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ON A SOLVABILITY CONDITION FOR A MIXED PROBLEM FOR SCHRÖDINGER EQUATION

In this paper we consider the unique classical solvability of the following mixed problem:

$$iu_t = u_{xx} + q(x)u,\ \ (t,x)\in Q = \{0<t<T,\ 0<x<1\} \qquad (1)$$

$$u(0,x) = \varphi(x),\ \ 0<x<1 \qquad (2)$$

$$U_1(u) \equiv a_1 u_x(t,0) + a_{10}u(t,0) + b_1 u_x(t,1) + b_{10}u(t,1) = 0$$

$$U_2(u) \equiv a_{20}u_x(t,0) + b_{20}u(t,1) = 0,\ \ 0<t<T \qquad (3)$$

where $u = u(t,x)$ -is the sought function, $q(x), \varphi(x) \in C[0,1]$-given complex functions, $a_1, a_{10}, a_{20}, b_1, b_{10}, b_{20}$-complex constants such that the boundary conditions *(3)* are strongly regular [1], [2], i.e.,

$$a_1 b_{20} + b_1 a_{20} \neq 0,\ \ a_1^2 \neq b_1^2,\ \ \ a_{20}^2 \neq b_{20}^2 \qquad (4)$$

1. Let's first consider the corresponding spectral problem:

$$y'' + q(x)y - \lambda^2 y = 0 \qquad (1.1)$$

$$U_i(y) = 0 \quad (i = 1,2) \qquad (1.2)$$

It's well known [2], [3] that under conditions *(4)* the problem *(1.1)*, *(1.2)* has a Green function $G(x,\xi,\lambda)$, which is analitic with respect to λ every wher, except for an encounterable set of poles $\{\lambda_\nu\}_{\nu=1}^{\infty}$. This set is divided into two series $\{\lambda_{\nu j}\}$ $(j = 1,2)$ for whose elements the following asymptotical expressions hold:

$$\lambda_{\nu j} = 2\nu\pi i + \alpha_j + O\left(\frac{1}{\nu}\right) \quad (j = 1,2; \nu \to \infty) \qquad (1.4)$$

where

$$\alpha_j = \ln_0 (a_1 b_{20} + b_1 a_{20})^{-1}\left[-(a_1 a_{20} + b_1 b_{20}) + (-1)^j \sqrt{(a_1^2 - b_1^2)(a_{20}^2 - b_{20}^2)}\right] \quad (1.5)$$

It is known as well [2] that for any function

$$f(x) \in D[0,1] = \{g(x) / g(x) \in C^2[0,1], U_i(g) = 0 \ \ (i = 1,2)\} \qquad (1.6)$$

the following expansion formula is true:

$$\sum_{\nu=1}^{\infty} \underset{\lambda_\nu}{res}\, \lambda \int_0^1 G(x,\xi,\lambda) f(\xi) d\xi = f(x) \qquad (1.7)$$

where the series is uniformly convergent with respect to $x \in [0,1]$.

For simplicity of notations, we shall suppose further that the poles of the Green function are simple.

2.Following [3], we define a linear operator $A_{\nu s}$ that maps $C[0,1]$ into $D[0,1]$ as

$$A_{\nu s}f(x) \equiv f_{\nu s}(x) \equiv \underset{\lambda_\nu}{res}\, \lambda^{2s+1} \int_0^1 G(x,\xi,\lambda) f(\xi) d\xi \qquad (2.1)$$

According to formula *(1.7)*, for any $f(x) \in D[0,1]$ we have:

$$\sum_{\nu=1}^{\infty} f_{\nu 0}(x) = f(x) \qquad (2.2)$$

Assume that the problem *(1)-(3)* has a classical solution $u(t,x)$. Then for any t the solution as a function of x belongs to the space $D[0,1]$ and therefore

$$\sum_{\nu=1}^{\infty} u_{\nu 0}(t,x) = u(t,x) \qquad (2.3)$$

On the other hand applying $A_{\nu s}$ to the equalities that we obtain upon substituting the solution into *(1),(2)*, we have

$$i\frac{du_{\nu s}(t,x)}{dt} = \underset{\lambda_\nu}{res}\, \lambda^{2s+1} \int_0^1 G(x,\xi,\lambda)\left[u_{\xi\xi}(t,\xi) + q(\xi)u(t,\xi)\right] d\xi\,,$$

$$u_{\nu s}(0,x) = \varphi_{\nu s}(x).$$

Thence using the properties of the Green function, we obtain:

$$i\frac{du_{\nu s}(t,x)}{dt} = u_{\nu s+1}(t,x), \quad u_{\nu s}(0,x) = \varphi_{\nu s}(x) \qquad (2.4)$$

But taking into account that λ_ν -is a simple pole of $G(x,\xi,\lambda)$, it's evident that

$$\underset{\lambda_\nu}{res}\left(\lambda^2 - \lambda_\nu^2\right)\lambda^{2s+1} \int_0^1 G(x,\xi,\lambda) u(t,\xi) d\xi \equiv 0,$$

thus

$$u_{\nu s+1}(t,x) = \lambda_\nu^2 u_{\nu s}(t,x).$$

Substituting the last expression into *(2.4)* we conclude that $u_{\nu s}(t,x)$ is a solution of the Cauchy problem:

$$i\frac{du_{\nu s}(t,x)}{dt} = \lambda_\nu^2 u_{\nu s}(t,s), \quad u_{\nu s}(0,x) = \varphi_{\nu s}(x) \qquad (2.5)$$

It's evident that the solution is unique and can be expressed as follows:

$$u_{\nu s}(t,x) = \underset{\lambda_\nu}{res}\, \lambda^{2s+1} e^{\ i\lambda^2 t} \int_0^1 G(x,\xi,\lambda)\varphi(\xi) d\xi \equiv 0 \qquad (2.6)$$

Substituting *(2.6)* into *(2.3)*, we get:

$$u(t,x) = \sum_{\nu} \underset{\lambda_\nu}{res}\, \lambda e^{-i\lambda^2 t} \int_0^1 G(x,\xi,\lambda)\varphi(\xi) d\xi \qquad (2.7)$$

This particularly shows that the problem *(1)-(3)* cannot have more than one classical solution.

So, we have proved the following

Lemma 2.1. *Suppoce the conditions (4) are satisfied and the problem (1)-(3) has a classical solution. Then the solution is unique and represented by formula (2.7).*

3. It's easy to see that if $\varphi(x)\in D[0,1]$ and the series *(2.7)* along with all the series obtained there from by differentiating twice with respect to x and once with respect to t are uniformly convergent, then $u(t,x)$ given by *(2.7)* is a classical solution of problem *(1)-(3)*.

It's well known [2], [3] that for other correct per I.G.Petrovsky equations, namely for wave equation and heat conduction equation, a finite set of conditions of the form

$$\left[\frac{d^2}{dx^2}+q(x)\right]^k \varphi(x)\in D[0,1] \qquad (k=0,1,\ldots,k_0) \tag{3.1}$$

ensure uniform convergence of such series, hence providing the existence of a solution.

In our case, we have to use a different approach. Since the asympthotical formula *(1.4)* shows that

$$\left|\exp\left(-i\lambda_{vj}^2 t\right)\right| = \exp\left[4\pi t v\left(\operatorname{Re}\alpha_j + O\left(\frac{1}{v}\right)\right)\right] \tag{3.2}$$

therefore, in the event at least one of the numbers $\operatorname{Re}\alpha_j \quad (j=1,2)$ is positive the uniform convergence of the series will reguire the following estimates to hold true:

$$\left|\underset{\lambda_{vj}}{res}\,\lambda^{2s+1}\frac{d^k}{dx^k}\int_0^1 G(x,\xi,\lambda)\varphi(\xi)d\xi\right| \le Ce^{-\varepsilon v} \qquad (k+2s\le 2) \tag{3.3}$$

where C and ε are positive numbers. But the last estimate ensues the applicatili by of the differential operation $\left[\dfrac{d^2}{dx^2}+q(x)\right]$ to the series in the right-hand side of the following expansion any number of times

$$\varphi(x)=\sum_v \underset{\lambda_v}{res}\,\lambda\int_0^1 G(x,\xi,\lambda)\varphi(\xi)d\xi \tag{3.4}$$

In other words

$$\left[\frac{d^2}{dx^2}+q(x)\right]^k \varphi(x)\in C[0,1] \qquad (k=0,1,2,\ldots).$$

Moreover, in this case *(3.4)* yields

$$U_i\left(\left[\frac{d^2}{dx^2}+q(x)\right]^k \varphi(x)\right)=0 \qquad (i=1,2; k=0,1,2,\ldots)$$

Thus, in this case the function $\varphi(x)$ must satisfy on infinite number of conditions similar to *(3.1)*. Thus for our purposes following inequalities are required to be true

$$\operatorname{Re}\alpha_j \le 0 \qquad (j=1,2) \tag{3.5}$$

Using the notation

$$\mu = -(a_1 b_{20} + b_1 a_{20})^{-1}(a_1 a_{20} + b_1 b_{20})$$

we can easily see from formula *(1.5)* that

$$\alpha_j = \ln_0\left[\mu + (-1)^j \sqrt{\mu^2 - 1}\right] \qquad (j = 1,2) \tag{3.6}$$

and $\alpha_2 = -\alpha_1$. Werefrom it follow that for the fulfilment of *(3.5)*. As necessary and sufficient that

$$\left|\mu - \sqrt{\mu^2 - 1}\right| = 1,$$

i.e.,

$$\left|\mu - \sqrt{\mu^2 - 1}\right| = e^{i\theta},$$

where θ-is some angle. From the last equality we obtain $\mu = \cos\theta$. Thus the enequalities *(3.5)* (that is $\mathrm{Re}\,\alpha_j = 0$) hold true if and only if

$$\mu = -(a_1 b_{20} + b_1 a_{20})^{-1}(a_1 a_{20} + b_1 b_{20}) \in [-1,1].$$

But the equalities $\mu = \pm 1$ cannot hold true because of conditions *(4)*.

Thus we're proved the following

Lemma 3.1. *Let conditions (4) be satisfied and* k_0 *be a fixed cardinal number. Then if for any function* $\varphi(x)$ *from class (3.1) problem (1)-(3) has a classical solution of from (2.7), whose derivatives can be calculated by termuise differentiation of series (2.7), the following inclusion must necessarily obtain:*

$$(a_1 b_{20} + b_1 a_{20})^{-1}(a_1 a_{20} + b_1 b_{20}) \in (-1,1) \tag{3.7}$$

4. Using the well known estimates of the Green function's derivatives for problem *(1.1)*, *(1.2)* and the fact that from *(3.7)* (that is $\mathrm{Re}\,\alpha_j = 0, \quad j = 1,2$) follows the inequality

$$\left|\exp\left(-i\lambda_\nu^2 t\right)\right| \le C,$$

we conclude that the uniform convergence of series *(2.7)* with respect to $(t,x) \in \overline{Q}$ along with termwise derivatives thereof will prove the following theorem:

Thoerem. *Assume that conditions (4) and (3.7) are satisfied and*

$$\left[\frac{d^2}{dx^2} + q(x)\right]^k \varphi(x) \in D[0,1] \qquad (k = 0,1) \tag{4.1}$$

Problem (1)-(3) has then a unique solution given by formula (2.7).

References

[1]. Birkhoff G.D., *Boundary value and expansion problems of ordinary linear differentical equations.* Am. Math. Soc., 9,4, 1908.

[2]. Naumark M.A., *Linear differential operators.* M., Nauka, 1969

[3]. Rasulov M.L., *Methods of contour integration.* North-Holland Pub. Comp.- Amsterdam, 1967.

Adel El-Kadi. **ŞREDİNGER TƏNLİYİ ÜÇÜN QARIŞIQ MƏSƏLƏNİN HƏLLİNİN VARLI ĞININ BİR ŞƏRTİ HAQQINDA**

Şredinger tənliyi üçün Q.D. Birkhof mənada güclü requlyar sərhəd şərtləri ilə birölçülü qarışıq məsələyə baxılır. Başlanbıc funksiyanın sonlu hamarlığa malik funksiyalar sinfindən olduğu halda məsələnin klassik həllinin varlığı üçün sərhəd şərtlərinin kompleks əmsallarının *(3.7)* şərtini ödəməsinin zəruriliyi göstərilir. Bu şərt ödəndikdə *(4.1)* sinfindən olan başlanğıc funksiyalar üçün qarışıq məsələnin yeganə klassik həllinin varlığı və bu həllin *(2.7)* düsturu ilə verilməsi isbat olunur.

Адел Эл-Кади. **ОБ ОДНОМ УСЛОВИИ РАЗРЕШИМОСТИ СМЕШАННОЙ ЗАДАЧИ ДЛЯ УРАВНЕНИЯ ШРЕДИНГЕРА**

Рассматривается смешанная задача для уравнения Шредингера при общих усиленно регулярных по Г.Д. Биркгофу граничных условиях. Доказывается, что для классической разрешимости задачи в классе начальных функций имеющих конечную гладкость необходимо чтобы комплексные коэффициенты граничных условий удовлетворяли соотношению *(3.7)*. А при выполнении этого условия и условия *(4.1)* доказано существование единственного классического решения, которое представимо формулой *(2.7)*.