UDC 517.956.4

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BOUNDARY PROPERTIES OF SOLUTIONS OF THE SECOND ORDER PARABOLIC EQUATIONS WITH NON-UNIFORM DEGREE DEGENERATION

We consider in $(n+1)$- dimensional Euclidean space R_{n+1} of points $(x,t)=(x_1,\dots,x_n,t)$ bounded domain D with parabolic boundary $\Gamma(D)$. Let

$$L=\sum_{i,k=1}^{n}a_{ik}(x,t)\frac{\partial^2}{\partial x_i\partial x_k}-\frac{\partial}{\partial t}$$

be the second order parabolic operator, which coefficients are determined in D. We consider the first boundary problem in D

$$\left.\begin{aligned} &Lu=0, \quad (x,t)\in D\\ &u\big|_{\Gamma(D)}=\varphi, \quad \varphi\in C[\Gamma(D)] \end{aligned}\right\}. \qquad (1)$$

Let us suppose, that $U_\varphi(x,t)$ is generalized Viner- Landis sense solution of the problem *(1)* (look [1]). Point $(x^0,t^0)\in\Gamma(D)$ is called regular with respect to the problem *(1)*, if for any function $\varphi(x,t)$, which is continuous on $\Gamma(D)$, following equality is implemented

$$\lim_{(x,t)\to(x^0,t^0)}U_\varphi(x,t)=\varphi(x^0,t^0) .$$

If L is the heat operator, then the criterion of regularity for any bounded domain in the terms of heat potentials has been defined by E.M. Landis [2]. From works, which are preceded to [2], we show at the well-known articles of Petrovsky I.G. [3] and Tikhonov A.N. [4]. In works of Novruzov A.A. [5-6]- results of E.M. Landis was transfered at the class of uniformly parabolic equations with continuous coefficients. (look also [7-8]). We show work of L.Evans and R.Gaviepy [9] for conditions of regularity of boundary point in terms of heat capacities.

In case, when operator L admits non-uniform degeneration in researching on regularity boundary point, we refer to work [10], where corresponding result has been received by implementation of the so- called "logarithm" condition on the speed of degeneration, which excepts equations with non-uniform degree degeneration from considering.

The aim of this note is to obtain sufficient condition of regularity of boundary point in terms of parabolic capacities for class of equations with non-uniform degree degeneration.

We suppose, that $(0,0)\in\Gamma(D)$ and on coefficients of operator L following conditions is carried out

$$\gamma\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2 \le \sum_{i,k=1}^{n}a_{ik}(x,t)\xi_i\xi_k \le \gamma^{-1}\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2 \quad , \qquad (2)$$

where $\gamma > 0$, $(x,t)\in D$, ξ is arbitrary n- dimensional vector and $\lambda_i(x,t)=$ $=\left(|x|_\alpha + \sqrt{|t|}\right)^{\alpha_i}$, $|x|_\alpha = \sum_{i=1}^{n}|x_i|^{\overline{\alpha_i}}$, $\overline{\alpha_i} = \dfrac{2}{2+\alpha_i}$, $\alpha_i \ge 0, i=1,\dots,n$.

Let us agree upon the next notation: for n-dimensional vector x^0, scalars t_1, t_2, positive numbers Rand K – $C_{x^0,R}^{t_1,t_2}(k)$ denotes cylinder

$$\left\{(x,t): \sum_{i=1}^{n}\frac{\left(x_i - x_i^0\right)^2}{R^{\alpha_i}} < (kR)^2 \ , \quad t_1 < t < t_2\right\}.$$

Let $C(t_1,t_2,k_1,k_2)$ be cylindrical layer $C_{0,R}^{t_1,0}(k_1)\setminus C_{0,R}^{t_2,0}(k_2)$, where $t_1 < t_2 < 0, k_1 > k_2 > 0$. In work of author [11] the following affirmation has been proved.

Lemma 1. *Let in $C(t_1,t_2,k_1,k_2)$ coefficients of operator L, satisfying to condition (2) be defined. Then there are positive constants ξ and β, depending only on coefficients of operator L, n, and also $\frac{|t_1|}{|t_2|}$ and $\frac{k_1}{k_2}$ such, that if*

$$G_R(x,y;t,\tau)=\begin{cases}(t-\tau)^{-s}\exp\left[-\dfrac{\sum_{i=1}^{n}\dfrac{(x_i-y_i)^2}{R^{\alpha_i}}}{4\beta(t-\tau)}\right], & t>\tau\\ 0 \ , & t\le\tau\end{cases}$$

then $\underset{(x,t)}{LG} \ge 0$ in $C(t_1,t_2,k_1,k_2)\setminus\{y,r\}$.

Let s and β be showed above constants corresponding to

$$\frac{|t_1|}{|t_2|} = 66 \quad , \qquad \frac{k_1}{k_2} = 17 .$$

Let E be B- set in $C(t_1,t_2,k_1,k_2)$. We call measure μ on E (s,β,R)-available, if

$$\int_E G_R(x,y;t,\tau)d\mu(y,\tau)\le 1 , \quad (x,t)\notin E .$$

Number $p_R(E)=\sup\mu E$, where superior bound is taken by all available measures, is being called (s,β,R)- capacity of set E.

We consider 3 cylinders

$$C_1 = C_{0,R}^{-66bR^2,0}(17)$$

$$C_2 = C_{0,R}^{-2bR^2,0}(9)$$

$$C_3 = C_{0,R}^{-65bR^2,-32bR^2}(9,5)$$

where $b = \dfrac{31}{256\beta s}$.

The following lemma is true [11]

Lemma 2. *Let in cylinder C_1 be situated domain D, which has limit points on $\Gamma(C_1)$ and interesects C_2. Let $U(x,t)$ be positive. L-subparabolic function in D, vanishing on $\Gamma(D)\cap C_1$. Then there is positive constant $\eta_1 = \eta_1(b,n)$ such that*

$$\sup_D U \ge \left(1+\eta_1 R^{-2s} p_R(B)\right)\sup_{D\cap C_2} U ,$$

where $B = C_3 \setminus D$.

Now we formulate corollary from this lemma, which we are going to use futher.

Let j be natural parameter

$R_j = 33^{-j/2}$, $\quad C_1^j = C_{0,R_j}^{-66R^2 j,0}(17)$, $\quad C_3^j = C_{0,R_j}^{-65bR^2 j,-32bR^2 j}(9{,}5)$, $\quad B^j = C_3^j \setminus D$, $p_{R_j}(B^j) = \gamma_j$.

Corollary. *If lemma 2 conditions are carried out, then for any natural j*

$$\sup_{D\cap C_1^j} U \ge \left(1+\eta_1 R_j^{-2s}\gamma_j\right)\sup_{D\cap C_1^{j+1}} U . \qquad (3)$$

Theorem 1. *Let D be bounded domain in R_{n+1}, in which coefficients of operator L, satisfying to condition (2), are defined. In order to point $(0,0)\in\Gamma(D)$ be regular with respect to the problem (1), it is sufficient that*

$$\sum_{j=1}^{\infty} R_j^{-2s}\gamma_j = \infty . \qquad (4)$$

Proof. For regularity of point $(0,0)$ it is enough to show the following: for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ will be found $\delta > 0$, such that for any $D' \subset D$, which completely lies in semi-space $t<0$ and L-subparabolic function $U(x,t)\le 1$ which is determined in D', from

$$U\Big|_{\Gamma(D')\cap O_{\varepsilon_1}(0,0)} \le 0$$

it follows

$$U\Big|_{D'\cap O_\delta(0,0)} \le \varepsilon_2$$

here $O_\delta(0,0)$ is δ-neighbourhood of point $(0,0)$. Suppose numbers $\varepsilon_1, \varepsilon_2$, subdomain D' and function $U(x,t)$ have been given. Denote through m_1 the least natural number, for which

$$66bR_{m_1} < \varepsilon_1 \ , \qquad 17R_{m_1} < \varepsilon_1 .$$

We assume, that there exists $m^* > m_1$ such that in intersection $C_1^{m^*} \cap D'$ will be found point (x^0, t^0) in which

$$U(x^0, t^0) > \varepsilon_2 . \qquad (5)$$

Let $D'_+ = \{(x,t) : (x,t) \in D', U(x,t) > 0\}$.

Taking into account, that $p_{R_j}\left(C_3^j \setminus D'_+\right) \geq p_{R_j}\left(B^j\right) = \gamma_j$, and consecutively applying inequality *(3)* for $j = m_1, \ldots, m^* - 1$, we get

$$1 \leq \sup_{D'_+ \cap C_1^{m_1}} U \geq \prod_{j=m_1}^{m^*-1}\left(1 + \eta_1 R_j^{-2s}\gamma_j\right) \sup_{D'_+ \cap C_1^{m^*}} U \ .$$

By using *(5)*, we find

$$\sum_{j=m_1}^{m^*-1} \ln\left(1 + \eta_1 R_j^{\ 2s}\gamma_j\right) \leq \ln \frac{1}{\varepsilon_2}$$

or

$$\sum_{j=m_1}^{m^*-1} R_j^{-2s}\gamma_j \leq a_1 \ln \frac{1}{\varepsilon_2} \ , \qquad (6)$$

where $a_1 = a_1(L, n)$.

According to condition *(4)* inequality *(6)* can be fulfilled if only $m^* \leq m_2(L, n, \varepsilon_1, \varepsilon_2)$.

Now it is enough to choose

$$\delta = \min\left\{17R_{m_2+1}, 66bR_{m_2+1}^2\right\}$$

and $U\big|_{D' \cap O_\delta(0,0)} \leq \varepsilon_2$. So theorem is proved

Lemma 3. *Let $U_\varphi(x,t)$ be generalized solution to the problem (1) in bounded domain $D \subset R_{n+1}$, $(0,0) \in \Gamma(D)$ and on coefficients of operator L condition (2) has been carried out. If for a certain $\varepsilon_0 > 0$ $U_\varphi\big|_{\Gamma(D) \cap O_{\varepsilon_0}(0,0)} = 0$, condition (4) is fulfilled, then*

$$U_\varphi(x,t) \leq A_1 \sup_{\Gamma(D)}|\varphi| \exp\left[-\eta_2 \sum_{j=1}^{\left[\ln \frac{1}{\rho(x,t)}\right]} R_j^{-2s}\gamma_j\right], \qquad (7)$$

where $\rho(x,t) = |x| + \sqrt{|t|}$ is sufficiently small, $A_1 = A_1(L,n)$, $\eta_2 = \eta_2(L,n)$, $A_1 > 0$, $\eta_2 > 0$.

Proof. Let $D^+ = \{(x,t) : (x,t) \in D\,, U_\varphi(x,t) > 0\}$. If D^+ is empty, then *(7)* is proved. Suppose, that D^+ isn't empty. Denote through m_0 the least natural number, for which $C_1^{m_0} \subset O_{\varepsilon_0}(0,0)$. Let (x,t) be any fixed point of D^+, and $\rho(x,t)$ is sufficiently small. Denote through $m_1 > m_0$ natural number such that

$$(x,t) \in C_1^{m_1}\ , \qquad (x,t) \notin C_1^{m_1+1} \ . \qquad (8)$$

Consecutively applying for $j = m_0, \ldots, m_1 - 1$ inequality *(3)* and using *(8)* we get

$$\sup_{\Gamma(D)}|\varphi| \geq U_\varphi(x,t) \prod_{j=m_0}^{m_1-1}\left(1 + \eta_1 R_j^{-2s}\gamma_j\right),$$

i.e.

$$U_\varphi(x,t) \le \sup_{\Gamma(D)}|\varphi| \exp\left[-\eta_3 \sum_{j=m_0}^{m_1-1} R_j^{-2s}\gamma_j\right], \qquad (9)$$

where $\eta_3 > 0$ depends only on coefficients of operator L and n. But according to conditions *(4)* for sufficiently big m_1, i.e. for small $\rho(x,t)$

$$\sum_{j=m_0}^{m_1-1} R_j^{-2s}\gamma_j = \sum_{j=1}^{m_1-1} R_j^{-2s}\gamma_j - \sum_{j=1}^{m_0-1} R_j^{-2s}\gamma_j \ge \frac{1}{2}\sum_{j=m_0}^{m_1-1} R_j^{-2s}\gamma_j ,$$

which together with *(9)* gives

$$U_\varphi(x,t) \le \sup_{\Gamma(D)}|\varphi| \exp\left[-\frac{\eta_3}{2} \sum_{j=1}^{m_1-1} R_j^{-2s}\gamma_j\right]. \qquad (10)$$

According to *(8)* one of two inequalities is carried out

i) either $\sqrt{|t|} \ge \sqrt{66b}\cdot\sqrt{33}^{\,-(m_1+1)}$;

ii) or $\sum_{i=1}^{n} \frac{x_i^2}{R_{m_1+1}^{\alpha_i}} \ge 17R_{m_1+1}^2$.

In both cases we can show, that

$$m_1 \ge a_2 \ln\frac{1}{\rho(x,t)},$$

where $a_2 > 0, a_2 = a_2(L,n)$.

So for small $\rho(x,t)$

$$m_1 - 1 \ge a_3 \ln\frac{1}{\rho(x,t)}, \quad (a_3 > 0,\, a_3 = a_3(L,n)).$$

Taking into account *(10)*we receive

$$U_\varphi(x,t) \le \sup_{\Gamma(D)}|\varphi| \exp\left[-\frac{\eta_3}{2} \sum_{j=1}^{\left[a_3 \ln\frac{1}{\rho(x,t)}\right]} R_j^{-2s}\gamma_j\right] \qquad (11)$$

Without loss of generality it is possible to assume, that $a_3 < 1$. Then

$$\sum_{j=1}^{\left[a_3 \ln\frac{1}{\rho(x,t)}\right]} R_j^{-2s}\gamma_j \ge \sum_{j=1}^{\left[\ln\frac{1}{\rho(x,t)}\right]} R_j^{-2s}\gamma_j - a_4, \qquad (12)$$

where $a_4 > 0, a_4 = a_4(L,n)$. Utiliring *(12)* in *(11)*, we obtain

$$U_\varphi(x,t) \le \sup_{\Gamma(D)}|\varphi| \exp\left[-\frac{\eta_3}{2} \sum_{j=1}^{\left[\ln\frac{1}{\rho(x,t)}\right]} R_j^{-2s}\gamma_j + \frac{\eta_3}{2}a_4\right]. \qquad (13)$$

Now it is enough to put $A_1 = \exp\left[\frac{\eta_3}{2}a_4\right]$, $\eta_2 = \frac{\eta_3}{2}$, and from *(13)* it follows demanded estimate *(9)* lemma is proved.

Remark. It is easy to see, that inequality *(9)* is also true by condition

$$U_{\varphi}\Big|_{\Gamma(D)\cap O_{s_0}(0,0)} \le 0 \ .$$

Now let us suppose, that boundary function $\varphi(x,t)$ of the problem *(1)* satisfies on $\Gamma(D)$ to Holdër's condition with a certain order $A>0$, i.e. for $(x,t)\in\Gamma(D)$, $(y,\tau)\in\Gamma(D)$

$$|\varphi(x,t)-\varphi(y,\tau)| \le H\left(|x-y|+\sqrt{|t-\tau|}\right)^{\lambda} \qquad (14)$$

Theorem 2. *Let in bounded domain $D\subset R_{n+1}$ coefficients of operator L, satisfying to condition (2) be defined. If on boundary function $\varphi(x,t)$ of the problem (1) condition (14) is implemented, then for generalized solution $U_{\varphi}(x,t)$ for sufficiently small $\rho(x,t)$ the following estimate is true*

$$|U_{\varphi}(x,t)-\varphi(0,0)| \le A_0 \exp\left[-\eta_0 \sum_{j=1}^{\left[\ln\frac{1}{\rho(x,t)}\right]} R_j^{-2s}\gamma_j\right], \qquad (15)$$

where A_0 and η_0 are positive constants, depending only on coefficients of operator L, n and function φ.

Proof. According to condition *(14)* for $(x,t)\in\Gamma(D)\cap\overline{C}_1^{\,j}$

$$|\varphi(x,t)-\varphi(0,0)| \le H[\rho(x,t)]^{\lambda} \ .$$

But

$$\rho(x,t)\le a_5 R_j = a_5 33^{-j/2} \ ,$$

where $a_5=a_5(L,n)$.

So, if

$$w_j = H a_5^{\lambda} 33^{-\lambda j/2} \ ,$$

then for $(x,t)\in\Gamma(D)\cap\overline{C}_1^{\,j}$

$$|\varphi(x,t)-\varphi(0,0)| \le w_j . \qquad (16)$$

We consider function

$$\vartheta(x,t)=U_{\varphi}(x,t)-\varphi(0,0)-w_j \ .$$

Let $d_j=R_j^{2s}\gamma_j$, $j=1,2,\ldots,n$. Suppose that set $D^+=\{(x,t):(x,t)\in D,\ \vartheta(x,t)>0\}$ isn't empty. If $M_j=\sup\limits_{D^+\cap C_1^j} U_{\varphi}$, $j=1,2,\ldots$, then by utiliring inequality (3) to function $\vartheta(x,t)$ we obtain

$$M_j-\varphi(0,0)-w_j \ge (1+\eta_1 d_j)(M_{j+1}-\varphi(0,0)-w_{j+1})-(1+\eta_1 d_j)(w_j-w_{j+1}) \ .$$

Thus, if $g(j)=M_j-\varphi(0,0)-w_j$, $j=1,2,\ldots$, then

$$g(j)\ge(1+\eta_1 d_j)g(j+1)-(1+\eta_1 d_j)(w_j-w_{j+1}) . \qquad (17)$$

Let (x,t) be any fixed point D^+, and $\rho(x,t)$ is sufficiently small. Denote through m_1 natural number such that

$$(x,t)\in C_1^{m_1} , \qquad (x,t)\notin C_1^{m_1-1} . \qquad (18)$$

Consecutively applying inequality *(17)* for $j=1, m_1-1$, we receive

$$g(1)\geq \prod_{j=1}^{m_1-1}(1+\eta_1 d_j)g(m_1)-\sum_{k=1}^{m_1-1}\prod_{j=1}^{k}(1+\eta_1 d_j)(w_j-w_{j+1}) . \qquad (19)$$

Let

$$a_k=\prod_{j=1}^{k}(1+\eta_1 d_j)(w_j-w_{j+1}) .$$

We show, that series $\sum_{k=1}^{\infty} a_k$ is convergent. Since for any natural j $\gamma_j \leq a_6 R_j^{25}$, $a_6=a_6(L,n)$, then $d_j\leq \eta_1 a_6$ and without loss of generality we can assume, that $\eta_1 a_6\leq 1$. Then $\frac{a_{k+1}}{a_k}\leq 2\cdot\frac{1-\frac{w_{k+2}}{w_{k+1}}}{\frac{w_k}{w_{k+1}}-1}$ and so

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}\leq 2\cdot\frac{1-33^{-\lambda/2}}{33^{\lambda/2}-1}=\frac{2}{33^{\lambda/2}} .$$

Again, we can assume that $33^{\lambda/2}>2$. Then

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}<1$$

i.e. $\sum_{k=1}^{\infty}a_k<\infty$. Taking into account the last in *(19)* we obtain

$$g(1)\geq\prod_{j=1}^{m_1-1}(1+\eta_1 d_j)g_{m_1}-a_7 ,$$

that by taking into account *(18)* and lemma *(3)* gives

$$U_\varphi(x,t)-\varphi(0,0)\leq A_2\exp\left[-\eta_4\sum_{j=1}^{\left[\ln\frac{1}{\rho(x,t)}\right]}d_j\right]+w_{m_1} . \qquad (20)$$

Here a_7, A_2 and η_4 are positive constants depending only on coefficiets of operator L, n and function φ. But otherwise according to *(18)*

$$w_{m_1}\leq a_8[\rho(x,t)]^{a_9} ,$$

where $a_i>0$, $a_i=a_i(L,n,\varphi)$, $i=8,9$. Moreover

$$[\rho(x,t)]^{a_9}\leq\exp\left[-\eta_5\sum_{j=1}^{\left[\ln\frac{1}{\rho(x,t)}\right]}d_j\right] , \qquad (21)$$

where $\eta_5>0$, $\eta_5=\eta_5(L,n,\varphi)$.

Denote $\eta_0=\min\{\eta_4,\eta_5\}$ and $A_0=A_i+a_8$. From *(20)* and *(21)* we get

$$U_\varphi(x,t)-\varphi(0,0)\le A_0\exp\left[-\eta_0\sum_{j=1}^{\left[\ln\frac{1}{\rho(x,t)}\right]}d_j\right] \qquad (22)$$

Estimate from below, which we can obtain by the same way gives now with *(22)* demanded estimate *(15)*. Theorem is proved.

In conclusion I would like to express my thanks to my supervisor prof. I.T. Mamedov for value consultations and encouragement.

References

[1]. Ландис Е.М. *Уравнения второго порядка эллиптического и параболического типа.* Москва: Наука, 1971, 288с.

[2]. Ландис Е.М. *Необходимое и достаточное условие регулярности граничной точки для задачи Дирихле для уравнения теплопроводимости.* ДАН СССР, 1969, т.185 №3, с.517-520.

[3]. Petrovsky I.G. *Zurersten-Rond werta uf gabe de Wavmeleitungsglechung.* Comp. Math., 1935, №1, p. 383-419.

[4]. Тихонов А.Н. *Об уравнении теплопроводности для нескольких переменных.* Бюлл. МГУ, 1938, секц. А, вып.9, с. 1-49.

[5]. Новрузов А.А. *О некоторых критериях регулярности граничных точек для линейных и квазилинейных параболических уравнений.* ДАН СССР, 1973, т. 209, №4, с. 785-787.

[6]. Новрузов А.А. *О модуле непрерывности решения задачи Дирихле для параболического уравнения 2-го порядка.* Матем. заметки, 1976, т.19, №4, с. 587-593.

[7]. Мамедов И.Т. *О регулярност и граничных точек для линейных и квазилинейных уравнений параболического типа.* ДАН СССР, 1975, т.223, №3, с. 559-561.

[8]. Мамедов И.Т. *О регулярности граничных точек для линейных уравнений параболического типа.* Матем заметки, 1976, т.20, №5, с. 717-723.

[9]. Evans L.C., Gariepy R.F. *Wiener's criterion for the heat equation.* Arch. Rath. Mech.Ann., 1982, v.78, №4, p.293-314.

[10]. Мамедов И.Т., Новрузов А.А. *О внутренней априорной оценке решений вырождающихся эллиптических уравнений 2-го порядка.* ДАН СССР, *1984*, т.278, №1, с.320-324.

[11]. Муштагов Ф.М. *Внутренняя априорная оценка нормы Гёльдера решений неравномерно вырождающихся параболических уравнений 2-го порядка.* Известия АН Азерб., сер. мат.; мех., 1996, т.XVII, №1-3, с.149-154.

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QEYRİ MÜNTƏZƏM QÜVVƏTLƏ CIRLAŞAN 2-Cİ TƏRTİB PARABOLİK TƏNLİKLƏRİN SƏRHƏD HİSSƏLƏRİ

Məqalədə 2-ci tərtibli qüvvə sürəti ilə qeyri-müntəzəm cırlaşan parabolik tənliklər üçün sərhəd nöqtəsinin reqularlıq şərti tapılıb. Reqular sərhəd nöqtəsində həllin kəsilməzlik modulu qiymətləndirilib.

Муштагов Ф.М.

ГРАНИЧНЫЕ СВОЙСТВА РЕШЕНИЙ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ 2-ГО ПОРЯДКА С НЕРАВНОМЕРНЫМ СТЕПЕННЫМ ВЫРОЖДЕНИЕМ

В статье найдено достаточное условие регулярности граничной точки относительно первой краевой задачи для класса параболических уравнений 2-го порядка с неравномерным степенным вырождением. Получена оценка модуля непрерывности решения в регулярной граничной точке.