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CUBIC FORMULA FOR CLASS OF WEAKLY SINGULAR SURFACE INTEGRALS

Abstract

In the paper a cubic formula a class of weakly singular surface integrals is constructed.

It is known that numerous problems of physics and mechanics(see[1]) are reduced to the solution of the weakly singular equation (WSIE).

Since in a many cases it is impossible to find the exact solution of WSIE, there arises an interest grounding the collection method for such equations (see [2]-[4]). To this end at first it is necessary to construct the cubic formula for these weakly singular integrals.

Let's consider the surface integral

$$A(x) = \int_S \frac{K(x,y)}{|x-y|^n} \rho(y)\, d\sigma_y, \ x \in S, \tag{1}$$

where $S \subset R^3$ Lyauponov's surface, n is a natural number, $K(x,y)$ is a continuous function on $S \times S$ and there exists a number, $\alpha \in (0,2)$ such that for any $x, y \in S$

$$|K(x,y)| \le M |x-y|^{n-\alpha}$$

(here and in the sequel, M denotes positive constant different at various in equalities), $\rho(x)$ is a continuous function on S.

Introduce the sequence $\{h\} \subset R$ of the values of discretization parameter h tending to zero, and partition S into elementary domains $S = \bigcup_{l=1}^{N(h)} S_l^h$

(1) for any $l \in \{1,2,...,N(h)\}$ S_l^h is closed and its set internal with respect to S points $\overset{\circ}{S}{}_l^h$ is not empty, for $mes\overset{\circ}{S}{}_l^h = mes S_l^h$ and $j \in \{1,2,...,N(h)\}$, $j \neq l$, $\overset{\circ}{S}{}_l^h \cap \overset{\circ}{S}{}_j^h = \varnothing$;

(2) for any $l \in \{1,2,...,N(h)\}$ S_l^h is a connected piece of the surface S with continuous boundary;

(3) for any $l \in \{1,2,...,N(h)\}$ $diam S_l^h \le h$;

(4) for any $l \in \{1,2,...,N(h)\}$ there exists a so called support point $x_l \in S_l^h$ such that:

(4.1) $r_l(h) \sim R_l(h)$ $\left(r_l(h) \sim R_l(h) \iff C_1 \le \frac{r_l(h)}{R_l(h)} \le C_2\right.$, where C_1 and C_2 positive constants independ of h), here $r_l(h) = \min\limits_{x \in \partial S_l^h} |x - x_l|$ and $R_l(h) = \max\limits_{x \in \partial S_l^h} |x - x_l|$;

(4.2) $R_l(h) \le \frac{d}{2}$, where $d-$ is a radius of standard sphere (see[5]);

(4.3) for any $j \in \{1,2,...,N(h)\}$ $r_j(h) \sim r_l(h)$.

Obviously, $r(h) \sim R(h)$, where $R(h) = \max\limits_{l=\overline{1,N(h)}} R_l(h)$, $r(h) = \min\limits_{l=\overline{1,N(h)}} r_l(h)$.

Let $S_d(x)$ and $\Gamma_d(x)$ -be the parts of the surface S and tangential plane $\Gamma(x)$, respectivel at the point $x \in S$ included interior the sphere $B_d(x)$ of radius d centered at the point x. Furthermore, let $\tilde{y} \in \Gamma(x)$ be the projection of the point $y \in S$.

Then

$$\left|x-\widetilde{y}\right| \le |x-y| \le C_1(S)\left|x-\widetilde{y}\right| \tag{2}$$

and

$$mesS_d(x) \le C_2(S)\, mes\Gamma_d(x), \tag{3}$$

where $C_1(S)$and $C_2(S)$ are positive constants dependent only on S (if S a sphere, then $C_1(S)=\sqrt{2}$ and $C_2(S)=2$). The following lemma is valid.

Lemma (see[6]). *There exist the constants $C_0'>0$ and $C_1'>0$ independent of h which for $\forall l,j\in\{1,2,...,N(h)\}$, $j\neq l$ and $\forall y\in S_j^h$ the following inequality is valid:*

$$C_0'|y-x_l| \le |x_j-x_l| \le C_1'|y-x_l|. \tag{4}$$

Theorem . *Let $\forall l,j\in\{1,2,...,N(h)\}$, $j\neq l$ and $\forall y\in S_j^h$ the function $K(x,y)$ statisfy the condition*

$$|K(x_l,y)-K(x_l,x_j)| \le M\,|x_j-y|^a\,|x_l-y|^b, \tag{5}$$

where $0<a\le 1$, $b\ge 0$ and $a+b>n-2$. Then the expression

$$A^{N(h)}(x_l)=\sum_{\substack{j=1\\ j\neq l}}^{N(h)}\frac{K(x_l,x_j)}{|x_l-x_j|^n}\rho(x_j)\,mesS_j^h \tag{6}$$

at the points $x_l, l=\overline{1,N(h)}$, is a cubic formula for integral (1), and

$$\max_{l=\overline{1,N(h)}}\left|r^{N(h)}(x_l)\right| \le M\cdot\left[\|\rho\|_\infty (R(h))^{a+b+2-n}+\omega(\rho,R(h))\right],$$

$$if\ \alpha\le 1,\ b<n-2,$$

$$\max_{l=\overline{1,N(h)}}\left|r^{N(h)}(x_l)\right| \le M\cdot\left[\|\rho\|_\infty (R(h))^{a}\left|\ln R(h)\right|+\omega(\rho,R(h))\right],$$

$$if\ \ \alpha<1,\ b=n-2,$$

$$\max_{l=\overline{1,N(h)}}\left|r^{N(h)}(x_l)\right| \le M\left[\|\rho\|_\infty (R(h))^{a}+\omega(\rho,R(h))\right],$$

$$if\ \ \alpha<1,\ b>n-2,$$

$$\max_{l=\overline{1,N(h)}}\left|r^{N(h)}(x_l)\right| \le M\cdot\left[\|\rho\|_\infty (R(h))^{a}\left|\ln R(h)\right|+\omega(\rho,R(h))\right],$$

$$if\ \ \alpha=1,\ b=n-2,$$

$$\max_{l=\overline{1,N(h)}}\left|r^{N(h)}(x_l)\right| \le M\cdot\left[\|\rho\|_\infty \left((R(h))^{a}+R(h)\left|\ln R(h)\right|\right)+\omega(\rho,R(h))\right],$$

$$if\ \ \alpha=1,\ b>n-2,$$

$$\max_{l=\overline{1,N(h)}}\left|r^{N(h)}(x_l)\right| \le M\cdot\left[\|\rho\|_\infty \left((R(h))^{2-a}+(R(h))^{a+b+2-n}\right)+\omega(\rho,R(h))\right],$$

$$if\ \ \alpha>1,\ b<n-2,$$

$$\max_{l=\overline{1,N(h)}}\left|r^{N(h)}(x_l)\right| \le M\cdot\left[\|\rho\|_\infty \left((R(h))^{2-a}+((R(h)^{a}\left|\ln R(h)\right|\right)+\omega(\rho,R(h))\right],$$

$$if \quad \alpha > 1, \ b = n-2,$$

$$\max_{l=\overline{1,N(h)}} \left| r^{N(h)}\left(x_l\right) \right| \le M \cdot \left[\|\rho\|_\infty \left((R(h)^a + (R(h)^{2-a} \right) + \omega\left(\rho, R\left(h\right)\right) \right],$$

$$if \quad \alpha > 1, \ b > n-2,$$

where $r^{N(h)}\left(x_l\right) = A\left(x_l\right) - A^{N(h)}\left(x_l\right),$ *and* $\omega\left(\rho, R\left(h\right)\right)$ *is a continuity modulus of the function* $\rho\left(x\right).$

Proof. Obviously,

$$r^{N(h)}\left(x_l\right) = \int\limits_{S_l^h} \frac{K\left(x_l, y\right)}{\left|x_l - y\right|^n} \rho\left(y\right) d\sigma_y + \sum_{\substack{j=1 \\ j \ne l}}^{N(h)} \int\limits_{S_j^h} \frac{K\left(x_l, y\right) - K\left(x_l, x_j\right)}{\left|x_l - y\right|^n} \rho\left(y\right) d\sigma_y +$$

$$+ \sum_{\substack{j=1 \\ j \ne l}}^{N(h)} \int\limits_{S_j^h} \left(\frac{1}{\left|x_l - y\right|^n} - \frac{1}{\left|x_l - x_j\right|^n} \right) K\left(x_l, x_j\right) \rho\left(y\right) d\sigma_y +$$

$$+ \sum_{\substack{j=1 \\ j \ne l}}^{N(h)} \int\limits_{S_j^h} \frac{K\left(x_l, x_j\right)}{\left|x_l - y_j\right|^n} \cdot \left(\rho\left(y\right) - \rho\left(x_j\right)\right) d\sigma_y =$$

$$= r_1^{N(h)}\left(x_l\right) + r_2^{N(h)}\left(x_l\right) + r_3^{N(h)}\left(x_l\right) + r_4^{N(h)}\left(x_l\right).$$

Applying the formula of reduction of the surface integral to double one, we get

$$\left| r_1^{N(h)}\left(x_l\right) \right| \le M \cdot \|\rho\|_\infty \int\limits_{S_l^h} \frac{1}{\left|x_l - y\right|^\alpha} d\sigma_y \le$$

$$\le M \|\rho\|_\infty \cdot C_2\left(S\right) \int\limits_0^{2\pi} \int\limits_0^{R(h)} \frac{1}{t^{\alpha - 1}} dt d\varphi = M \cdot \|\rho\|_\infty \left(R(h)\right)^{2-\alpha}.$$

Let $y \in S_j^h$ and $j \ne l,$ taking unto account (5) we have:

$$\left| r_2^{N(h)}\left(x_l\right) \right| \le M \cdot \left(R(h)\right)^a \|\rho\|_\infty \int\limits_{S \backslash S_l^h} \frac{1}{\left|x_l - y\right|^{n-b}} d\sigma_y \le$$

$$\le M \cdot \|\rho\|_\infty \left(R(h)\right)^{a+b+2-n}, \quad if \quad b < n-2,$$

$$\left| r_2^{N(h)}\left(x_l\right) \right| \le M \cdot \|\rho\|_\infty \left(R(h)\right)^a \cdot \left|\ln R\left(h\right)\right|, \quad if \quad b = n-2$$

$$\left| r_2^{N(h)}\left(x_l\right) \right| \le M \cdot \|\rho\|_\infty \left(R(h)\right)^a, \quad if \quad b > n-2.$$

Furthermore, taking into account the inequality

$$\left| \left(\frac{1}{\left|x_l - y\right|^n} - \frac{1}{\left|x_l - x_j\right|^n} \right) \cdot K\left(x_l, x_j\right) \right| \le M \cdot \frac{\left|x_j - y\right|^n}{\left|x_l - y\right|^{1+\alpha}},$$

we have

$$\left|r_3^{N(h)}(x_l)\right| \le M \cdot R(h)\,\|\rho\|_\infty \int\limits_{S\setminus S_l^h} \frac{1}{|x_l-y|^{1+\alpha}} d\sigma_y \le M\cdot\|\rho\|_\infty R(h),\ if\ \alpha<1,$$

$$\left|r_3^{N(h)}(x_l)\right| \le M \cdot \|\rho\|_\infty R(h)\cdot|\ln R(h)|\,,if\ \alpha=1,$$

$$\left|r_3^{N(h)}(x_l)\right| \le M \cdot \|\rho\|_\infty \quad (R(h))^{2-a}\ if\ \alpha>1.$$

Obviously

$$\left|r_4^{N(h)}(x_l)\right| \le M\cdot\omega(\rho,R(h))\cdot \int\limits_{S\setminus S_l^h} \frac{1}{|x_l-y|^{\alpha}} d\sigma_y \le M\cdot\omega(\rho,R(h))\,.$$

As a result, summing up the obtained estimations for the expressions $r_1^{N(h)}(x_l)\,, r_2^{N(h)}(x_l)\,,\ \ r_3^{N(h)}(x_l)$ and $r_4^{N(h)}(x_l)$,we get the proof of the theorem.

Example 1. Let's consider the acoustic exponential of the simple layer

$$L(x)=\int\limits_S \Phi_k(x,y)\cdot\rho(y)\,d\sigma_y,\quad x\in S,$$

where $S\subset R^3-$ is Lyapunov's surface $\overrightarrow{n}(y)-$ is the external unit normal at the point $y\in S,\ \Phi_k(x,y)=\frac{\exp(ik|x-y|)}{4\pi|x-y|}-$ is a fundamental solution of the Helmholtz equation, $k-$ is a wave number, $\operatorname{Im}k\ge 0$ and $\rho(y)-$ is a continuous function on S.

Since

$$|\Phi_k(x,y)|\le\frac{1}{|x-y|},\quad \forall x,y\in S,\ x\ne y,$$

and $\forall l,j\in\{1,2,...,N(h)\}\,,\ j\ne l\ and\ \forall y\in S_j^h$

$$|K(x_l,y)-K(x_l,x_j)|=\frac{1}{4\cdot\pi}\left|e^{i\cdot k|x_l-y|}-e^{i\cdot k|x_l-x_j|}\right|\le M\cdot|x_j-y|\,,$$

then the expression

$$L^{N(h)}(x_l)=\sum_{\substack{j=1\\ j\ne l}}^{N(h)}\Phi_k(x_l,x_j)\cdot\rho(x_j)\,mesS_j^h$$

at the points $x_l, l=\overline{1,N(h)}$ is a cubic formula for the integral $L(x)$. Since $n=1,\alpha=1$ and $b=0$, then

$$\max_{l=\overline{1,N(h)}}\left|L(x_l)-L^{N(h)}(x_l)\right|\le M\cdot(R(h)\cdot|\ln R(h)|+\omega(\rho,R(h)))\,.$$

Example 2. Let's consider the acoustic potential of the double layer

$$G(x)=\int\limits_S\frac{\partial\Phi_k(x,y)}{\partial n(y)}\cdot\rho(y)\,d\sigma_y,\ x\in S,$$

where $S\subset R^3$ is Lyapunov's surface with the exponent $\beta\in(0,1]$.

It is easy to calculate

$$\frac{\partial\Phi_k(x,y)}{\partial n(y)}=-\frac{(xy,n(y))\cdot(1-i\cdot k\cdot|x-y|)\cdot e^{ik|x-y|}}{4\cdot\pi\cdot|x-y|^3},$$

so

$$\left|\frac{\partial\Phi_k(x,y)}{\partial n(y)}\right|\le\frac{M}{|x-y|^{2-\beta}},\ \forall x,y\in S,\ x\ne y.$$

Furthermore, $\forall l,j\in\{1,2,...,N(h)\},\ j\ne l\ and\ \forall y\in S_j^h$

$$|K(x_l,y)-K(x_l,x_j)|=\Big|(x_ly,n(y))\cdot(1-i\cdot k\cdot|x_l-y|)\cdot e^{ik|x_l-y|}-$$

$$-(x_lx_j,n(x_j))\cdot(1-i\cdot k\cdot|x_l-x_j|)\cdot e^{ik|x_l-x_j|}\Big|\le$$

$$\le\left|((x_ly,n(y))-(x_lx_j,n(x_j)))\cdot(1-i\cdot k\cdot|x_l-y|)\cdot e^{ik|x_l-y|}\right|+$$

$$+\left|(1-i\cdot k\cdot|x_l-y|)\cdot e^{ik|x_l-y|}-(1-i\cdot k\cdot|x_l-x_j|)\cdot e^{ik|x_l-x_j|}\right|\times$$

$$\times|(x_lx_j,n(x_j))|=|((x_jy,n(y))+(x_lx_j,n(y)-n(x_j)))\times$$

$$\times(1-i\cdot k\cdot|x_l-y|)\cdot e^{ik|x_l-y|}\Big|+$$

$$+\left|(1-i\cdot k\cdot|x_l-y|)\cdot e^{ik|x_l-y|}-(1-i\cdot k\cdot|x_l-x_j|)\cdot e^{ik|x_l-x_j|}\right|\times$$

$$\times|(x_lx_j,n(x_j))|\le M\cdot\left(|y-x_j|^{1+\beta}+|y-x_j|^{\beta}\cdot|x_l-x_j|+|y-x_j|\times\right.$$

$$\left.\times|x_l-x_j|^{1+\beta}\right)\le M\cdot|x_j-y|^{\beta}|x_l-y|.$$

The expression

$$G^{N(h)}(x_i)=\sum_{\substack{j=1\\ j\ne l}}^{N(h)}\frac{\partial\Phi_k(x_l,x_j)}{\partial n(x_j)}\rho(x_j)\,mesS_j^h$$

at the points $x_l,l=\overline{1,N(h)}$ is a cubic formula for the integral $G(x)$. Since $n=3,\alpha=2-\beta,a=b$ and $b=1$, the following estimation is valid

$$\max_{l=\overline{1,N(h)}}\left|G(x_l)-G^{N(h)}(x_l)\right|\le M\cdot\left[\|\rho\|_\infty\cdot(R(h))^\beta\,|\ln R(h)|+(\rho,R(h))\right].$$

Example 3. Lets consider a normal derivative of the acoustic potential of a simple layer:

$$T(x)=\frac{\partial}{n(x)}\left(\int_S\Phi_k(x,y)\cdot\rho(y)d\sigma_y\right),\ x\in S,$$

where $S\subset R^3$ is Lapunov's surface with the exponent β.

It is known that

$$T(x)=\int_S\frac{\partial\Phi_k(x,y)}{\partial n(x)}\rho(y)\,d\sigma_y=$$

$$= \int_S \frac{(xy, n(x)) \cdot \left((1 - i \cdot k \cdot |x-y|) \cdot e^{ik|x-y|}\right)}{4 \cdot \pi |x-y|^3} \rho(y) d\sigma_y, \ x \in S,$$

and

$$\left| \frac{\partial \Phi_k(x,y)}{\partial n(x)} \right| \leq \frac{M}{|x-y|^{2-\beta}}.$$

Since $\forall l, j \in \{1, 2, ..., N(h)\}, \ j \neq l \ and \ \forall y \in S_j^h$

$$|K(x_l, y) - K(x_l, x_j)| = \Big|(x_l y, n(x_l)) \cdot (1 - i \cdot k \cdot |x_l - y|) \cdot e^{ik|x_l - y|} -$$

$$- (x_l x_j, n(x_l)) \cdot (1 - i \cdot k \cdot |x_l - x_j|) \cdot e^{ik|x_l - x_j|}\Big| \leq$$

$$\leq \Big|(x_l y, n(x_l)) \cdot (1 - i \cdot k \cdot |x_l - y|) \cdot e^{ik|x_l - y|}\Big| +$$

$$+ \Big|(1 - i \cdot k \cdot |x_l - y|) \cdot e^{ik|x_l - y|} - (1 - i \cdot k \cdot |x_l - x_j|) \cdot e^{ik|x_l - x_j|}\Big| \times$$

$$\times |(x_l x_j, n(x_l))| = ((x_j y, n(x_l)) - n(x_j)) + (x_j y, n(x_j))) \times$$

$$\times (1 - i \cdot k \cdot |x_l - y|) \cdot e^{ik|x_l - y|}\Big| +$$

$$+ \Big|(1 - i \cdot k \cdot |x_l - y|) \cdot e^{ik|x_l - y|} - (1 - i \cdot k \cdot |x_l - x_j|) \cdot e^{ik|x_l - x_j|}\Big| \times$$

$$\times |(x_l x_j, n(x_l))| \leq M \cdot (|y - x_j| \, |x_l - x_j|^\beta + |y - x_j|^{1+\beta} + |y - x_j| \cdot |x_l - x_j|^{1+\beta}) \leq$$

$$\leq M \cdot |x_j - y| \, |x_l - y|^\beta,$$

then the expressions

$$T^{N(h)}(x_l) = \sum_{\substack{j=1 \\ j \neq l}}^{N(h)} \frac{\partial \Phi_k(x_l, x_j)}{\partial n(x_l)} \rho(x_j) \, mes S_j^h$$

at the points $x_l, l = \overline{1, N(h)}$ is a cubic formula for the integral $T(x)$ .Since $n = 3, \alpha = 2 - \beta, a = 1$ and $b = \beta$, the following estimations are valid:

$$\max_{l = \overline{1, N(h)}} \left| T(x_l) - T^{N(h)}(x_l) \right| \leq M \left[\|\rho\|_\infty (R(h))^\beta + \omega(\rho, R(h)) \right],$$

$$if \ \ 0 < \beta < 1,$$

$$\max_{l = \overline{1, N(h)}} \left| T(x_l) - T^{N(h)}(x_l) \right| \leq M \cdot [\|\rho\|_\infty (R(h)) \cdot |\ln R(h)| + \omega(\rho, R(h))],$$

$$if \ \ \beta = 1.$$

Example 4 Let's consider the surface integral

$$F(x) = \int_S \left(\frac{\partial}{\partial n(x)} \frac{\partial(\Phi_k(x,y) - \Phi_0(x,y))}{\partial n(y)} \right) \cdot \rho(y) \, d\sigma_y, \quad x \in S$$

where $S \subset R^3$ is Lyapunov's surface with the exponent β and $\Phi_0(x,y) = \frac{1}{4\cdot\pi\cdot|x-y|}$.

It is easy to show that

$$\frac{\partial}{\partial n(x)}\left(\frac{\partial(\Phi_k(x,y)-\Phi_0(x,y))}{\partial n(y)}\right) = \frac{K(x,y)}{4\cdot\pi\cdot|x-y|^5},$$

where

$$K(x,y) = (yx, n(x))(xy, n(y)) \cdot \left(\left(3 - 3\cdot i\cdot k\,|x-y| - k^2\cdot|x-y|^2\right)e^{ik|x-y|} - 3 + \right.$$

$$\left. + (n(y), n(x)) \cdot (1 - i\cdot k\,|x-y|)\cdot e^{ik|x-y|} - 1\right) \cdot |x-y|^2 .$$

Since for any $x, y \in S$

$$\left|(1 - i\cdot k\,|x-y|)\cdot e^{ik|x-y|} - 1\right| \le M\cdot|x-y|^2 , \qquad (7)$$

then

$$\left|\frac{\partial}{\partial n(x)}\left(\frac{\partial(\Phi_k(x,y)-\Phi_0(x,y))}{\partial n(y)}\right)\right| \le \frac{M}{|x-y|}.$$

Obviously, then function

$$\varphi(t) = \frac{(1 - i\cdot k\cdot t)\cdot e^{ikt} - 1}{t}$$

is continuously differentiable on the integral $[t_1, t_2]$, where, $t_1 > 0, t_2 > 0$. Therefore, taking info account inequality (7) and applying the Lagrange theorem for the function $\varphi(t)$, we get $\forall l, j \in \{1, 2, ..., N(h)\}$, $j \ne l$ and $\forall y \in S_j^h$ the following estimations is valid:

$$\left|((1 - i\cdot k\cdot|x_l - y|)\cdot e^{ik|x_l-y|} - 1) - ((1 - i\cdot k\cdot|x_l - x_j|)\cdot e^{ik|x_l-x_j|} - 1)\right| =$$

$$= ||x_l - y|\cdot\varphi(|x_l - y|) - |x_l - x_j|\cdot\varphi(|x_l - x_j|)| =$$

$$= |(|x_l - y| - |x_l - x_j|)\cdot\varphi(|x_l - y|) + |x_l - x_j|\cdot(\varphi(|x_l - y|) - \varphi(|x_l - x_j|))| \le$$

$$\le M\cdot(|y - x_j|\cdot|x_l - y| + |y - x_j|\cdot|x_l - x_j|).$$

As a result we have:

$$|K(x_l, y) - K(x_l, x_j)| = |(yx_j, n(x_l))\cdot(x_l y, n(y)) \times$$

$$\times \left(3 - 3\cdot i\cdot k\cdot|x_l - y| - k^2\cdot|x_l - y|^2\right)\cdot e^{ik|x_l-y|} - 3)\Big| +$$

$$+ |(x_j x_l, n(x_l))\cdot((x_j y, n(y)) + (x_l x_j, n(y) - n(x_j)))\times$$

$$\times \left((3 - 3\cdot i\cdot k\cdot|x_l - y| - k^2\cdot|x_l - y|^2\right)\cdot e^{ik|x_l-y|} - 3)\Big| +$$

$$+ (x_j x_l, n(x_l))\cdot(x_l x_j, n(x_j))\cdot((3\cdot((1 - i\cdot k\cdot|x_l - y|)\cdot e^{ik|x_l-y|} - 1) - 3\times$$

$$\times \left((1 - i\cdot k\cdot|x_l - x_j|)\,e^{ik|x_l-x_j|} - 1\right)) + k^2\cdot|x_l - y|^2\cdot e^{ik|x_l-y|} -$$

$$- k^2\,|x_l - x_j|^2\cdot e^{ik|x_l-x_j|})\Big| + |(n(y) - n(x_j), n(x_l))\times$$

$$\times\left((1-i\cdot k\cdot|x_l-y|\cdot e^{ik|x_l-y|}-1\right)\cdot|x_l-y|^2\Big|+$$

$$\Big|\left(((1-i\cdot k\cdot|x_l-y|\cdot e^{ik|x_l-y|}-1\right)-\left(\left(1-i\cdot k\cdot|x_l-x_j|\cdot e^{ik|x_l-x_j|}-1\right)\right)\times$$

$$\times(n(x_j),n(x_l))\,|x_l-y|^2\Big|+|(n(x_j),n(x_l))\times$$

$$\times\left(\left(1-i\cdot k\cdot|x_l-x_j|\cdot e^{ik|x_l-x_j|}-1\right)\right)\cdot\left(|x_l-y|^2-|x_l-x_j|^2\right)\Big|\le$$

$$\le M\cdot\left(|y-x_j|\cdot|x_l-y|^{3+\beta}+|y-x_j|^{\beta}\cdot|x_l-y|^4+\right.$$

$$+|y-x_j|\cdot|x_l-y|^2\cdot|x_l-x_j|+|y-x_j|\cdot|x_l-y|\times$$

$$\times|x_l-x_j|^2+|y-x_j|^{\beta}\cdot|x_l-y|^2\cdot|x_l-x_j|^{2+\beta}+|y-x_j|\times$$

$$\left.\times|x_l-x_j|^3\right)\le M\cdot|x_j-y|^{\beta}\cdot|x_l-y|^{4+\beta}.$$

So , the expression

$$F^{N(h)}(x_l)=\sum_{\substack{j=1\\ j\neq l}}^{N(h)}\frac{\partial}{\partial n(x_l)}\left(\frac{\partial(\Phi_k(x_l,x_j)-\Phi_0(x_l,x_j)}{\partial n(x_j)}\right)\rho(x_j)\,mesS_j^h$$

at the point $x_l, l=\overline{1,N(h)}$ is a cubic formula for the integral $F(x)$. Since $n=5,\ \alpha=1, a=\beta$ and $b=4-\beta$, the following estimations are valid:

$$\max_{l=\overline{1,N(h)}}\left|F(x_l)-F^{N(h)}(x_l)\right|\le M\left\lfloor\|\rho\|_\infty(R(h))^{\beta}+\omega(\rho,R(h))\right\rfloor,\quad \text{if}\ \ 0<\beta<1;$$

$$\max_{l=\overline{1,N(h)}}\left|F(x_l)-F^{N(h)}(x_l)\right|\le M\cdot\left[\|\rho\|_\infty(R(h))\cdot|\ln R(h)|+\omega(\rho,R(h))\right],$$

$$if\ \ \beta=1.$$

References

[1]. Kolton D., Kress R. *Methods of integral equation in scattering theory* M:,Mir, 1987, 311 p. (Russian)

[2]. Abdullayev F.A.,Khalilov E.H. *Groundg of the collocation method for a class of boundary integral equations* Diff.Uravn. 2004, vol.40 No 1, pp82-86 (Russian)

[3]. Mustafa N., Khalilov E.H. *The collection method for solution of boundary integral equations* . Applicable analysis, 2009, vol.88, No 12, pp.1665-1675 .

[4]. Khalilov E.H. *On approximate solution of a boundary integral equation of mixed problem for Helmholz eqation* // Proceeding of Institute of Mathematics and Mehanics of NASA , vol 31(39), Baku, 2009 , pp105-110.

[5]. Vladimirov B.S. *Equations of mathematical physics* M:, Nauka, 1976, 527 p.(Russian)

[6]. Kustov Yu.A. Musayev B.I. *Cubic formula for a two-dimensional singular integral and its application* Dp. VINITI No 4281-81 60 p. (Russian).

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