-p(\cdot))}}(\widetilde{B}(x,2^{j+1}t)\backslash \widetilde{B}(x,2^jt))} \times$$

$$\times \left\| (2^jt)^{\frac{\lambda(x)}{p(\cdot)}} f \right\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,2^{j+1}t)\backslash \widetilde{B}(x,2^jt))} \le$$

$$\le C\,\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}} \sum_{j=0}^{\infty} (2^{j+1}t)^{-\frac{n-\lambda(x)}{p(x)}} (2^{j+1}t+|x|)^{-\frac{\gamma}{p(x)}} \le$$

$$\le Ct^{-\frac{n-\lambda(x)}{p(x)}} |x|^{-\frac{\gamma}{p(x)}} \,\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}}.$$

Hence by estimate () (with $\nu(x) \equiv 0$ ), we get

$$\|Tf_2\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))} \le Ct^{-\frac{n-\lambda(x)}{p(x)}} |x|^{-\frac{\gamma}{p(x)}} \,\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}} \,\|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot),|\cdot|^\gamma}(\Omega)} \le$$

$$\le Ct^{\frac{\lambda(x)}{p(x)}} \,\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}}. \tag{26}$$

From (25) and (26) we have

$$\|Tf\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}} \le C\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}}.$$

The boundedness of the operator $T^*$ follows from the known estimate

$$T^*f(x) \le c[M(Tf)(x) + Mf(x)],$$

from Theorem 3.2 and Theorem 5.2.

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