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THE FIRST BOUNDARY PROBLEM FOR THE SECOND ORDER PARABOLIC EQUATIONS WITH NON-UNIFORM DEGREE DEGENERATION

Let D be bounded domain with twice smooth boundary dD,, which is situated, in the n- dimensional Euclidean spase E_n of points $x=(x_1,\dots,x_n)$, $0\in D$. Let further $Q_T=D\times(-T_1,T_2)$ be the cylindrical domain, which is situated in $(n+1)$- dimensional Euclidean space R_{n+1} of points $(x,t)=(x_1,\dots,x_n,t)$, $T_1>0$, $T_2>0$. We consider in Q_T the first boundary problem

$$LU=\sum_{i,k=1}^{n} a_{ik}(x,t)U_{ik}-U_t=f(x,t) \tag{1}$$

$$U\big|_{\gamma(Q_T)}=0 \tag{2}$$

where $\gamma(Q_T)$ is parabolic boundary of the cylinder Q_T. It is well- known, if the matrix $\|a_{ik}(x,t)\|$ is uniformly positive defined in Q_T and coefficients $a_{ik}(x,t)$ are continuous in $\overline{Q}_T$ the problem *(1) - (2)* is uniqely solvable in the space $W_{p,0}^{2,1}$ for all $f(x,t)\in L_p(Q_T)$, $1<p<\infty$ ([1,2]). In case of $p=2$ the corresponding result has been taken place for the class of equations with discontinuous coefficients ([3]) satisfying to condition of Cordes type. In [4] the unique solvability of the problem *(1) - (2)* for non- uniformly degenerating parabolic equations has been proved. In this case the speed of degeneration is satisfied to so- called "logarithm" condition, so none of equations with non- uniform degree degeneration can be considered. The aim of this work is to find class of the second order parabolic equations with non- uniform degree degeneration for which the unique solvability of the problem *(1) - (2)* takes place.

We suppose that the following conditions on the coefficients of the operator L have been carried out

$$\gamma_1\sum_{i=1}^{n}\lambda_i(x,t)\zeta_i^2\le\sum_{i,k=1}^{n}a_{ik}(x,t)\zeta_i\zeta_k\le\gamma_2\sum_{i=1}^{n}\lambda_i(x,t)\zeta_i^2, \tag{3}$$

$$\frac{a_{ik}(x,t)}{\sqrt{\lambda_i(x,t)\lambda(x,t)}}\in C(\overline{Q}_T) \tag{4}$$

where $\gamma_1 > 0, \gamma_2 < \infty, \lambda_i(x,t) = \left(|x|_\alpha + \sqrt{|t|}\right)^{\alpha_i}, |x|_\alpha = \sum_{i=1}^{n} |x_i|^{\overline{\alpha}_i}, \overline{\alpha}_i = \frac{2}{2+\alpha_i}, \alpha_c \ge 0,$

$i = 1, \ldots, n.$

Moreover we shall suppose, if $\alpha^+ = \max\{\alpha_i\}, \alpha^- = \min\{\alpha_i$, then

$$\alpha^+ < 2\left(1 + \frac{1}{n}\right), \qquad (5)$$

$$\frac{2n}{n+1}\alpha^+ - \alpha^- < 2 \qquad (6)$$

Let us note, that from the condition *(5) - (6)*, we receive

$$\frac{\alpha^+ n}{2} < n+1, \quad \frac{2\alpha^+ n}{2+\alpha^-} < n+1$$

So that, if

$$p_1 = \max\left\{\frac{\alpha^+ n}{2}, \frac{2\alpha^+ n}{2+\alpha^-}\right\},$$

then $p_1 < n+1$.

Denote

$$q_0 = \frac{(n+1)^2}{n+1-p_1}.$$

Further throughout the paper it is supposed, that p is any number from interval $\left(1, \frac{n+1}{2\beta}\right)$, where $\beta = \max\left\{\frac{1}{2}, \frac{2}{2+\alpha^+}\right\}$ and $q \in (q_0, \infty)$.

We denote through $C_{0,R}^{t_1,t_2}(K)$ – cylinder $\left\{(x,t) : \sum_{i=1}^{n} \frac{x_i^2}{R^{\alpha_i}} < k^2 R^2, t_1 < t < t_2\right\}$ and through $Ш_{x^0,R}^{t_1,t_2}$ - cylinder $\{(x,t) : |x - x_0| < R, t_1 < t < t_2$. Let $R > 0$ be so little, that closore of the cylinder $C_{0,R}^{-4R^2,4R^2}(4)$ is contented in $\overline{Q}_T$. Let us consider the cylindrical layer

$$A = C_{0,R}^{-4R^2,4R^2}(4) \setminus C_{0,R}^{-\frac{1}{2}R^2,\frac{1}{2}R^2}\left(\frac{1}{2}\right)$$

Don't particularly announce we shall suppose that all constants utilized in this work positive. Moreover for briefness we shall denote $\operatorname{ess\,sup}_{Q_T}$ through $\sup_{Q_T}(U)$.

Lemma 1. Suppose that conditions (3) - (6) on coefficients of the operator L have been fulfilled. In this case, if $U(x,t) \in C_0^\infty(A)$ and R is coefficiently little, then there exists constant $C_1 = C_1(L, n, p)$ such that

$$\sum_{i=1}^{n}\left[\lambda_i(x,t)\lambda_k(x,t)\right]^{p/2}\left|U_{ik}\right|^p dxdt+\int_A\left|U_t\right|^p dxdt\le C_1\int_A\left|LU\right|^p dxdt$$

(7) **Proof.** We make transformation of variables $Y_i=\frac{x_i}{R^{\alpha_i}/2},\ i=1,n,\ \tau=t$. Then A transforms in the usual cylindrical layer $\widetilde{A}=П_{0,4R}^{-4R^2,4R^2}\setminus П_{0,\frac{R}{2}}^{-\frac{1}{2}R^2,\frac{1}{2}R^2}$ and the operator L - in the operator

$$\tau=\sum_{i=1}^{n}\frac{a_{ik}(x,t)}{R^{(\alpha_i+\alpha_k)/2}}\cdot\frac{\partial^2}{\partial y_i\partial y_k}-\frac{\partial}{\partial\tau}$$

According to the condition *(3)*, the operator L is uniformly parabolic in $\widetilde{A}$ and also its parabolicity constants don't depend on R. Moreover from *(4)* we get that module of continuity of the Loperator's coefficients doesn't also depend on R. Then according to the apriori estimate [5]

$$\int_{\widetilde{A}}\sum_{i=1}^{n}\left|\widetilde{u}_{y_iy_k}\right|^p dyd\tau+\int_{\widetilde{A}}\left|\widetilde{u}_\tau\right|^p dyd\tau\le C_2(L,n,p)\int_{\widetilde{A}}\left|L\widetilde{u}\right|^p dyd\tau,$$

where $\widetilde{u}(y,\tau)=u\left(\ldots,R^{\alpha_i/2}y_i,\ldots,\tau\right)$.

Retuning to variables (x,t), we get

$$\int_A\sum_{i=1}^{n}R^{\alpha_i\cdot p/2}R^{\alpha_k\cdot p/2}\left|u_{ik}\right|^p dydt+\int_{\widetilde{A}}\left|u_t\right|^p dydt\le C_2\int_{\widetilde{A}}\left|Lu\right|^p dxdt \qquad (8)$$

Now, it is enough to take into account, that in the case $(x,t)\in A$, $\lambda_i(x,t)\le$ $\le C_3(L,n)R^{\alpha_i}$, $i=1,n$ and then from *(8)* we obtain demanding estimate *(7)*. Lemma is proved.

Lemma 2. *Let the previous lemma conditions has been implemented. Then*

$$\int_A\left[\left|u\right|\right]^p+\sum_{i=1}^{n}\left[d_i(x,t)\right]^{p/2}\left|u_i\right|^p+\sum_{i,k=1}^{n}\left[\lambda_i(x,t)\lambda_k(x,t)\right]^{p/2}\left[\left|u_{ik}\right|^p+\left|u_t\right|^p\right]dxdt\le$$
$$\le C_4(L,n,p)\int_A\left|Lu\right|^p dxdt$$

The proof of this lemma is grounded on the estimate [7] and Steklov-Fridrix type inequatily. Let $A_1=C_{0,R}^{-2R^2,2R^2}(2)\setminus C_{0,R}^{-R^2,R^2}(1)$.

Lemma 3. *Let the lemma 1 conditions have carried out and* $U(x,t)\in C^\infty(\overline{A})$. *Then*

$$\int_A\left[\left|u\right|^p\right]+\sum_{i=1}^{n}\left[\lambda_i(x,t)\right]^{p/2}\left|u_i\right|^p+\sum_{i,k=1}^{n}\left[\lambda_i(x,t)\lambda_k(x,t)\right]^{p/2}\left[\left|u_{ik}\right|^p+\left|u_t\right|^p\right]dxdt\le$$
$$\le C_5\int_A\left|Lu\right|^p+C_6\int_A\sum_{i=1}^{n}\left[\lambda_i(x,t)\right]^{p/2}\left|u_i\right|^p dxdt+C_7\int_A\left|u\right|^p dxdt,$$

where constant C_5 *only depends on coefficients of the operator* L, n *and* p, *and* C_6, C_7 *- moreover depend onp.*

The proof of this lemma is grounded at the application of the previous lemma to the function

$$v(x,t) = u(x,t)\cdot \eta(x,t),$$

where $\eta(x,t)=1$ in $A_1, \eta(x,t)\in C_0^\infty(A), 0\le \eta(x,t)\le 1$.

We denote through $W_{p,\Lambda}^{2,1}(D)$ the Banach space of the functions $u(x,t)$ defined in D, with finite norm

$$\|u\|_{W_{p,\Lambda}^{2,1}}(D)=\left[\int_D\left[|u|^p+\sum_{i=1}[\lambda_i(x,t)]^{p/2}|u_i|^p+\sum_{i,k=1}^{n}[\lambda_i(x,t)\lambda_k(x,t)]^{p/2}|u_{ik}|^p+\right.\right.$$
$$\left.\left.+|u_t|^p\right]dxdt\right]^{1/p}$$

Corollary if the lemma 1 conditions have been fulfilled, then for any function $u(x,t)\in W_{p,\Lambda}^{2,1}(A)$ and for all $\varepsilon>0$ the following estimate is true

$$\|u\|^p_{W_{p,\Lambda}^{2,1}(A_1)}\le C_8\int_A|Lu|^p dxdt+\varepsilon\|u\|^p_{W_{p,\Lambda}^{2,1}(A)}+C(\varepsilon)\cdot C_9\int_A\frac{|u|^p}{[\rho(x,t)]L\rho p}dxdt,$$

(9)

where C_8 and C_9 constants, depending only on coefficients of the operator L, n and $p, \rho(x,t)=\sum_{i=1}^{n}|x_i|+|t|$. We can deduce the estimate *(9)* from the lemma 3, utilizing interpolation inequality.

Let $C^0=C_{0,R}^{-4R^2,4R^2}(4), C^1=C_{0,R}^{-2R^2,2R^2}(2)$.

Lemma 4. *In the conditions of the lemma 1 for* $u(x,t)\in W_{p,\Lambda}^{2,1}(C_0)$ *and for all* $\varepsilon>0$ *the following inequality is true*

$$\|u\|^p_{W_{p,\Lambda}^{2,1}(C_1)}\le C_{10}\int_{C^0}|Lu|^p dxdt+\varepsilon\|u\|^p_{W_{p,\Lambda}^{2,1}(C^0)}+C(\varepsilon)C_{11}\int_{C^0}\frac{|u|^p}{[\rho(x,t)]^{2\beta\rho}}dxdt,$$

where $C_{10}=C_{10}(L,n,p), C_{11}=C_{11}(L,n,p)$.

For proof it is enough to put $R=2^{-k}, k=0,1,\ldots,$ in the inequality *(9)* and sum up received estimates by k from 0 to ∞.

Denote through $\overset{\circ}{W}{}_{p,\Lambda}^{2,1}(Q_T)$ subspace of $W_{p,\Lambda}^{2,1}(Q_T)$, in which the infinitely differentible functions vanishing on $\gamma(Q_T)$ make a dense set.

Theorem 1. *Let* $u(x,t)\in W_{p,\Lambda}^{2,1}(Q_T)$. *Then, these exist constants* C_{12} *and* C_{13}, *depending only on coefficients of the operator* L, n *and* p, *such, that*

$$\|u\|^p_{W_{p,\Lambda}^{2,1}(Q_T)}\le C_{12}\int_{Q_T}|Lu|^p dxdt+C_{13}\int_{Q_T}\frac{|u|^p}{[\rho(x)t]^{2\beta p}}dxdt$$

The proof of this is grounded on the "local straighting of boundary" (look [6]) met had and the lemma 4.

Lemma 5. *Let* $u(x,t)\in \overset{\circ}{W}{}^{2,1}_{p,\Lambda}(Q_T)$ *and* $q>q_0$. *Then*

$$\sup_{Q_T}|u|\le C_{14}(q,n,L,Q_T)\|Lu\|_{L_q(Q_T)}, \tag{10}$$

Proof. According to the N.V Krylovapriori estimate [7]

$$\sup_{Q_T}|u|\le C_{15}(L,n,Q_T)\left\|(\det a)^{\frac{-1}{n+1}}|Lu|\right\|_{L_{n+1}(Q_T)}, \tag{11}$$

where $a=\|a_{ij}(x,t)\|$.

But

$$\det a\ge C_{16}(L,n)\prod_{i=1}^{n}\lambda_i(x,t)\ge C_{17}(L,n)[\rho(x,t)]^{p_1}. \tag{12}$$

Taking (12) into account in (11), we get

$$\sup_{Q_T}|u|\le C_{17}(L,n,Q_T)\left(\int_{Q_T}\frac{|Lu|^{n+1}}{[\rho(x,t)]^{p_2}}dxdt\right)^{\frac{1}{n+1}}\le\left(\int_{Q_T}|Lu|^{(n+1)s}dxdt\right)^{\frac{1}{s(n+1)}}\times$$

$$\times\left(\int_{Q_T}[\rho(x,t)]^{-\frac{p_2 s}{s-1}}dxdt\right)^{\frac{s-1}{s}}, \tag{13}$$

where s is an arbitrary number from the seminfinite interval $\left(\frac{n+1}{n+1-p_1},\infty\right)$. It is easy to see, that $\frac{p_1 s}{s-1}<n+1$. That's why from (13) we infer

$$\sup_{Q_T}|u|\le C_{18}(L,N,Q_T,s)\int_{Q_T}|Lu|^{(n+1)s}dxdt \tag{14}$$

Now itis enough to choose s, so that $(n+1)s=q$, and then from *(14)* we get demanded estimate *(10)*.

Lemma is proved.

Corollary 1. *In the conditions of the lemma*

$$\int_{Q_T}\frac{|u|^p dxdt}{[\rho(x,t)]^{2\beta p}}\le C_{19}(L,n,Q_T,q,p)\|Lu\|^p_{L_q(Q_T)}$$

For proof it is enough to notice, that acconing to the choice $p\ 2\beta p<n+1$, and then apply the lemma 5.

Corollary 2. *In the conditions of the lemma*

$$\int_{Q_T}|Lu|^p dxdt\le C_{20}(l,n,Q_T,q,p)\|Lu\|^p_{L_q(Q_T)}$$

For proof it is enough to apply Holder's inequality. Now we denote through $\dot{V}^{2,1}_{p,\Lambda}(Q_T)$ closure of the set of the smooth functions, defined in Q_T vanishing on $\gamma(Q_T)$, by the norm

$$\|u\|_{\dot{V}^{2,1}_{p,\Lambda}(Q_T)} = \sup_{Q_T}|u| + \left(\int_{Q_T}\left[\sum_{i=1}^{n}[\lambda_i(x,t)]^{p/2}|u_i|^p + |u_t|^p + \right.\right.$$

$$\left.\left. + \sum_{i,k=1}^{n}[\lambda_i(x,t)\lambda_k(x,t)]^{p/2}|u_{it}|^p\right]dxdt\right)^{1/p} .$$

From proved lemmas we get the next theorem

Theorem 2. *Let* $u(x,t)\in \dot{V}^{2,1}_{p,\Lambda}(Q_T), p\in\left(1,\dfrac{n+1}{2\beta}\right)$ *and* $q\in(q_0,\infty)$. *Then*

$$\|u\|_{\dot{V}^{2,1}_{p,\Lambda}(Q_T)} \le C_{21}(L,n,Q_T,p,q)\|Lu\|Lq(Q_T) .$$

Now we prove the main theorem of this work.

Theorem 3. *Let* p *and* q *satisty the conditions of the previous theorem. Then the first boundary problem*

$$Lu = f, \quad U\Big|_{\gamma(Q_T)} = 0 \tag{15}$$

has unique generalized (almost every where) solution $u(x,t)\in\dot{V}^{2,1}_{p,\Lambda}$ *for all* $f(x,t)\in L_q(Q_T)$.

In this case

$$\|u\|_{\dot{V}^{2,1}_{p,\Lambda}(Q_T)} \le C_{21}\|f\|_{L_q(Q_T)} . \tag{16}$$

Proof. Let $h>0$ be sufficiently small number. In this case, if

$$\lambda_i^h(x,t) = \max\{\lambda_i(x,t), h\} \quad i=1,\dots,n$$

then there is sufficiently small neigh bourhood $O_1(h)$ of the point $(0,0)$ such, that $\lambda_i^h(x,t)=\lambda_i(x,t)$ for $(x,t)\in Q_T\setminus O_1(h)$.

Let $\|a_{ik}^h(x,t)\|$ be matrix such that $a_{ik}^h(x,t)\in C(\overline{Q}_T)$, $j_1\sum_{i=1}^{n}\lambda_i^h(x,t)\zeta_i^2\le$ $\le\sum_{i,k=1}^{n}a_{ik}^h(x,t)\zeta_i\zeta_k\le j_2\sum_{i=1}^{n}\lambda_i^h(x,t)\zeta_i^2$ and there is neughbourhood $O_2(h)$ of the point $(0,0)$, out of which

$$a_{ik}^h(x,t) = a_{ik}(x,t) .$$

Let further

$$L^h = \sum_{i,k=1}^{n}a_{ik}^h(x,t)\frac{\partial L}{\partial x_i\partial x_k} - \frac{\partial}{\partial x} .$$

We consider auxiliary problem

$$L^h u^h = f, \quad (x,t)\in Q_T, \quad U^h\Big|_{\gamma(Q_T)} = 0 \tag{17}$$

As the operator L^h is non-degenerated for all $h>0$, the problem *(17)* has unique generalized solution $U^h(x,t)\in W_q^{2,1}(Q_T)$ (look [5]).

But $\dot{W}_q^{2,1}(Q_T)\subset \dot{W}_p^{2,1}(Q_T)\subset \dot{W}_{p,\Lambda}^{2,1}(Q_T)$.

Moreover, according to the lemma 5

$$\sup_{Q_T}\left|U^h\right|\le C_{14}\left\|f^hU^h\right\|_{L_q(Q_T)}=C_{14}\|f\|_{L_q(Q_T)}<\infty \quad ,$$

i.e. $U^h\in \dot{V}_{p,\Lambda}^{2,1}(Q_T)$. Orter wise by the theorem 2

$$\left\|U^h\right\|_{\dot{V}_{p,\Lambda}^{2,1}(Q_T)}\le C_{21}\|f\|_{L_q(Q_T)}=C_{22}<\infty \ .$$

Thus the family $\{U^h(x,t)$ is bounded by the norm $\dot{V}_{p,\Lambda}^{2,1}(Q_T)$ uniformly with respect to h.i.e. this family is weakly compact in this space. It means, that there exist sequence $h_\gamma\to\infty\ (\gamma\to\infty)$ and function $U(x,t)\in \dot{V}_{p,\Lambda}^{2,1}(Q_T)$, such that

$$\left(LU^{h\gamma},\varphi\right)\to\left(LU,\varphi\right) \qquad (\gamma\to\infty)$$

for all $\varphi\in C_0^\infty(Q_T)$. Here

$$\left(LU,\varphi\right)=\int_{Q_T}LU(x,t)\varphi(x,t)dxdt \quad .$$

Now we show that

$$\left(LU,\varphi\right)=\left(f,\varphi\right) \quad . \tag{18}$$

Really

$$\left(LU^{h\gamma},\varphi\right)=\left(\left(L-L^{h\gamma}\right)U^{h\gamma},\varphi\right)+\left(L^{h\gamma}U^{h\gamma},\varphi\right)=\left(\left(L-L^{h\gamma}\right)U^{h\gamma},\varphi\right)+\left(f,\varphi\right),$$

and *(18)* will be proved, if we show, that

$$\lim_{\gamma\to\infty}\left(\left(L-L^{h\gamma}\right)U^{h\gamma},\varphi\right)=0, \tag{19}$$

for all $\varphi(x,t)\in C_0^\infty(Q_T)$. But the operetors L and $L^{h\gamma}$ coinside out of the neighbourhood $O_2^{h\gamma}$ of the point (0,0). So that

$$\left|\left(\left(L-L^{h\gamma}\right)U^{h\gamma},\varphi\right)\right|\le\int_{O_2^{h\gamma}}\sum_{sat}^{h}\left|a_{ct}^{h\gamma}(x,t)-a_{ct}(x,t)\right|\left|U_{ch}^{h\gamma}(x,t)\right|\left|\varphi(x,t)\right|dxdt\le$$

$$\le C_{23}\int_{O_2^{h\gamma}}\sum_{sat}^{h}\sqrt{\lambda_t^{h\gamma}(x,t)\lambda_t^{h\gamma}(x,t)}\left|U_{ct}^{h\gamma}(x,t)\right|\left|\varphi(x,t)\right|dxdt\le C_{25}\left|f\right|_{Lq}\left|\left(O_2^{h\gamma}\right)\right|\left|\varphi\right\|_{Lp'}\left(O_2^{h\gamma}\right)$$

Wese constants C_{23},C_{24},C_{25} don't depend on γ and $\dfrac{1}{p}+\dfrac{2}{p'}=1$. Hence we get the limit eguality *(19)*. These fore from *(18)* we get, that $LU=f$ almost everywhere in Q_T. Moreover $U(x,t)\in V_{P\Lambda}^{2,1}(Q_T)$. Thus $U(x,t)$ is generalired solutionto the to the boundary problem *(15)*. Now the estimate *(15)* is a corollary of the theorem 2, but unigueness of the solution to the problem *(15)* can be obtained from the estimate *(16)*. Theorem is proved.

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İKİNCİ TƏRTİBLİ QÜVVƏ SÜR'ƏTİ İLƏ QEYRİ-MÜNTƏZƏM ĞIRLAŞAN PARABOLİK TƏNLİKLƏR ÜÇUN BİRİNCİ SƏRHƏD MƏSƏLƏSİ

Məqalədə bir sinif qüvvə sürəti ilə qeyri- müntəzəm qırlaşan parabolik tənliklər üçün Sobolev fəzasında birinci sərhəd məsələsi hərlinin varlığı və yeganəliyi isbat olunmuşdur.

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ПЕРВАЯ КРАЕВАЯ ЗАДАЧА ДЛЯ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ 2-ГО ПОРЯДКА С НЕРАВНОМЕРНЫМ СТЕПЕННЫМ ВЫРОЖДЕНИЕМ

В статье доказывается однозначная разрешимость первой краевой задачи для класса параболических уравнений 2-го порядка с неравномерным степенным вырождением.