Farman I. MAMEDOV, Sakina H. ISMAILOVA,
Firana M. MAMEDOVA, Mushviq J. ALIYEV

ON SOME LOG REGULARITY CONDITIONS AND THE BOUNDEDNESS OF WEIGHTED HARDY OPERATOR IN $L^{p(\cdot)}(0,l)$

Abstract

We present two type non-coincide log-conditions on sufficiency and necessity for Hardy operator $Hf(x)=\int_0^x f(t)dt$ to be bounded in variable exponent Lebesgue space $L^{p(\cdot)}(0,l)$.

1. Introduction

Many authors put the log regularity condition on the exponent function $p(.)$ to study the boundedness of different integral operators in $L^{p(\cdot)}(0,l)$ spaces. In this paper, we focus on the explore the log-regularity conditions that ensures validity or absence of variable exponent Hardy inequality. More precisely, we present here two type close log conditions for sufficiency and necessity, respectively.

The main object of study of this paper is the following general weighted inequality for the Hardy operator $Hf(x)=\int_0^x f(t)dt$

$$\left\|\sigma(\cdot)^{\frac{1}{p(\cdot)}}W(\cdot)^{-1}Hf(\cdot)\right\|_{L^{p(\cdot)}(0,l)}\leq C\left\|\omega(\cdot)^{\frac{1}{p(\cdot)}}f(\cdot)\right\|_{L^{p(\cdot)}(0,l)};\ \ f\geq 0, \tag{1.1}$$

in the norms of the variable exponent Lebesgue space $L^{p(\cdot)}(0,l)$, where the constant $C>0$ does not depend on a measurable function f, $W(x)=\int_0^x\sigma(t)dt$; $\sigma(t)=\omega(t)^{-\frac{1}{p(t)-1}}$, $l>0$ (see, e.g. [15] for this inequality in the case of constant exponent $p(.)$).

The investigation of variable exponent Lebesgue spaces is motivated by the modeling of electrorheological fluids [23], [25] and the study of regularity properties of related partial differential equations with nonstadardt growth condition (see, [1], [21], [11], [5]). Parallel, the study of classical integral operators of harmonic analyze were elaborated in the variable exponent settings (see, the monographs [6], [8]).

The variable exponent Hardy inequality were studied in the works [24], [14], [10], [12]; subsequently in [9], [22], [17], [20], [13], [2] was proved a necessary and sufficient condition for the power weighted case. We refer to the works [7], [18], [19], [16] concerning necessary and sufficient conditions in the case of general weights and [3], [4] in the case of monotone functions f.

2. Main results and the notation

As to the basic properties of spaces $L^{p(.)}$ we refer to the monographs [6], [8]. Throughout this paper, it is assumed that $p(x)$ is a measurable function in the finite interval $(0,l)$, taking its values in the interval $[1,\infty)$ with $p^+=\sup\{p(x):x\in(0,l)\}<\infty$. The space of functions $L^{p(.)}(0,l)$ is introduced as the class of measurable functions $f(x)$ in $(0,l)$ which have a finite

$$I_{p(.)}(f)=\int_0^l|f(x)|^{p(x)}\,dx$$

modular; put also $I_{p(.);E}(f)=\int_E |f(x)|^{p(x)}dx$). A norm in $L^{p(.)}(0,l)$ is given in the form

$$\|f\|_{L^{p(.)}(0,l)}=\inf\left\{\lambda>0: I_{p(.)}\left(\frac{f}{\lambda}\right)\leq 1\right\}.$$

By $\|f\|_{p(.);E}$ we denote the $p(.)$ norm over set E.

For $1<p^-,\,p^+<\infty$ the space $L^{p(.)}(0,l)$ is a reflexive Banach space. The relation between the modular and the norm is expressed by the inequalities (see, [6], [8])

$$\|f\|^{p^+}_{L^{p(.)}(0,l)}\leq I_p(f)\leq \|f\|^{p^-}_{L^{p(.)}(0,l)},\quad \|f\|_{L^{p(.)}(0,l)}\leq 1 \tag{2.1}$$

$$\|f\|^{p^-}_{L^{p(.)}(0,l)}\leq I_p(f)\leq \|f\|^{p^+}_{L^{p(.)}(0,l)},\quad \|f\|_{L^{p(.)}(0,l)}> 1. \tag{2.2}$$

These inequalities enable us to derive our norm estimates passing to the relevant modular inequalities.

For the function $p(x)$ with $1\leq p(x)<\infty$ the $p'(x)$ denotes the conjugate function of $p(x)$, $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$ and $p'=\infty$ if $p=1$. We denote by $C,C_1,C_2,...$ various positive constants whose values may vary at each appearance. The characteristic function of the set E is denoted by χ_E.

Throughout the paper, we assume that $p:(0,l)\to[1,\infty)$ and $\omega:(0,l)\to(0,\infty)$ be a measurable function such that $\sigma=\omega^{-\frac{1}{p(x)-1}}\in L(0,a)$ for all $a\in(0,l)$ and $W(x)=\int_0^x\sigma(t)dt,\ 0<x<l$.

Following main results are obtained in this paper.

Theorem 2.1. *Let p be a measurable function on finite interval $(0,l)$ such that $1<p^-,p^+<\infty$ and it holds the condition:*

$$\limsup_{x\to 0}|p(x)-p(0)|\ln\frac{1}{W(x)}<\infty;$$

then it holds the inequality (1.1) for every positive and measurable function f.

For the $x\in(0,l)$ and $k\in\mathbb{N}$ denote by $x_{(k)}=\sup\left\{y: W(y)\leq\frac{1}{2^k}W(x)\right\}$, i.e. $W(x_{(k)})=\frac{1}{2^k}W(x)$. Also for $k\in\mathbb{N}$ and $x\in(0,\frac{l}{2^k})$ we put $W(x^{(k)})=2^kW(x)$.

In particular, we put by $x_{(1)}$ and $x^{(1)}$ the numbers such that $2W(x_{(1)})=W(x)=\frac{1}{2}W(x^{(1)})$

Theorem 2.2. *Let p be increasing on $(0,l)$; then the condition*

$$\limsup_{x\to 0}\left[p(x)-p(x_{(1)})\right]\ln\frac{1}{W(x)}<\infty \tag{2.3}$$

is necessary for the inequality (1.1) to hold.

Inserting in this assertion $\omega(x)=1$ we get the following necessary condition for the inequality

$$\left\|x^{-1}Hf(\cdot)\right\|_{L^{p(.)}(0,l)}\leq C\left\|f(\cdot)\right\|_{L^{p(.)}(0,l)};\ \ f\geq 0, \tag{2.4}$$

to hold.

Corollary 2.3. *Let p be increasing on $(0,l)$; then the condition*

$$\limsup_{x\to 0}\left[p(x)-p\left(\frac{x}{2}\right)\right]\ln\frac{1}{x}<\infty \tag{2.5}$$

is necessary for the inequality (2.4) to hold.

3. Proof of Theorems 2.1 and 2.2

Proof of Theorem 2.1. To prove Theorem 2.1 we use the following assertion that follows from [19].

Let v and ω be positive measurable functions on the finite interval $(0,l)$ such that $v(.)$, $\omega(.)^{-\frac{1}{p(.)-1}} \in L^1(0,a)$ for any $a \in (0,l)$.

Corollary 3.1. *Let the measurable functions $q(.)$ and $p(.)$ are such that $p^- > 1$, $q(0) \geq p(0)$. Let $q(.), p()$ satisfy (2.5). Then it holds the inequality*

$$\|v^{\frac{1}{q(.)}} Hf(.)\|_{L^{q(.)}(0,l)} \leq C\|\omega^{\frac{1}{p(.)}} f(.)\|_{L^{p(.)}(0,l)}; \tag{3.1}$$

iff

$$\sup_{t\in(0,l)} \left(\int_t^l v(s)ds\right)^{\frac{1}{q(0)}} \left(\int_0^t \omega^{-\frac{1}{p(s)-1}} ds\right)^{\frac{1}{p'(0)}} < \infty \tag{3.2}$$

Apply the inequality (3.1) of Corollary 3.1 in the case $q(.) = p(.)$ and $v(x) = W(x)^{-p(x)}\sigma(x)$, $\sigma(x) = \omega(x)^{-\frac{1}{p(x)-1}}$. Verify the condition (3.2) under this settings. It follows from the condition (2.5) that

$$\int_t^l W^{-p(s)}\sigma(s)ds \sim \int_t^l W^{-p(0)}dW(s) \sim W(t)^{1-p(0)}, \quad 0 < t < l.$$

Therefore, it is nothing to check the condition (3.2).

This campletes the proof of Theorem 2.1

Proof of Theorem 2.2. Consider a test function $f_0(x) = \chi_{a_{(1)},a}(x)\dfrac{\sigma(x)}{W(x)^{\frac{1}{p(x)}}}$.

Then it follows that,

$$I_{p(.)}\left(\omega^{\frac{1}{p(.)}} f_0(.)\right) = \int_0^l \left(\chi_{a_{(1)}(x),a}\frac{\sigma(x)}{W(x)^{\frac{1}{p(x)}}}\right)^{p(x)} dx =$$

$$= \int_{a_{(1)}}^a \frac{\sigma(x)^{p(x)}}{W(x)}\omega(x)dx = \int_{a_{(1)}}^a \frac{\sigma(x)}{W(x)}dx = \int_{\frac{1}{2}W(a)}^{W(a)} \frac{dW}{W} = \ln 2 \leq 1.$$

This and (2.1) imply $\|\omega(.)^{\frac{1}{p(.)}}\|_{p(.)} \leq 1$.

Insert the function f_0 into the inequality (1.1). It follows that

$$\|W(.)^{-1}\sigma(.)^{\frac{1}{p(.)}} Hf_0(.)\|_{p(.)} \leq C.$$

This implies

$$I_{p(.)}\left(W^{-1}\sigma^{\frac{1}{p}} Hf_0\right) \leq C_1.$$

is a positive measurable function such that $\|\omega(.)^{\frac{1}{p(.)}} f(.)\|_{p(.)} \leq 1$ (see, [6], [8]). Therefore,

$$C_1 \geq \int_0^l W^{-p(x)}\sigma(x)\Big(\int_0^x \chi_{a_{(1)},a}(t)\frac{\sigma(t)}{W(t)^{\frac{1}{p(t)}}}dt\Big)^{p(x)} dx \geq$$

$$\geq \int_{a^{(1)}}^{a^{(2)}} W^{-p(x)}\sigma(x)\Big(\int_{a_{(1)}}^{a} \chi_{a_{(1)},a}(t)\frac{\sigma(t)}{W(t)^{\frac{1}{p(t)}}}dt\Big)^{p(x)}dx. \tag{3.3}$$

For $t \in (a_{(1)}, a)$ by using increasing of the function $p(.)$ we have

$$W(t)^{\frac{1}{p(t)}} = \left(\frac{W(t)}{W(l)}\right)^{\frac{1}{p(t)}} W(l)^{\frac{1}{p(t)}} \leq \left(\frac{W(a)}{W(l)}\right)^{\frac{1}{p(a)}} W(l)^{\frac{1}{p(t)}} \leq$$

$$\leq W(a)^{\frac{1}{p(a)}} W(l)^{\frac{1}{p(t)}-\frac{1}{p(a)}} \leq \left[1 + W(l)^{\frac{1}{p(0)}-\frac{1}{p(l)}}\right] W(a)^{\frac{1}{p(a)}} = C_2 W(a)^{\frac{1}{p(a)}}.$$

Therefore,

$$\int_{a_{(1)}}^{a} \frac{\sigma(t)dt}{W(t)^{\frac{1}{p(t)}}} \geq \frac{1}{C_2} W(a)^{-\frac{1}{p(a)}} \left(\int_{a_{(1)}}^{a} \sigma(t)dt\right) = \frac{1}{2C_2} W(a)^{\frac{1}{p'(a)}}.$$

From the last inequality and (3.3) it follows that

$$C_1 \geq \frac{1}{2C_2}\int_{a^{(1)}}^{a^{(2)}} W(a)^{\frac{p(x)}{p'(a)}} W(x)^{-p(x)}\sigma(x)dx \geq \frac{1}{2C_2}\int_{a^{(1)}}^{a^{(2)}} W(a)^{1-\frac{p(x)}{p(a)}}\sigma(x)dx. \tag{3.4}$$

For $x \in (a^{(1)}, a^{(2)})$ we have

$$\left(\frac{1}{W(a)}\right)^{\frac{p(x)}{p(a)}-1} \geq \left(\frac{1}{W(a)}\right)^{\frac{p(a^{(1)})}{p(a)}-1}.$$

Therefore, it follows from (3.4) that

$$C_1 \geq \frac{\ln 2}{2C_2}\left(\frac{1}{W(a)}\right)^{\frac{p(a^{(1)})}{p(a)}-1},$$

i.e.

$$\left[p(a^{(1)}) - p(a)\right] \ln \frac{1}{W(x)} \leq \ln\left(2C_2C_1 \ln 2\right) p(l).$$

This implies (2.5) and thereby completes the proof of Theorem 2.2.

The first three authors were partially supported by the Science Development Foundation under the President of Azerbaijan Republic under the grant EIF-2012-2(6)-39/09/1.

The author is grateful to the Scientific and Technological Research Council of Turkey for a scholarship (TÜBİTAK-BİDEB, 2012)

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Farman I. Mamedov, Sakina H. Ismailova,
Firana M. Mamedova, Mushviq J. Aliyev
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, B.Vahabzade str., AZ 1141, Baku, Azerbaijan
E-mails: farman-m@mail.ru; g.sakina@rambler.ru;
sunny.fira@list.ru; a.mushfiq@rambler.ru
Tel.: (99412) 539 47 20 (off.).

Received June 20, 2013; Revised September 17, 2013.